Laplace's Equation
The Fundamental Solution and Green's Function

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We have covered some elementary initial properties[1] of **harmonic functions**, that is, functions satisfying Laplace's PDE $\Delta u = 0$. These mostly followed from the **mean value properties** and included the (strong) maximum principle and higher (interior) regularity. Now we complement this discussion with some observations about the natural boundary value problem, or **Dirichlet problem**, for Laplace's equation:

$$\begin{cases} \Delta u = 0 & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g. \end{cases} \tag{1}$$

Here the set $\mathcal{U}$ is an open (often bounded) subset of $\mathbb{R}^n$, the operator is, of course, the Laplace operator

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

and $\partial\mathcal{U}$ is the boundary of the domain $\mathcal{U}$ defined by

$$\partial\mathcal{U} = \overline{\mathcal{U}} \cap \overline{\mathcal{U}^c}$$

as usual. We are looking, at least initially, for a **classical solution** $u : \overline{\mathcal{U}} \to \mathbb{R}$ with

$$u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}}).$$

1 Boundary Values

We have no trouble making sense of **continuous boundary values** $g : \partial\mathcal{U} \to \mathbb{R}$ since $\partial\mathcal{U}$ is a metric space with the inherited distance from the Euclidean space $\mathbb{R}^n$ containing $\mathcal{U}$ and $\partial\mathcal{U}$. If we want higher regularity, however, then generally we may want to impose additional regularity on the set $\partial\mathcal{U}$ requiring $\partial\mathcal{U}$ to be a differentiable or C^1 curve of $\mathcal{U} \subset \mathbb{R}^2$, a smooth surface if $\mathcal{U} \subset \mathbb{R}^3$ and some kind of smooth hypersurface if $\mathcal{U} \subset \mathbb{R}^n$ for $n > 3$. One way to avoid all the technicalities of such a discussion (at least in part) is to simply require the boundary values g to be defined with certain regularity on a larger (full dimension) set containing $\partial\mathcal{U}$. For example, we could consider $g : \mathbb{R}^n \to \mathbb{R}$ with $g \in C^1(\mathbb{R}^n)$ or $g \in C^\infty(\mathbb{R}^n)$ and then it is understood that the function g appearing in (1) is a restriction of such a boundary value function g to $\partial\mathcal{U}$. Thus, to make (1) properly stated we should write

$$u\big|_{\partial\mathcal{U}} = g\big|_{\partial\mathcal{U}}.$$

This is what we should do if we want to be careful and proper. In practice, this is almost never done, though it is often understood that g is defined in a full dimension set containing $\partial\mathcal{U}$.

In particular, one very important special instance of this is going to be considered below, so at least we will have mentioned it, and you will know what's going on.

[1] These may be found in the notes on "Integration and the Divergence."

2 Fundamental Solution

The function $\Phi : \mathbb{R}^2 \backslash \{\mathbf{0}\} \to \mathbb{R}$ given by

$$\Phi(\mathbf{x}) = -\frac{1}{2\pi} \ln |\mathbf{x}|$$

is called the **fundamental solution** of Laplace's equation for $\mathbb{R}^2$. Notice this function is not defined on all of $\mathbb{R}^2$, but it has a **singularity** at the origin $\mathbf{x} = \mathbf{0}$. In the punctured plane, however, it is easy to see that

$$\Delta\Phi(\mathbf{x}) \equiv 0 \qquad \text{for } \mathbf{x} \neq \mathbf{0}.$$

The function $u(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{w})$ is also harmonic in $\mathbb{R}\backslash\{\mathbf{w}\}$ with singularity (translated to) $\mathbf{w}$.

Similarly, the function $\Phi : \mathbb{R}^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ by

$$\Phi(\mathbf{x}) = \frac{1}{n(n-2)\omega_n} \frac{1}{|\mathbf{x}|^{n-2}}$$

is the fundamental solution for Laplace's equation for $n \geq 3$.

In all cases, $\Phi \in L^1_{loc}(\mathbb{R}^n)$, so given a function $f \in C^0_c(\mathbb{R}^n)$ the convolution integral

$$v(\mathbf{x}) = \int_{\mathbf{w}\in\mathbb{R}^n} f(\mathbf{w})\Phi(\mathbf{x} - \mathbf{w})$$

defines a function $v \in C^2(\mathbb{R}^n)$. Of course, you have to prove the regularity of v, but it's not so difficult, and then you will find

$$-\Delta(f * \Phi) = f.$$

This is pretty nice: The fundamental solution of Laplace's equation gives us a bunch[2] of solutions of Poisson's equation. These solutions are not immediately connected to any particular boundary values in any way, but we'll make a connection in the next section.

3 The Boundary Value Problem for Laplace's Equation

Now, say we have $g \in C^2(\mathbb{R}^n)$, and we want to solve (1). A first observation is that if we could solve the boundary value problem

$$\begin{cases} -\Delta v = f & \text{on } \mathcal{U} \\ v\big|_{\partial\mathcal{U}} \equiv 0 \end{cases} \tag{2}$$

for Poisson's equation for all $f \in C^0(\overline{\mathcal{U}})$, then we can solve (1). To see this, set

$$f = \Delta g.$$

Since $g \in C^2(\mathbb{R}^n)$, we know $f \in C^0(\mathbb{R}^n) \subset C^0(\overline{\mathcal{U}})$. Thus if v is the solution of (2), for this choice of f, then $u = v + g$ satisfies

$$\Delta u = \Delta v + \Delta g = -f + f = 0$$

and

$$u\big|_{\partial\mathcal{U}} = v\big|_{\partial\mathcal{U}} + g\big|_{\partial\mathcal{U}} = g.$$

The key observation associated with the Green's function is that one does not need to be able to solve (2) for every $f \in C^0(\overline{\mathcal{U}})$, or equivalently, one does not need to be able to solve (1) for every $g \in C^2(\mathbb{R}^n)$.

[2] Important technical term.

4 Green's Function

The Green's function is a function **associated with a particular domain** $\mathcal{U}$ and depending on $2n$ variables. More precisely, the Green's function is a function $G : \mathcal{U} \times \mathcal{U} \to \mathbb{R} \cup \{\infty\}$ given by

$$G(\mathbf{x}, \mathbf{w}) = \Phi(\mathbf{x} - \mathbf{w}) - \phi(\mathbf{x}, \mathbf{w}) = \Phi(\mathbf{x} - \mathbf{w}) - h(\mathbf{x})$$

where $h(\mathbf{x}) = \phi(\mathbf{x}, \mathbf{w})$ is a **harmonic corrector function** satisfying the boundary value problem

$$\begin{cases} \Delta h = 0 \quad \text{on } \mathcal{U} \\ h(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{w}) \quad \text{for } \mathbf{x} \in \partial\mathcal{U}. \end{cases} \tag{3}$$

The claim is: If you can solve the boundary values problems (3), then you can solve (1) for every $g \in C^0(\mathbb{R}^n)$. In fact, the formula for the solution is

$$u(\mathbf{x}) = -\int_{\mathbf{w} \in \partial\mathcal{U}} g(\mathbf{w})\ DG(\mathbf{x}, \mathbf{w}) \cdot \mathbf{n} \tag{4}$$

where

$$DG = \left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_n} \right)$$

and $\mathbf{n}$ is the outward unit normal to $\mathcal{U}$ along $\partial\mathcal{U}$. Of course, this requires that $\partial\mathcal{U}$ be regular enough to have a well-defined outward unit normal at least as a domain of integration (i.e., a set of measure zero consisting of edges and corners and such is okay).

Notice the main point: If you can solve (1) for

$$g \in \{\Phi(\mathbf{x} - \mathbf{w}) : \mathbf{w} \in \mathcal{U}\},$$

then you can solve (1) for all $g \in C^0(\partial\mathcal{U})$.

Exercise 1 *For what class of inhomogeneities f does one need to be able to solve (2) in order to construct the Green's function for a domain $\mathcal{U}$?*

Of course, it requires a (careful) computation to show the function u given in (4) performs the feat we have ascribed to it, namely that by this formula we obtain a solution $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ of (1). Without too much more work one can also prove the following generalization:

Theorem 1 *(solution of the Dirichlet problem for Poisson's equation) If $\mathcal{U}$ is an open bounded subset of $\mathbb{R}^n$ with C^2 boundary and*

1. *$f \in C^0(\mathcal{U})$, and*

2. *$g \in C^0(\partial\mathcal{U})$,*

then

$$v(\mathbf{x}) = -\int_{\mathbf{w} \in \partial\mathcal{U}} g(\mathbf{w})\ DG(\mathbf{x}, \mathbf{w}) \cdot \mathbf{n} - \int_{\mathbf{w} \in \mathcal{U}} f(\mathbf{w})\ G(\mathbf{x}, \mathbf{w})$$

satisfies $v \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ and

$$\begin{cases} -\Delta v = f \quad \text{on } \mathcal{U} \\ v\big|_{\partial\mathcal{U}} = g. \end{cases} \tag{5}$$