Mollifiction
(rough draft)

John McCuan

April 14, 2020

The mollifiction we will discuss[1] is based on the non-negative symmetric mollifier (also sometimes called the standard bump function) $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Using β, we define $\mu_1 : \mathbb{R} \to \mathbb{R}$ by

$$\mu_1(x) = \frac{\beta(x)}{\int_{\mathbb{R}} \beta}.$$

More generally, for $\delta > 0$, we define $\mu_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\mu_\delta(x) = \frac{1}{\delta}\, \mu_1\left(\frac{x}{\delta}\right).$$

Given $u \in L^1_{loc}(\mathbb{R})$, the **mollification** of u is given by $\mu_\delta * u : \mathbb{R} \to \mathbb{R}$ by

$$\mu_\delta * u(x) = \int_{\xi \in \mathbb{R}} \mu_\delta(\xi) u(x - \xi).$$

The function $\mu_\delta * u$ is called a **convolution** of μ_δ and u.

The construction above may be generalized to higher dimensions as follows: We start with $\beta : \mathbb{R}^n \to \mathbb{R}$ by

$$\beta(\mathbf{x}) = \begin{cases} e^{-\frac{1}{1-|\mathbf{x}|^2}}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \geq 1. \end{cases}$$

[1] In other contexts this may be called symmetric (or standard) mollification. The basic idea can be extended to a general mollifier $\mu \in C^\infty_c(\mathbb{R})$ with $\int \mu = 1$.

$\mu_1 : \mathbb{R}^n \to \mathbb{R}$ by

$$\mu_1(\mathbf{x}) = \frac{\beta(\mathbf{x})}{\int_{\mathbb{R}^n} \beta}.$$

For $\delta > 0$, we define $\mu_\delta : \mathbb{R}^n \to \mathbb{R}$ by

$$\mu_\delta(\mathbf{x}) = \frac{1}{\delta^n} \mu_1 \left(\frac{\mathbf{x}}{\delta} \right).$$

Given $u \in L^1_{loc}(\mathbb{R}^n)$, the **mollification** of u is given by $\mu_\delta * u : \mathbb{R}^n \to \mathbb{R}$ by

$$\mu_\delta * u(\mathbf{x}) = \int_{\xi \in \mathbb{R}} \mu_\delta(\xi) u(\mathbf{x} - \xi).$$

1 Important Preliminary Observations

I will state these observations for $n = 1$ and leave the generalizations to $\mathbb{R}^n$ as exercises.

1.1 Regularity and Support

The standard bump function satisfies

$$\beta \in C_c^\infty(\mathbb{R}) \qquad \text{with} \qquad \operatorname{supp} \beta = [-1, 1].$$

The standard mollifier μ_1 satisfies

$$\mu_1 \in C_c^\infty(\mathbb{R}) \qquad \text{with} \qquad \operatorname{supp} \mu_1 = [-1, 1].$$

More generally

$$\mu_\delta \in C_c^\infty(\mathbb{R}) \qquad \text{with} \qquad \operatorname{supp} \mu_\delta = [-\delta, \delta].$$

All of these functions are non-negative and even. Furthermore,

$$\int_{\mathbb{R}} \mu_1 = 1.$$

In fact,

$$\int_{\mathbb{R}} \mu_\delta = \frac{1}{\delta} \int_{x \in \mathbb{R}} \mu_1 \left(\frac{x}{\delta} \right) = \int_{\xi \in \mathbb{R}} \mu_1 (\xi) = 1.$$

We have used the change of variables $\xi = x/\delta$.

The mollification $\mu_\delta * u$ satisfies $\mu_\delta * u \in C^\infty(\mathbb{R})$. Also, if u has compact support (or essential compact support), then $\mu_\delta * u \in C_c^\infty(\mathbb{R})$.

Exercise 1 *Determine the support of $\mu_\delta * u$ when u is non-negative. Consider also the case $u : \mathbb{R}^n \to \mathbb{R}$.*

Note that

$$\mu_\delta * u(x) = \int_{\xi \in B_\delta(0)} \mu_\delta(\xi) u(x - \xi).$$

Also, the commutativity of the convolution is key to seeing the regularity of the mollification:

$$\mu_\delta * u(x) = u * \mu_\delta(x) = \int_{\xi \in \mathbb{R}} u(\xi)\mu_\delta(x - \xi) = \int_{\xi \in B_\delta(x)} u(\xi)\mu_\delta(x - \xi).$$

Exercise 2 *Verify the commutativity of the convolution using the change of variables $\eta = x - \xi$. Consider also the case when $\mu_\delta, u : \mathbb{R}^n \to \mathbb{R}$.*

The commutativity allows one to differentiate under the integral sign:

$$\frac{d}{dx}\mu_\delta * u = \frac{d}{dx}\int_{\xi \in \mathbb{R}} u(\xi)\mu_\delta(x - \xi) = \int_{\xi \in \mathbb{R}} u(\xi)\mu_\delta'(x - \xi) = \mu_\delta' * u.$$

1.2 Approximation and Convergence

The integral functional associated with μ_δ is $M_\delta : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$M_\delta[\phi] = \int \mu_\delta \phi.$$

As distributions

$$\lim_{\delta \searrow 0} M_\delta = \delta_0$$

where δ_0 is the Dirac delta distribution (or evaluation functional) given by $\delta_0[\phi] = \phi(0)$.

If $u \in C^0(\mathbb{R})$, then

$$\lim_{\delta \searrow 0} \mu_\delta * u(x) = \lim_{\delta \searrow 0} \int_{\xi \in \mathbb{R}} \mu_\delta(\xi) u(x - \xi) = u(x) = \delta_x[u].$$

More generally, if $u \in C^k(\mathbb{R})$, then for any compact set $K \subset \mathbb{R}$

$$\lim_{\delta \searrow 0} \|\mu_\delta * u - u\|_{C^k(K)} = 0.$$

That is, $\mu_\delta * u$ converges to (and approximates) u in $C^k(K)$.

For any $u \in L^1_{loc}(\mathbb{R})$,

$$\lim_{\delta \searrow 0} \mu_\delta * u(x) = u(x) \qquad \text{at every Lebesgue point } x \text{ of } u.$$

2 Some Elementary Computations

2.1 Mollification of a constant

If $u \equiv c$ is constant, then $\mu_\delta * u \equiv c$.

2.2 Mollification of an affine function

If $u(x) = x$, then

$$\mu_\delta * u(x) = \int_{\xi\in\mathbb{R}} (x-\xi)\mu_\delta(\xi) = x\int \mu_\delta - \int \xi\mu_\delta(\xi) = x.$$

Notice that the symmetry of the mollifier μ_δ is required here to conclude

$$\int \xi\mu_\delta(\xi) = 0.$$

Explicitly, using the change of variables $\eta = -\xi$, we have

$$\int \xi\mu_\delta(\xi) = \int_{-\delta}^{0} \xi\mu_\delta(\xi)\, d\xi + \int_0^\delta \xi\mu_\delta(\xi)\, d\xi = \int_\delta^0 \eta\mu_\delta(-\eta)\, d\eta + \int_0^\delta \xi\mu_\delta(\xi)\, d\xi = 0.$$

The symmetry leading to the generalization of this result to higher dimensions is rather interesting.

2.3 Mollification of a quadratic function

If $u(x) = x^2$, then

$$\mu_\delta * u(x) = \int_{\xi\in\mathbb{R}} (x-\xi)^2\mu_\delta(\xi) = x^2 - 2x\int \xi\mu_\delta(\xi) + \int \xi^2\mu_\delta(\xi) = x^2 + c$$

where

$$c = \int \xi^2\mu_\delta(\xi) > 0.$$

Exercise 3 *Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x,y) = x^2 - y^2$, then $\mu_\delta * u(x,y) = x^2 - y^2$.*

Note that we have shown the mollification of every (classically) harmonic function $u : \mathbb{R} \to \mathbb{R}$ satisfies $\mu_\delta * u = u$. Notice also that $u(x, y) = x^2 - y^2$ is harmonic on $\mathbb{R}^2$. Furthermore, if $u \in C^2(\mathbb{R})$ is harmonic, then we an differentiation under the integral sign directly to see

$$\Delta \mu_\delta * u = \mu_\delta * \Delta u = 0,$$

so the mollification $\mu_\delta * u$ is also harmonic.

Exercise 4 *Is it true that $\mu_\delta * u = u$ for every harmonic function $u \in C^2(\mathbb{R}^2)$?*

3 Less elementary computations

We begin with the solution of the exercise just stated above.

3.1 Mollification of a harmonic function

Recall that a harmonic function u satisfies $\Delta u = 0$ and also the mean value property:

$$u(x) = \frac{1}{2\pi r} \int_{\xi \in \partial B_r(x)} u(\xi) \qquad \text{for every } r > 0.$$

With this in mind, we compute using a poloar version of Fubini's theorem

$$\begin{aligned} \mu_\delta * u(x) &= \int_{\xi \in \mathbb{R}^2} \mu_\delta(x - \xi)\, u(\xi) \\ &= \int_{\xi \in B_\delta(x)} \mu_\delta(x - \xi)\, u(\xi) \\ &= \int_0^\delta \left(\int_{\xi \in \partial B_r(x)} \mu_\delta(x - \xi)\, u(\xi) \right) dr. \end{aligned}$$

It appears that the factor $\mu_\delta(x - \xi)$ in the integrand, because it depends on $\xi \in \partial B_r(x)$, cannot be taken out of the inside integral (as a constant independent of ξ). However, recall the symmetry of μ_δ according to which if $|x - \xi| = r$, then

$$\mu_\delta(x - \xi) = \mu_\delta(|x - \xi|\mathbf{e}_1) = \mu_\delta(r\mathbf{e}_1)$$

is, in fact, independent of ξ for $\xi \in \partial B_r(x)$. Thus, we may continue:

$$\begin{aligned}
\mu_\delta * u(x) &= \int_0^\delta \left(\int_{\xi \in \partial B_r(x)} \mu_\delta(r\mathbf{e}_1)\, u(\xi) \right) dr \\
&= \int_0^\delta \mu_\delta(r\mathbf{e}_1) \left(\int_{\xi \in \partial B_r(x)} u(\xi) \right) dr \\
&= \int_0^\delta \mu_\delta(r\mathbf{e}_1)\ (2\pi r u(x))\ dr \\
&= u(x) \int_0^\delta \mu_\delta(r\mathbf{e}_1) \left(\int_{\xi \in \partial B_r(x)} 1 \right) dr \\
&= u(x) \int_0^\delta \left(\int_{\xi \in \partial B_r(x)} \mu_\delta(r\mathbf{e}_1) \right) dr \\
&= u(x) \int_0^\delta \left(\int_{\xi \in \partial B_r(x)} \mu_\delta(x-\xi) \right) dr \\
&= u(x) \int_{\xi \in B_\delta(x)} \mu_\delta(x-\xi) \\
&= u(x) \int_{\xi \in \mathbb{R}^2} \mu_\delta(x-\xi) \\
&= u(x). \qquad \square
\end{aligned}$$

I guess that last computation has taken us out of the realm of "elementary." It gives us, however, a proof of a result called **Weyl's lemma** which states that any classical solution $u \in C^2(\mathbb{R}^2)$ of Laplaces equation satisfies $u \in C^\infty(\mathbb{R}^2)$. I prefer to think of the assertion of the exercise above as the fact that *a harmonic function is left invariant by mollification.*

Exercise 5 *Generalize the exercise above (and Weyl's lemma) to higher dimensions and to the case $u \in C^2(U)$ for U an open subset of $\mathbb{R}^n$.*

3.2 Mollification of weak derivatives

The following computation gives what is often called the fact that *mollification commutes with taking weak derivatives.*[2] I have always found this description a bit opaque. I prefer to say the following:

[2]Incidentally, I don't think this clever observation is explicitly in the standard texts *Partial Differential Equations* by Evans or *Second Order Elliptic Partial Differential Equations* by Gilbarg

The mollification of a weak derivative is the classical deriviative of the mollification:

$$D^\alpha(\mu_\delta * u) = \mu_\delta * D^\alpha u.$$

Here we are taking a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ or order k and assuming $u \in W^{k,p}(\mathbb{R}^n)$ so that the derivative $D^\alpha u$ appearing on the right is a weak derivative of order α. Of course, this one works in lower dimensions, but I'll give the proof in $\mathbb{R}^n$. We recall the defining condition for weak derivatives:

$$\int u\, D^\alpha \phi = (-1)^{|\alpha|} \int D^\alpha \phi\, u \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n). \tag{1}$$

Recall also that the order of the derivative is $k = |\alpha| = \alpha_1 + \cdots + \alpha_n$.

In the following computation, I will use the notation D_x^α to distinguish the α derivative with respect to x as opposed to D_ξ^α denoting the same derivative but with respect to the variable ξ.

$$\begin{aligned} D^\alpha(\mu_\delta * u)(x) &= D^\alpha \int_{\xi\in\mathbb{R}^n} \mu_\delta(x-\xi)\, u(\xi) \\ &= \int_{\xi\in\mathbb{R}^n} D_x^\alpha \mu_\delta(x-\xi)\, u(\xi) \\ &= \int_{\xi\in\mathbb{R}^n} (-1)^{|\alpha|} D_\xi^\alpha \mu_\delta(x-\xi)\, u(\xi) \\ &= (-1)^{|\alpha|} \int_{\xi\in\mathbb{R}^n} D_\xi^\alpha \mu_\delta(x-\xi)\, u(\xi). \end{aligned}$$

Note that $\phi(\xi) = \mu_\delta(x-\xi)$ satisfies $\phi \in C_c^\infty(\mathbb{R}^n)$ so that the integrand now has the form associated with the weak adjoint derivative operator in (1). Thus, we continue the computation:

$$\begin{aligned} D^\alpha(\mu_\delta * u)(x) &= (-1)^{|\alpha|} \int_{\xi\in\mathbb{R}^n} D^\alpha \phi(\xi)\, u(\xi) \\ &= (-1)^{|\alpha|}(-1)^{|\alpha|} \int_{\xi\in\mathbb{R}^n} \phi(\xi)\, D^\alpha u(\xi) \\ &= \int_{\xi\in\mathbb{R}^n} \mu_\delta(x-\xi)\, D^\alpha u(\xi) \\ &= \mu_\delta * D^\alpha u(x). \qquad \square \end{aligned}$$

and Trudinger, but both certainly use it implicitly. I first learned the explicit statement from Leon Simon.