Commentary on Körner's Lemma 53.2

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April 16, 2025

A friend of mine Ed Bueler has called the book *Fourier Analysis* by T.W. K orner the "best math book ever written." In view of such an accolade it perhaps makes sense to express some devotional thoughts about Körner's exposition in the manner not unlike a theologian might write about a passage in the bible. I've chosen for various reasons to focus on Lemma 53.2 in Körner's Chapter 53. Rather than discuss Körner's presentation specifically, I will just attempt to express what he has written in my own words and from my own point of view.

This lemma concerns complex valued functions, and I will start with a brief review of some definitions and properties of complex functions. I will assume some things about continuity and differentiability especially for real valued functions. Each complex valued function $f : \mathbb{R} \to \mathbb{C}$ has associated with it two real valued functions with values given by the real and imaginary parts of f. Let us use the notation $f = f_1 + if_2$ with $f_1 = \operatorname{Re}(f)$ and $f_2 = \operatorname{Im}(f)$.

Differentiability for a complex valued function of this sort means simply that the derivatives of f_1 and f_2 exist and there is a well-defined complex function of the same sort with values at each $x \in \mathbb{R}$ given by $f_1'(x) + if_2'(x)$, and we denote this function by $f' : \mathbb{R} \to \mathbb{C}$ and write

$$f' = f_1' + if_2'.$$

Integrability extends pretty much the same way: If f_1 and f_2 are continuous and $a, b \in \mathbb{R}$ with $a < b$, then

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx + i\int_a^b f_2(x)\,dx.$$

The notion of **continuity** of course makes sense for $f : \mathbb{R} \to \mathbb{C}$ directly, and it is easy to check that f is continuous if and only if the real and imaginary parts f_1 and f_2 are continuous. We denote the collection of all continuous real valued functions on $\mathbb{R}$ by $C^0(\mathbb{R})$ and more generally the collection of all continuous real valued functions with domain any subset $A \subset \mathbb{R}$ by $C^0(\mathbb{R})$. You may recall the definition of continuity at a point for a function $g : A \to \mathbb{R}$ is the following: For any $x_0 \in A$ and any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|g(x) - g(x_0)| < \epsilon \qquad \text{whenever} \qquad x \in A \quad \text{and} \quad |x - x_0| < \delta.$$

Similarly, we denote the collection of all continuous complex valued functions on any set $A \subset \mathbb{R}$ by $C^0(A \to \mathbb{C})$. In the case where A is all of $\mathbb{R}$ or more generally when U is an **open** subset of $\mathbb{R}$, we denote the collection of all differentiable functions $f : U \to \mathbb{C}$ with $f' \in C^0(U \to \mathbb{R})$ by $C^1(U \to \mathbb{C})$.

Note that the various classes of functions $f : \mathbb{R} \to \mathbb{C}$ are very different from the functions $f : \mathbb{C} \to \mathbb{C}$ comprising the main objects of study in complex analysis. In particular the notion of **complex differentiability** for a function $f : U \to \mathbb{C}$ where U is an open subset of $\mathbb{C}$ is quite distinct from the definition of $C^1(U \to \mathbb{C})$ where U is an open subset of $\mathbb{R}$.

The **fundamental theorem of calculus** holds for complex valued functions in the sense that the following hold:

(definite integral version) If $a, b \in \mathbb{R}$ with $a < b$ and $f \in C^1(\mathbb{R} \to \mathbb{C})$ then

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

and

(indefinite integral version) If $a, b \in \mathbb{R}$ with $a < b$ and $g \in C^0([a, b] \to \mathbb{C})$ then $f : (a, b) \to \mathbb{C}$ with values given by

$$f(x) = \int_a^x g(\xi)\, d\xi$$

satisfies $f \in C^1((a, b) \to \mathbb{C})$ with $f'(x) = g(x)$ for each $x \in (a, b)$.

These assertions are also easy to check using the corresponding assertions for real valued functions.

With these preliminaries out of the way, I think I am in a position to address some of the preliminaries of Körner's lemma: We begin with a sequence of functions

$$\{f_j\}_{j=1}^{\infty} \subset C^1(\mathbb{R} \to \mathbb{C}),$$

and we start with the assumption that for each $x \in \mathbb{R}$ the sequence of complex numbers

$$\{f_j(x)\}_{j=1}^{\infty}$$

converges to some value defining a function $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = \lim_{j \nearrow \infty} f_j(x).$$

The idea is that, at the moment, we do not know anything whatsoever about the continuity or differentiability of the function f. We only know that the so-called **pointwise limit** of the functions f_j exists. We assume also, however, that the derivatives converge pointwise to some function $g : \mathbb{R} \to \mathbb{C}$ so that

$$g(x) = \lim_{j \nearrow \infty} f_j'(x),$$

and we assume furthermore that this pointwise convergence of the derivatives is **uniform on compact subsets**. One simple way to say this is, as Körner says it, that for any $a, b \in \mathbb{R}$ with $a < b$, the pointwise convergence is uniform on the interval $[a, b]$. Explicitly, for any $\epsilon > 0$, there exists some $N > 0$ so that

$$|f_j'(x) - g(x)| < \epsilon \qquad \text{for all } j > N \text{ and } a \le x \le b.$$

We make two preliminary assertions based on this assumption.[1] The first is $g \in C^0(\mathbb{R} \to \mathbb{C})$, that is, g is continuous. To see this, fix some $x_0 \in \mathbb{R}$ and any $\epsilon > 0$. By the uniform convergence of f_j' to g on the compact interval $[x_0 - 1, x_0 + 1]$ there exists some $N > 0$ for which

$$|f_N'(\xi) - g(\xi)| < \frac{\epsilon}{3} \qquad \text{whenever} \qquad \xi \in [x_0 - 1, x_0 + 1].$$

We know also that the function f_N' is continuous and continuous at x_0 in particular so that there is some δ satisfying $0 < \delta < 1$ with

$$|f_N'(x) - f_N'(x_0)| < \frac{\epsilon}{3} \qquad \text{whenever} \qquad |x - x_0| < \delta.$$

Thus, if $|x - x_0| < \delta$, then

$$\begin{aligned} |g(x) - g(x_0)| &= |g(x) - f_N'(x) + f_N'(x) - f_N'(x_0) + f_N'(x_0) - g(x_0)| \\ &\le |g(x) - f_N'(x)| + |f_N'(x) - f_N'(x_0)| + |f_N'(x_0) - g(x_0)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

[1]Körner states a somewhat more general result, his Theorem 53.3, along with some kind of supposedly sage remark labeled as a "proof."

This is what it means for g to be continuous at x_0 and hence to have $g \in C^0(\mathbb{R} \to \mathbb{C})$.

The second preliminary assertion is about the convergence of integrals. Without giving the general assertion of Körner's Theorem 53.3, let us start with the specific observation that for each $x > 0$

$$\lim_{j \nearrow \infty} \int_0^x f_j'(\xi)\, d\xi = \int_0^x g(\xi)\, d\xi. \tag{1}$$

This has most of the content we will need. This limiting assertion follows from the uniform convergence of f_j' to g. Specifically, remembering that $x > 0$ is fixed here we have for any $\epsilon > 0$, some N for which

$$|f_j'(\xi) - g(\xi)| < \frac{\epsilon}{2x} \qquad \text{whenever} \qquad j > N \quad \text{and} \quad 0 \le \xi \le x.$$

Therefore,

$$\begin{aligned}
\left| \int_0^x f_j'(\xi)\, d\xi - \int_0^x g(\xi)\, d\xi \right| &= \left| \int_0^x f_j'(\xi) - g(\xi)\, d\xi \right| \\
&\le \int_0^x |f_j'(\xi) - g(\xi)|\, d\xi \\
&\le \int_0^x \frac{\epsilon}{2x}\, d\xi \\
&= \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

With these preliminaries out of the way, Körner's Lemma 53.2 makes the interesting and useful assertion(s) $f \in C^1(\mathbb{R} \to \mathbb{C})$ and $f' = g$. In view of our first preliminary assertion that g is continuous, it is enough to show the derivative $f'(x)$ exists at every fixed $x \in \mathbb{R}$ and $f'(x) = g(x)$. To this end observe that by the definite integral version of the fundamental theorem of calculus

$$f_j(\xi) = \int_0^\xi f_j'(t)\, dt + f_j(0) \qquad \text{for every } \xi > 0. \tag{2}$$

Thinking of ξ as fixed for a moment, we can take the limit as $j \nearrow \infty$ in (2) using the preliminary assertion concerning the convergence of the integral above (or Körner's Theorem 53.3) to find

$$f(\xi) = \int_0^\xi g(t)\, dt + f(0).$$

Taking this as an identity and applying the indefinite integral version of the fundamental theorem of calculus we conclude

$$f'(\xi) = g(\xi) \qquad \text{for each } \xi > 0.$$

In particular, we have shown that for every fixed $x > 0$ the function f is differentiable at x and $f'(x) = g(x)$. This looks very much like what we said we needed to prove. The only deficiency is that our argument is restricted to $x > 0$. Thus, what we have really shown is

$$f\big|_{(0,\infty)} \in C^1((0,\infty) \to \mathbb{C}) \qquad \text{and} \qquad \frac{d}{dx} f\big|_{(0,\infty)} = g\big|_{(0,\infty)}.$$

Perhaps the simplest way to handle showing $f'(x)$ exists and $f'(x) = g(x)$ when $x \le 0$, is to show (1) holds under this more general assumption. One also needs to adapt the statement of the fundamental theorem of calculus above, specifically the indefinite integral version, to situations in which the upper limit of integration x happens to be lower than the lower limit of integration, that is

$$f(x) = \int_a^x g(\xi)\, d\xi$$

with $x \le a$. The basic meaning of such an integral is already well-known and/or easy to define:

$$\int_a^x g(\xi)\, d\xi = -\int_x^a g(\xi)\, d\xi,$$

and of course,

$$\int_a^a g(\xi)\, d\xi = 0,$$

but notice the argument above justifying (1) uses prominently the (positive) tolerance $\epsilon/(2x)$, so clearly some different argument needs to be made when $x \le 0$. Rather than attempting a more general phrasing of my commentary above, I will finish with an exercise someone will hopefully find interesting enough to undertake.

Exercise 1 Show carefully and in detail that $f'(x)$ exists and $f'(x) = g(x)$ when $x \le 0$ by considering two cases $x = 0$ and $x < 0$.