Fundamental Solutions and Green's Functions For the Laplace Operator

John McCuan

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We have studied extensively the Green's function for the trivial ordinary differential equation $-u'' = f$ and the two point boundary value problem

$$\begin{cases} -u'' = f, \ x \in (a, b) \\ u(a) = u(b) = 0 \end{cases}$$

with homogeneous boundary values in particular. In fact, I think (almost) the very first homework assignment I gave in this class was to solve this equation for some specific inhomogeneities f. Now, I'm going to cast all our complicated manipulations in one dimension into a less trivial setting, namely the setting of the Laplace operator

$$\Delta : C^2(\overline{U}) \to C^0(\overline{U})$$

where U is a bounded open domain in $\mathbb{R}^n$ with smooth C^2 boundary.

1 Fundamental Solutions

For each $n = 1, 2, 3, \ldots$ there is a fundamental solution. Each is determined up to an additive constant as a solution, satisfying certain symmetry and regularity requirements, of the distributional partial differential equation "$-\Delta\Phi = \delta_0$," that is

$$\int_{\mathbb{R}^n} \Phi\,(-\Delta\phi) = \phi(0) \qquad \text{for every } \phi \in C_c^\infty(\mathbb{R}^n).$$

The symmetry requirement is that $\Phi(\mathbf{x}) = \Phi_0(|\mathbf{x}|)$ for some function $\Phi_0 : (0, \infty) \to \mathbb{R}$. The regularity requirement is that $\Phi_0 \in C^2(0, \infty)$. In one dimension, the symmetry condition amounts to the requirement that $\Phi_0 = \Phi_0(x)$ is even, and we have seen $\Phi(x) = -|x|/2$. Each solution will be singular at the origin in $\mathbb{R}^n$.

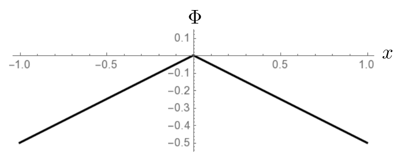

Figure 1: The fundamental solution in one dimension

Exercise 1 *Show* $\Phi(x) = -|x|/2$ *is the unique fundamental solution (up to an additive constant) when* $n = 1$.

In higher dimensions, the fundamental solutions associated with the Laplace operator are these:

$$\Phi(\mathbf{x}) = -\frac{1}{2\pi} \ln |\mathbf{x}|, \qquad (n = 2) \tag{1}$$

$$\Phi(\mathbf{x}) = \frac{1}{n(n-2)\omega_n} \frac{1}{|\mathbf{x}|^{n-2}}, \qquad (n > 2). \tag{2}$$

where ω_n is the "volume," i.e., n dimensional Lebesgue measure, of the unit ball $B_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$ in $\mathbb{R}^n$. You know $\omega_1 = 2$, $\omega_2 = \pi$, and $\omega_3 = 4\pi/3$. You may not know that $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ in general. But now you know. You also may not have noticed that the $n-1$ dimensional Hausdorff measure (this means counting measure when $n = 1$, length when $n = 2$, area when $n = 3$ etc.) of the boundary of the unit ball is $n\omega_n$, but that is indeed the case, and now you know. We'll use this below. Up until this point, I've been using $\mathbf{x}$ to denote points in $\mathbb{R}^n$. I'm not going to switch and use x, ξ, etc. We'll just have to remember that $x = (x_1, x_2, \ldots, x_n)$ has multiple components.

Now, we consider the function $\Phi(x - \xi)$ where we translate the singularity to a point $\xi \in U$. This gives us some nice smooth boundary values to consider on ∂U as indicated in Figure 4. In particular, we define $w = w(x, \xi)$ as the solution of the boundary value problem

$$\begin{cases} \Delta w = 0, \; x \in U \\ w\Big|_{x \in \Omega} = \Phi(x - \xi). \end{cases} \tag{3}$$

This function $w = w(x, \xi)$ may be called the **corrector** for the fundamental solution. It is obvious from the symmetry that $\Phi(x - \xi) = \Phi(\xi - x)$, but it is not obvious at all that the corrector is symmetric. But it is true.

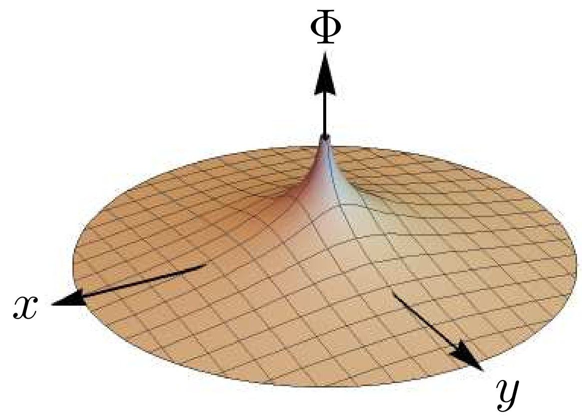

Figure 2: The fundamental solution in two dimensions

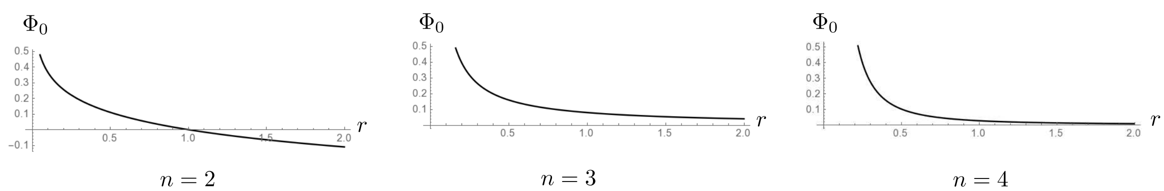

Figure 3: The profiles of fundamental solutions in $n = 2$, $n = 3$, and $n = 4$ dimensions

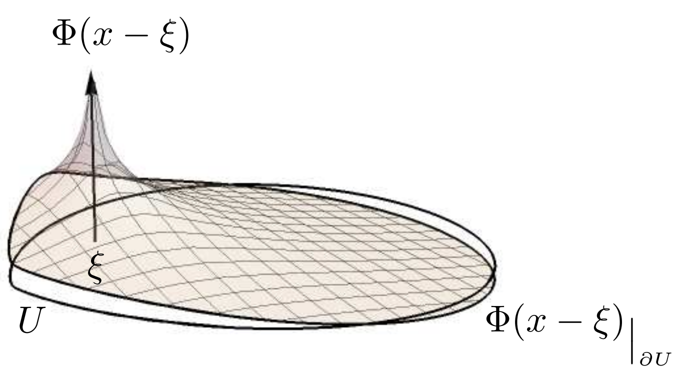

Figure 4: Boundary values obtained by translating the fundamental solution

Theorem 1 $w(x,\xi) = w(\xi, x)$.

Consequently, the Green's function

$$G(x,\xi) = \Phi(x-\xi) - w(x,\xi) \tag{4}$$

shares the same symmetry. We will prove this symmetry property later.

Recall that the main expectation of a Green's function is that it is an **integral kernel** which can be used to write down a formula for a solution of a certain problem. In this case, we claim

$$u(x) = \int_{\xi\in U} f(\xi)\, G(x,\xi) \tag{5}$$

solves

$$\begin{cases} -\Delta u = f, \;\; x \in U \\ u\Big|_{x\in\Omega} = 0. \end{cases} \tag{6}$$

In fact, we will show more. The usual approach to showing this result depends on something called **Green's formula**. We have discussed the divergence theorem

$$\int_U \operatorname{div} \mathbf{v} = \int_{\partial U} \mathbf{v} \cdot n$$

and the generalization arising from the product rule $\operatorname{div}(u\mathbf{v}) = Du \cdot \mathbf{v} + u \operatorname{div} \mathbf{v}$. In particular, when $\mathbf{v} = Dv$ is the gradient of a function, then we obtain the identity

$$\int_U u\, \Delta v + \int_U Du \cdot Dv = \int_{\partial U} u\, Dv \cdot n.$$

Note the quantity $D_n v = Dv \cdot n$ is called the **outward normal derivative** of v. Green's formula takes this one step further by switching the roles of u and v and then subtracting:

$$\int_U (u\Delta v - v\Delta u) = \int_{\partial U} (u\, Dv - v\, Du) \cdot n. \tag{7}$$

In order to apply Green's formula, we replace U with $U_\epsilon = U \backslash B_\epsilon(\xi)$ where $B_\epsilon(\xi) \subset\subset U$ and we take $v(x) = G(x,\xi)$. This yields

$$\int_{U_\epsilon} (u\Delta G - G\Delta u) = \int_{\partial U_\epsilon} (u\, DG - G\, Du) \cdot n. \tag{8}$$

The first term on the left

$$\int_{U_\epsilon} u\Delta G$$

vanishes since both the fundamental solution $\Phi(x-\xi)$ and the corrector $w(x,\xi)$ are harmonic in x for $x \in U_\epsilon$. Furthermore, if we assume u is a (classical) solution of (6), then the second integral on the left becomes

$$\int_{x \in U_\epsilon} f(x)\, G(x,\xi).$$

In view of the symmetry, this may also be written as

$$\int_{x \in U_\epsilon} f(x)\, G(\xi,x)$$

matching our proposed formula for the solution value $u(\xi)$ given in (5). More generally, assuming we have a function $u \in C^2(\overline{U})$ satisfying $-\Delta u = f$, we have

$$\int_{U_\epsilon} f(x)\, G(\xi,x) = \int_{\partial U_\epsilon} (u\, DG - G\, Du) \cdot n.$$

The boundary integrals on the right include integrals around ∂U as well as around $\partial B_\epsilon(\xi)$ with the unit normal $n = -(x-\xi)/|x-\xi|$ pointing into $B_\epsilon(\xi)$. For example,

$$\int_{\partial U_\epsilon} u\, DG \cdot n = \int_{\partial U} u\, DG \cdot n + \int_{\partial B_\epsilon(\xi)} u\, DG \cdot n.$$

Decomposing G further as $G(x,\xi) = \Phi(x-\xi) - w(x,\xi)$ the second integral on the right may be written as

$$\int_{\partial B_\epsilon(\xi)} u\, DG \cdot n = \int_{\partial B_\epsilon(\xi)} u\, D\Phi \cdot n \int_{\partial B_\epsilon(\xi)} u\, Dw \cdot n.$$

The second of these integrals has bounded integrand $u\, Du \cdot n$ and, therefore, satisfies

$$\lim_{\epsilon \searrow 0} \int_{\partial B_\epsilon(\xi)} u\, Dw \cdot n = 0.$$

Calculating for $n \geq 2$, we see

$$D\Phi(\mathbf{x}) = -\frac{1}{2\pi} \frac{\mathbf{x}}{|\mathbf{x}|^2}, \qquad (n=2) \tag{9}$$

$$\Phi(\mathbf{x}) = -\frac{1}{n\omega_n} \frac{\mathbf{x}}{|\mathbf{x}|^n}, \qquad (n>2). \tag{10}$$

Consequently, the first integral becomes

$$\begin{aligned}\int_{\partial B_\epsilon(\xi)} u\, D\Phi\cdot n &= \frac{1}{n\omega_n}\int_{\partial B_\epsilon(\xi)} u\,\frac{x-\xi}{\epsilon^n}\cdot\frac{x-\xi}{\epsilon}\\ &= \frac{1}{n\omega_n}\int_{\partial B_\epsilon(\xi)}\frac{u}{\epsilon^{n-1}}\\ &= \frac{1}{\mu\partial B_\epsilon(\xi)}\int_{\partial B_\epsilon(\xi)} u.\end{aligned}$$

Notice this is an average value so that

$$\lim_{\epsilon\searrow 0}\int_{\partial B_\epsilon(\xi)} u\, D\Phi\cdot n = u(\xi).$$

We now consider the last term

$$-\int_{\partial U_\epsilon} G\, Du\cdot n$$

in (8). Since $G\equiv 0$ on ∂U, we have here only

$$-\int_{\partial U_\epsilon} G\, Du\cdot n = \int_{\partial B_\epsilon(\xi)} w\, Du\cdot n - \int_{\partial B_\epsilon(\xi)} \Phi\, Du\cdot n.$$

The first integrand on the right is bounded, so

$$\lim_{\epsilon\searrow 0}\int_{\partial B_\epsilon(\xi)} w\, Du\cdot n = 0.$$

The growth rate of the fundamental solution also gives

$$\lim_{\epsilon\searrow 0}\int_{\partial B_\epsilon(\xi)} \Phi\, Du\cdot n = 0.$$

In fact,

$$\left|\int_{\partial B_\epsilon(\xi)} \Phi\, Du\cdot n\right| \le \sup|Du|\Phi(\epsilon)n\omega_n\epsilon^{n-1}.$$

Returning once again to the second term

$$\int_{x\in U_\epsilon} f(x)\, G(x,\xi) = \int_{x\in U_\epsilon} f(x)\,\Phi(x-\xi) - \int_{x\in U_\epsilon} f(x)\, w(x,\xi)$$

in (8) we may restrict to $B_r(\xi)$ with $\epsilon < r$ and $\Phi(x-\xi) > 0$ on $B_r(\xi)$ to calculate

$$\lim_{\epsilon\searrow 0}\int B_r(\xi)\backslash B_\epsilon(\xi)\Phi(x-\xi) < \infty.$$

Consequently, combining these calculations in the limit, we have

$$\int_{x\in U_\epsilon} f(x)\, G(\xi,x) = \int_{\partial U} u\, DG\cdot n + u(\xi).$$

Exchanging the roles/names of x and ξ, we arrive at our final formula:

$$u(x) = \int_{\xi\in U_\epsilon} f(\xi)\, G(x,\xi) - \int_{\xi\in\partial U} u(\xi)\, DG(x,\xi)\cdot n.$$

If $u \equiv 0$ on ∂U, then the second integral on the right vanishes. If u takes other boundary values, the formula we have still holds, so that we have a formula for classical solutions $u \in C^2(\overline{U})$ satisfying

$$\begin{cases} -\Delta u = f, \; x\in U \\ u\Big|_{x\in\Omega} = g, \end{cases} \tag{11}$$

namely

$$u(x) = \int_{\xi\in U_\epsilon} f(\xi)\, G(x,\xi) - \int_{\xi\in\partial U} g(\xi)\, DG(x,\xi)\cdot n.$$

It remains to show the symmetry $G(x,\xi) = G(\xi,x)$ of the Green's function. To see this, note that $u(x) = G(x,\xi)$ is harmonic in $U\backslash\{\xi\}$ with a singularity at $x = \xi$ while $v(\xi) = G(\xi,x)$ is harmonic in $U\backslash\{x\}$ with a singularity at $\xi = x$. We may apply Green's formula integrating with respect to some variable other than x or ξ:

$$\int_{\eta\in U_\epsilon} u\Delta v - v\Delta u) = \int_{\partial U_\epsilon} (u\, Dv - v\, Du)\cdot n.$$

using the approach above with $U_\epsilon = U\backslash(B_\epsilon(x)\cup B_\epsilon(\xi))$. We evidently get

$$\begin{aligned} 0 = \int_{\eta\in\partial B_\epsilon(x)} &[G(\eta,\xi)\, DG(\eta,x) - G(\eta,x)\, DG(\eta,\xi)] \\ &+ \int_{\eta\in\partial B_\epsilon(\xi)} [G(\eta,\xi)\, DG(\eta,x) - G(\eta,x)\, DG(\eta,\xi)]. \end{aligned}$$

Taking the limit as $\epsilon \searrow 0$ as above, we see that exactly two limits do not vanish:

$$0 = G(x,\xi) - G(\xi,x).$$