

Poisson's Equation

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1 Preview

I want to present to you a proof of the following existence and uniqueness theorem for weak solutions of the homogeneous boundary value problem for Poisson's equation:

Theorem 1 *Given a bounded open subset $U \subset \mathbb{R}^n$ and any $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of*

$$\begin{cases} \Delta u = f \text{ on } U, \\ u|_{\partial U} \equiv 0. \end{cases}$$

In order to understand the statement fully, you need to know what it means to be a **weak solution** of the problem. In order to know that, you need basically two things:

1. The space of functions $H_0^1(U)$.
2. Integration on $U \subset \mathbb{R}^n$.

Both of these things are relatively easy and will be covered either elsewhere or below. For now, if you don't know one or both of them, do not worry. Let me go ahead and give the formulation and make some comments. Again, all details that are not clear should become clear at some point.

2 Weak Formulation

Definition 1 *(weak formulation of the homogeneous boundary value problem for Poisson's PDE) Let U be an open subset of \mathbb{R}^n and $f \in L^2(U)$. We say $u \in H_0^1(U)$*

is a **weak solution** of

$$\begin{cases} \Delta u = f \text{ on } U, \\ u|_{\partial U} \equiv 0. \end{cases}$$

if

$$-\int_U Du \cdot D\phi = \int_U f\phi \quad \text{for all } \phi \in C_c^\infty(U). \quad (1)$$

Recall that $L^2(U)$ is an inner product space with inner product

$$\langle u, v \rangle_{L^2} = \int_U uv.$$

Therefore, the condition for a weak solution u given in (1) may be written as

$$B[u, \phi] = -\langle f, \phi \rangle_{L^2} \quad \text{for all } \phi \in C_c^\infty(U)$$

where $B : H_0^1(U) \times H_0^1(U) \rightarrow \mathbb{R}$ is the bilinear form given by

$$B[u, v] = \int_U Du \cdot Dv. \quad (2)$$

You may not know it yet, but the space $H_0^1(U)$ is a subspace of the space $W^{1,2}(U)$ of L^2 functions with first order weak derivatives in L^2 . In fact, $W^{1,2}(U)$ is also called $H^1(U)$ because $W^{1,2}(U)$ is a **Hilbert space**. Again if you don't know what it means to be a Hilbert space, don't worry. You will soon. Just think of it as a really really nice function space based on integration. But the really important point here is that functions in $H_0^1(U)$, which is also called $W_0^{1,2}(U)$, have weak derivatives. This will become more natural in time, but the derivatives appearing in (1) and (2) are **weak derivatives**. Let me pause to remind you about how weak derivatives work:

3 Weak Derivatives

Definition 2 We say $u \in W^{1,2}(U) = H^1(U)$ has a weak derivative $g_j \in L^2(U)$ for some particular index $j \in \{1, 2, 3, \dots, n\}$ (corresponding to a standard unit direction \mathbf{e}_j) if

$$-\int_U u D_j \phi = \int_U g_j \phi = \langle g_j, \phi \rangle_{L^2} \quad \text{for all } \phi \in C_c^\infty(U). \quad (3)$$

A weak derivative $g_j \in L^2(U)$, when it exists, is denoted by $D_j u$. (Yes, this is the same notation used for classical derivatives, but you'll get used to it.)

The definition is almost precisely the same in $W^{1,p}(U)$:

Definition 3 We say $u \in W^{1,p}(U)$ has a weak derivative $g_j \in L^p(U)$

$$-\int_U u D_j \phi = \int_U g_j \phi = \langle g_j, \phi \rangle_{L^2} \quad \text{for all } \phi \in C_c^\infty(U). \quad (4)$$

We write $g_j = D_j u$.

Remember that the functional $\mathcal{G} : C_c^\infty(U) \rightarrow \mathbb{R}$ by $\mathcal{G}[\phi] = \langle g, \phi \rangle_{L^2}$ is the “operator version” of the function g .

Exercise 1 Remind yourself, using the fundamental lemma of the calculus of variations, that knowing $g \in L^1_{loc}(U)$ and knowing $\mathcal{G} : C_c^\infty(U) \rightarrow \mathbb{R}$ are the same thing.

What is happening on the left side in (3) and (4) is a touch complicated, so let’s discuss that for a moment. You see the **functional** $\mathcal{G} : C_c^\infty(U) \rightarrow \mathbb{R}$ above which gives an operator version of the function $g : U \rightarrow \mathbb{R}$. This kind of representation is often/usually used somewhat informally, but for clarity, let’s be a little more formal about it. Let us denote by \mathfrak{F} the collection (or maybe more properly *some* collection) of functionals $\mathcal{F} : C_c^\infty(U) \rightarrow \mathbb{R}$. Remember $C_c^\infty(U)$ represents the set of **test functions**, the idea being that

Whatever $\mathcal{F} \in \mathfrak{F}$ is representing, you get that information from the values of \mathcal{F} on test functions.

This is a super important idea, and your main example is the one given above for functions. I’ll repeat: If $g \in L^1_{loc}(U)$ is a real valued function, then g has a representative in \mathfrak{F} given by $\mathcal{G} \in \mathfrak{F}$ with $\mathcal{G}[\phi] = \int g\phi$. This is called an **integral functional**. In practice, if we want to nail things down, we need to **specify a norm** on the set of test functions $C_c^\infty(U)$. There are lots of different choices for this norm, but once we choose a norm, then we can say more precisely that we want the functionals in \mathfrak{F} to be those which are **continuous** with respect to that norm. We also want them to be **linear**. In this way, the test functions $C_c^\infty(U)$ become a **normed space** (usually a **subspace** of some other normed space—think $C_c^\infty(U) \subset L^2(U)$). Then, in summary, \mathfrak{F} is the collection of continuous linear functionals on $C_c^\infty(U)$, once we specify the norm.

The integral functionals like \mathcal{G} associated to functions g are essentially always examples of continuous linear functionals in \mathfrak{F} , but the integral functionals corresponding to functions are not the only kind of continuous linear functionals in \mathfrak{F} . In

any case, with some formal notion of \mathfrak{F} in hand, let's tackle the left sides of (3) and (4):

The operator on the left $L_j : L_{loc}^1 \rightarrow \mathfrak{F}$ gives the value of a functional $\mathcal{D}_j \in \mathfrak{F}$. This operator L_j is the operator corresponding to **weak differentiation**. The form used here is motivated by integration by parts,¹ and this is sometimes called a **weak adjoint operator**:

$$\mathcal{D}_j[\phi] = - \int_U u D_j \phi.$$

So this is a little complicated. L_j is “weak differentiation,” which means $L_j[u]$ is supposed to somehow represent differentiating u . Of course, u doesn't have a classical derivative, so the “differentiation” is expressed in terms of a functional (motivated by integration by parts): $L_j u = \mathcal{D}_j : C_c^\infty(U) \rightarrow \mathbb{R}$ with

$$\mathcal{D}_j[\phi] = - \int_U u D_j \phi \quad (\text{with } D_j \phi \text{ appearing here being a classical derivative}).$$

4 Weak Poisson's Equation

For the weak formulation of $\Delta u = f$, we can imagine multiplying by a smooth test function ϕ and integrating by parts (again, this really means using the divergence theorem):

$$\int_U \Delta u \phi = \int_U f \phi.$$

That is,

$$- \int_U Du \cdot D\phi = \langle f, \phi \rangle_{L^2} = \mathcal{F}[\phi]. \tag{5}$$

The expression on the left now, if u has classical derivatives, has an integrand

$$Du \cdot D\phi = \sum_{j=1}^n D_j u D_j \phi = \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \right) \left(\frac{\partial \phi}{\partial x_j} \right).$$

¹Actually, you may need to brush up on your multivariable calculus/integration to fully understand this condition. It's not quite just integration by parts as you know it from 1-D calculus. It's multivariable integration by parts using the **divergence theorem**, which we will review relatively soon. But at least it looks like 1-D integration by parts.

We observe that we could replace the classical derivatives of u with weak derivatives $g_j = D_j u \in L^1_{loc}$ in this last expression. In fact, the expression

$$B[u, \phi] = \int_U Du \cdot D\phi$$

makes perfectly good sense for $u \in W^1_{loc}(U)$. However, at this point, we're going to make two restrictions, or you can think of it as one restriction, to the space $H^1_0(U)$. The restrictions work together in sort of a complicated way, though they are essentially easy to state:

1. We choose to work with the L^2 norm. Essentially, we restrict u to $H^1(U) = W^{1,2}(U)$. This gives us an inner product structure which is crucial for the proof.
2. We restrict further to the subspace $H^1_0(U) = W^{1,2}_0(U)$ which is the **closure** with respect to the $W^{1,2}$ norm of the subspace $C^\infty_c(U) \subset H^1(U)$. This does two things. First it takes care of the boundary condition, or at least we can think of this restriction as imposing a weak version of the boundary condition

$$u|_{\partial U} \equiv 0.$$

But second, and almost equally importantly for us, this condition turns out to make the natural extension of our bilinear form B which started out as $B : H^1_0(U) \times C^\infty_c(U) \rightarrow \mathbb{R}$ by

$$B[u, \phi] = - \sum_{j=1}^n L_j[D_j u][\phi] = - \sum_{j=1}^n \mathcal{D}_{jj}[\phi] = \sum_{j=1}^n \int_U D_j u D_j \phi$$

and extends to $B : H^1_0(U) \times H^1_0(U) \rightarrow \mathbb{R}$ by

$$B[u, v] = \int_U Du \cdot Dv$$

into an **inner product** in its own right.

There are more than a few comments to make about this, and there are more than a few questions for you to ask. We'll get to those in due time. For now, I want to point out that the difficult part is in what we've done above and in the details and questions associated with it. Once we have chosen and understood the correct spaces, framework, and operators, the proof is relatively easy. The proof is so "easy" that I'm going to give it now.

5 Proof of Theorem 1

There are three steps.

1. The bilinear form $B : H_0^1(U) \times H_0^1(U) \rightarrow \mathbb{R}$ by

$$B[u, v] = \int_U Du \cdot Dv$$

is an inner product. (We need to show this.)

2. The continuous linear functional $\mathcal{F} : C_c^\infty(U) \rightarrow \mathbb{R}$ by

$$\mathcal{F}[\phi] = \int_U f\phi = \langle f, \phi \rangle_{L^2}$$

extends to a continuous linear functional on $H_0^1(U)$ by the same formula. This is essentially obvious since $H_0^1(U) \subset L^2(U)$, though something does need to be checked with the norms. Note that we're using the $W^{1,2}$ norm on the test functions and for the closure $H_0^1(U)$. Let's call this extension $\mathcal{F}_0 : H_0^1(U) \rightarrow \mathbb{R}$ by

$$\mathcal{F}_0[v] = \int_U fv = \langle f, v \rangle_{L^2}$$

and note that $-\mathcal{F}_0 : H_0^1(U) \rightarrow \mathbb{R}$ is also a continuous linear functional, so that our equation (i.e., weak formulation of the boundary value problem) now reads

$$B[u, \phi] = -\mathcal{F}_0[\phi] \quad \text{for all } \phi \in C_c^\infty(U).$$

3. The **Riesz representation theorem** (in this special case) says that given any continuous linear functional $-\mathcal{F}_0 : H_0^1(U) \rightarrow \mathbb{R}$, there exists a unique $u \in H_0^1(U)$ such that

$$B[u, v] = -\mathcal{F}_0[v] \quad \text{for all } v \in H_0^1(U).$$

In particular,

$$B[u, \phi] = -\mathcal{F}_0[\phi] \quad \text{for all } \phi \in C_c^\infty(U). \quad \square$$

There is a little bit to understand about continuous linear functionals and the Riesz representation theorem, but that material is really easy (and fun).

6 The Weak Laplace Operator

As a final note, let me mention that the expression appearing on the left in (5) and generally associated with the bilinear form $B[u, \phi]$, or more properly $-B[u, \phi]$ given by

$$\Delta^w[\phi] = - \int_U D_j u D_j \phi$$

may be considered a **weak Laplace operator**. Precisely, we consider $\mathcal{L} : H^1(U) \rightarrow \mathfrak{F}$ by $\mathcal{L}[u] = \Delta^w$ with $\Delta^w[\phi]$ given above. This is what it “means” to take the Laplacian of a function $u \in H^1(U)$, or this is how we think about the Laplacian for these functions. Basically, $u \in H^1(U)$ does not have even two weak derivatives; this function u certainly does not have a classical Laplacian. It only has one weak derivative (or more properly weak derivatives of order one). But, as the theorem illustrates, it is still sometimes useful to try to talk about the Laplacian of such a function.

Exercise 2 *What is the domain and codomain of Δ^w above?*

Exercise 3 *Consider the general second order linear partial differential operator $L : C^2(U) \rightarrow C^0(U)$ given in divergence form by*

$$L[u] = \sum_{j=1}^n D_j [a_{ij} D_j u] + \sum_{j=1}^n b_j D_j u + cu$$

where the coefficients satisfy $a_{ij} \in C^1(U)$ and $b_j, c \in C^0(U)$. Formulate a weak version of this operator for functions $u \in H^1(U)$. What (relaxed) regularity may be assumed on the coefficients?

7 Regularity

Theorem 2 (regularity) *If $f \in C^\infty(\overline{U})$ and U is a bounded open subset of \mathbb{R}^2 with C^∞ boundary, then the weak solution $u \in H_0^1(U)$ given by Theorem 1 has*

$$u \in C^\infty(\overline{U}).$$