Lagrange Multipliers For Farmers
Assignment 7 Problem 4

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I'm going to try to compute the largest volume contained by a cylinder with a conical roof, assuming the total surface area of both the cylinder and the roof is fixed. I'll tackle the problem both for a conical roof of fixed pitch/slope, and for the pitch giving the max volume. The set up is according to Figure 1 where I've drawn the cross section.

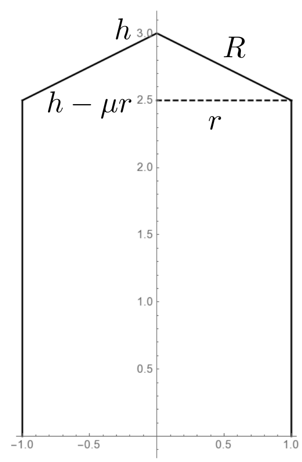

Figure 1: Cross section of a silo

1 Area

The area of the cylinder is $2\pi r(h - \mu r)$. The area A_{cone} of the conical roof satisfies

$$\frac{A_{cone}}{\pi R^2} = \frac{2\pi r}{2\pi R}$$

where $2\pi r$ is the circumference of the cylinder and R is the distance from the vertex of the cone to the base circle, that is, the length of the pitch of the roof. When the cone is flattened out, it is a sector of a circle of radius R and the ratio of the arc of that sector to the total circumference $2\pi R$, which is the ratio on the right gives the ratio of the area of the cone to the area of that circle. See Figure 2.

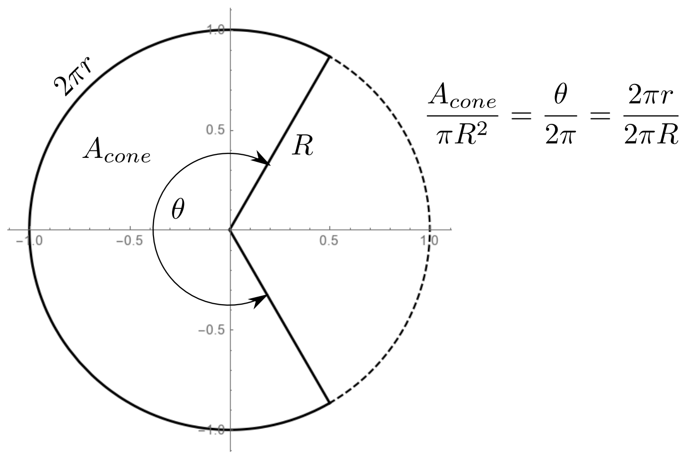

Figure 2: The conical roof of the silo, flattened into a plane

Note that $R = \sqrt{\mu^2 + 1}\, r$. Thus,

$$A_{cone} = \frac{\pi r R^2}{R} = \pi r R = \pi \sqrt{\mu^2 + 1}\, r^2.$$

Thus, the total area is

$$A_{total} = 2\pi r(h - \mu r) + \pi \sqrt{\mu^2 + 1}\, r^2 = \pi \left[2hr + \left(\sqrt{\mu^2 + 1} - 2\mu \right) r^2 \right],$$

and the constraint takes the form

$$2hr + \left(\sqrt{\mu^2 + 1} - 2\mu \right) r^2 = \frac{A}{\pi} \tag{1}$$

where A is some fixed constant.

2 Volume

The height of the cone is μr and the volume is

$$V_{cone} = \frac{1}{3}\pi r^2 \, \mu r = \frac{\pi}{3}\, \mu r^2.$$

Adding the volume $\pi r^2(h - \mu r)$ of the cylinder, we get

$$V_{total} = \frac{\pi}{3}(3hr^2 - 2\mu r^3).$$

Thus, we can minimize

$$v(h, r; \mu) = 3hr^2 - 2\mu r^3 \qquad \text{subject to} \qquad a(h, r; \mu) = 2hr + \left(\sqrt{\mu^2 + 1} - 2\mu\right) r^2 = \frac{A}{\pi}.$$

Notice that the volume function v is zero when $r = 0$, and for fixed μ, there is a natural maximum radius corresponding to the edge of the conical roof meeting the ground when $h - \mu r = 0$ and $\pi a(\mu r, r; \mu) = A_{cone} = A$. That is, when

$$\sqrt{\mu^2 + 1}\, r^2 = \frac{A}{\pi}$$

or

$$r_{end} = \frac{1}{\sqrt{\pi}} \frac{\sqrt{A}}{\sqrt[4]{\mu^2 + 1}} \qquad \text{and} \qquad h_{end} = \frac{\mu}{\sqrt{\pi}} \frac{\sqrt{A}}{\sqrt[4]{\mu^2 + 1}}.$$

Also, (1) is easily solved for h as a function of r:

$$h(r) = h(r; \mu) = \frac{1}{r} \left[\frac{A}{\pi} + \left(2\mu - \sqrt{\mu^2 + 1}\right) r^2 \right].$$

We have plotted in Figure 3 the value of the volume under the area constraint as a function of r for $0 \le r \le r_{end}$ with $A = 5$ and $\mu = 1/2$. The result is as expected. One guesses some convexity can be shown by a calculation.

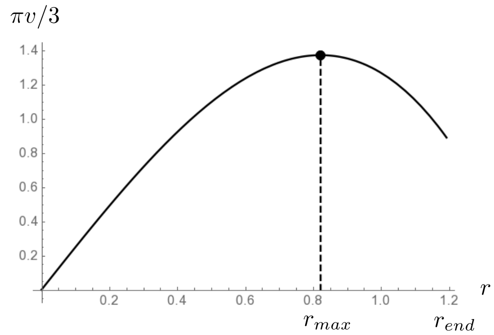

Figure 3: Volume as a function of radius where $v = v(h(r), r; 1/2)$ with $h(r)$ determined by the constraint. In this plot I've fixed $A = 5$ and I've also plotted the point given by the Lagrange multipliers solution given below in this case with $\mu = 1/2$.

3 Lagrange multipliers with μ fixed

We consider the system of equations $\nabla v = \lambda \nabla a$. This means we have

$$3r^2 = \lambda\, 2r$$
$$6hr - 6\mu r^2 = \lambda \left[2h + 2\left(\sqrt{\mu^2+1} - 2\mu\right) r\right].$$

We also have the constraint:

$$2hr + \left(\sqrt{\mu^2+1} - 2\mu\right) r^2 = \frac{A}{\pi}.$$

Writing $\lambda = 3r/2$ from the first equation, the second equation becomes

$$2h - 2\mu r = h + \left(\sqrt{\mu^2+1} - 2\mu\right) r.$$

Therefore,

$$h = \sqrt{\mu^2+1}\, r.$$

The constraint then reads

$$2\sqrt{\mu^2+1}\, r^2 + \left(\sqrt{\mu^2+1} - 2\mu\right) r^2 = \frac{A}{\pi}.$$

That is,

$$r_{max} = \sqrt{\frac{A}{\pi\left(3\sqrt{\mu^2+1} - 2\mu\right)}} \qquad \text{and} \qquad h_{max} = \sqrt{\frac{(\mu^2+1)A}{\pi\left(3\sqrt{\mu^2+1} - 2\mu\right)}}.$$

4 Optimal Pitch

If we maximize also over μ, then we have another equation corresponding to

$$\frac{\partial v}{\partial \mu} = \lambda \frac{\partial a}{\partial \mu}.$$

That is,

$$-2r^3 = \lambda \left(\frac{\mu}{\sqrt{\mu^2+1}} - 2 \right) r^2.$$

The optimal choices for $\lambda = 3r/2$ and $r = r_{max}$ remain unchanged:

$$-2r_{max} = \frac{3}{2} r_{max} \left(\frac{\mu}{\sqrt{\mu^2+1}} - 2 \right).$$

That is,

$$-4 = \frac{3\mu}{\sqrt{\mu^2+1}} - 6 \qquad \text{or} \qquad 3\mu = 2\sqrt{\mu^2+1} > 0.$$

Therefore, $\mu = \mu_{max} = 2/\sqrt{5}$ gives the optimal pitch. The corresponding dimensions of the silo are

$$r_{max} = \sqrt{\frac{A}{\pi}} \frac{1}{\sqrt[4]{5}} \qquad \text{and} \qquad h_{max} = \frac{3}{5^{3/4}} \sqrt{\frac{A}{\pi}}.$$

5 Verification

We note, finally, that we can use the constraint (1) to express h as a function of r and μ:

$$h(r, \mu) = \frac{1}{2r} \left[\frac{A}{\pi} - \left(\sqrt{\mu^2+1} - 2\mu \right) r^2 \right].$$

This allows us to plot the volume as a function of r and μ and verify the formulas for r_{max}, h_{max} and μ_{max} give above at least visually.

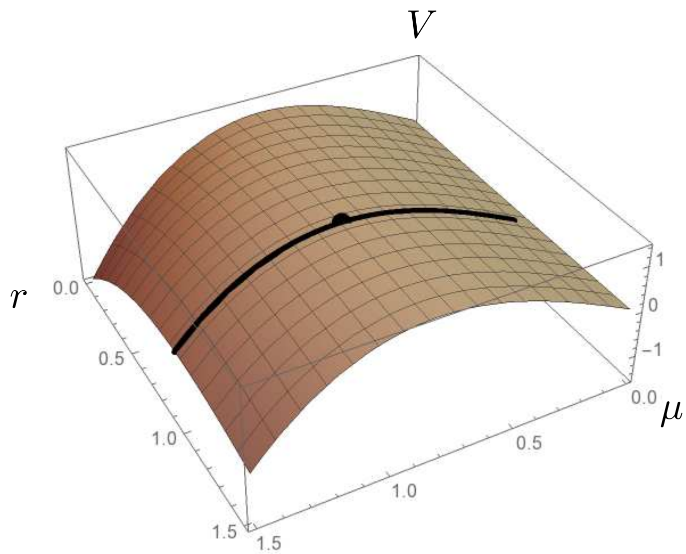

Figure 4: Volume as a function of radius and μ where $v = v(h(r,\mu), r; \mu)$ with $h(r,\mu)$ determined by the constraint. In this plot I've fixed $A = 5$ and I've also plotted the restriction of the volume $\pi v/3$ to the line $\mu = \mu_{max} = 2/\sqrt{5}$ and the point given by the Lagrange multipliers solution $(r_{max}, \mu_{max}, \pi v_{max}/3)$.