Introduction to Differential Equations

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The ultimate goal of this course is to present an introduction to elementary aspects of partial differential equations (PDEs). The hope is that this introduction will put the student in a position to have some background to tackle much harder questions concerning the study of very complicated systems of partial differential equations like the Navier-Stokes equations, about which very little is known. While very little is known about the Navier-Stokes equations, from a mathematical or theoretical point of view, there is a great industry in studying these equations numerically, and many engineers have a pretty deep understanding of the behavior of solutions which are presumed (though not mathematically proven) to exist. The perceived (though not mathematically proven) properties of solutions of these equations are usually based on or motivated by an understanding of solutions of simpler PDEs which we do know things about. There are various lists of these simpler equations all of which include the following:

1. Laplace's equation,

2. the heat equation, and

3. the wave equation.

The list in our textbook *Mathematical Methods in the Physical Sciences* by Mary Boas is somewhat longer including Poisson's equation, the Helmholtz equation, and the Schrödinger equation. I have always been somewhat disappointed by the introductory material intended to motivate the study of partial differential equations found in textbooks on the subject. The best, it seems, anyone can come up with is to give such a list, which I guess is okay. At any rate, others have tried other approaches, especially involving derivations of certain PDE, but I can't say that I have found the effort to be any real improvement over a list.

Elliptic, Parabolic, and Hyperbolic

Before I make some attempt at my own approach to an introduction, let me mention something that all accounts of elementary PDE should include which will (at least eventually) serve as a framework for what one can really realistically expect to understand about partial differential equations: The three equations listed above, Laplace's equation, the heat equation, and the wave equation are all equations that are **second order**, that are **linear**, and that have **constant coefficients**. These are terms of classification for PDEs which will be explained in more detail later. For now, it's adequate to simply remember the classifying words:

Linear, second order, constant coefficient PDEs

Every linear, second order, constant coefficent PDE has associated with it a "type" determined by the coefficients. Again, if you don't know at the moment how to identify the "coefficients," don't worry about that, it will be explained later. The important thing to note is that there is a reasonably large class of PDEs called **linear, second order, constant coefficient PDEs,** and these PDEs are of three well-defined types called **elliptic**, **parabolic**, and **hyperbolic**. The three PDEs on our list are representative of the three types:

Laplace's equation is elliptic.

The heat equation is parabolic.

The wave equation is hyperbolic.

Associated with every type, there is a list of properties of solutions, but those lists are **very** different for each type. So that is your **framework** for this course, and from it you may compose a list of objectives:

1. Learn how to recognize a linear, second order, constant coefficient PDE.

2. Learn how to determine the type (elliptic, parabolic, or hyperbolic) of such a PDE.

3. Learn some basic properties of, and know what to expect of, solutions of each of these types.

We will also include/learn some other things along the way, including most notably how to find some solutions of such PDE in relatively special cases, and how to prove some of the properties, or understand more deeply why those properties hold.

ODEs

Let me now back up, and try to offer some motivation for how to think about PDEs from the very beginning. I'll start with ordinary differential equations (ODEs) about which students in this course should already know something. Let's start with a real valued function of one variable defined by

$$f(x) = x^2. \tag{1}$$

If we write $y = f(x)$, as is customary in calculus, then we can compute derivatives $y' = 2x$ and $y'' = 2$. Thus, we have arrived at an ODE:

$$y'' = 2. \tag{2}$$

At some level, the simplest way to view the subject of ODEs is as the process of reversing this process of differentiation. That is, we forget about (1), start with (2), and ask the question: What can we say about a function $f(x)$ with the property that $y'' = 2$ when $y = f(x)$? Of course, it's somewhat interesting that one does not get back to (1), or at least one does not get back there directly.

Exercise 1 *If $f(x) = e^x$ and $y = f(x)$, then $y'' = e^x$ and $y'' = y$. Find the solutions of these two ODEs.*

These kind of exercises lead one to ask what happens if one writes down anything of the form

$$y'' = F(y', y, x) \tag{3}$$

that looks "reasonable." It doesn't take too long to realize that not every reasonable looking ODE is one you would find easily by starting with a well-known function $y = f(x)$ and differentiating. For example,

$$y'' = \sin(x^2) \qquad \text{and} \qquad y'' = e^{-x^2}. \tag{4}$$

Exercise 2 *While the ODEs given in (4) may be viewed as having no well-known function as a solution, show that if one is willing to consider the following functions as "well-known," then they can certainly be solved:*

$$f_1(x) = \int_0^x \int_0^\xi \sin(t^2) \, dt d\xi.$$

$$f_2(x) = \int_0^x \int_0^\xi e^{-t^2} \, dt d\xi.$$

Exercise 3 *Use mathematical software (e.g., Matlab or Mathematica) to plot the functions* $y = f_1(x)$ *and* $y = f_2(x)$ *given in the previous exercise as well as their derivatives* $y' = f_1'(x)$ *and* $y' = f_2'(x)$.

Exercise 4 *Consider* $y' = f_1'(x)$ *given by*

$$f_1'(x) = \int_0^x \sin(t^2)\, dt.$$

Show the following:

1. f_1' *is an odd function.*

2. $f_1'(0) = f_1'(\sqrt{\pi}) = f_1'(\sqrt{2\pi}) = 0$.

3. f_1' *is increasing for* $0 < x < \sqrt{\pi}$ *and decreasing for* $\sqrt{\pi} < x < \sqrt{2\pi}$.

4. $0 < f_1'(\sqrt{2\pi}) < f_1'(\sqrt{\pi})$.

If you can show these things, you can show that $f_1'(x) > 0$ *for* $x > 0$, *and it can also be shown that*

$$\lim_{x \to \infty} f_1'(x) = \int_0^\infty \sin(t^2)\, dt = \frac{\sqrt{2\pi}}{4} \approx 0.6267.$$

Exercise 5 *Show that* $y = f_1(x)$ *satisfies the following:*

1. f_1 *is an even function.*

2. f_1 *is increasing for* $0 < x < \sqrt{3\pi}$.

Exercise 6 *Consider* $y' = f_2'(x)$ *given by*

$$f_2'(x) = \int_0^x e^{-t^2}\, dt.$$

Show the following:

1. f_2' *is an odd function.*

2. $f_2'(0) = 1$.

3. f_2' *is increasing for* $0 < x$ *with* $0 < f_2'(x) < 1$.

4

4. f_2' is bounded above and

$$\lim_{x\to\infty} f_2'(x) = \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2} \approx 0.8862.$$

Exercise 7 *Show that $y = f_2(x)$ satisfies the following:*

1. f_2 *is an even function.*

2. f_2 *is increasing for $0 < x$.*

If one continues to study these "reasonable" ODEs, one accumulates some other useful facts and distinctions. Among these one distinguishes **linear** ODEs, which are those involving a **linear operator**[1] $\mathcal{L}$ having the form

$$\mathcal{L}[y] = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_1 y' + a_0 y.$$

The operator is said to be of n-**th order** and **nonsingular** at a point $x = x_0$ if $a_n(x_0) \neq 0$. The n-th order linear ODE then has the form

$$\mathcal{L}[y] = f(x)$$

where the function f is called the **inhomogeneity**, and there is a nice theory of solutions which includes the following assertion:

Theorem 1 *If (a, b) is any open interval on which the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ and the function f are continuous and the leading coefficient a_n does not vanish, then there exists a particular solution y_p (which is n times continuously differentiable and for which $\mathcal{L}y_p = f$), and every other solution of the equation has the form*

$$y = y_p(x) + y_h(x) \tag{5}$$

*where y_h is in the n-dimensional vector space of n times continuously differentiable functions $Y_h = \{y : \mathcal{L}[y] = 0\}$. In fact, given any $x_0 \in (a, b)$ and any values $y_0, y_0', \ldots, y_0^{(n-1)}$, **there exists a unique solution** $y = y(x)$ of the form (5) such that*

$$y(x_0) = y_0, \ y'(x_0) = y_0', \ \ldots, \ y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

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[1] A linear ordinary differential operator to be exact.

This is called the **existence and uniqueness theorem for linear ODE**. There is a generalization of it called the **general existence and uniqueness theorem for ODEs**, but it is easier to state and understand in terms of **equivalent systems**.

Theorem 2 *Assuming F is a continuous function of $n+1$ variables, the solution set of the n-th order ODE*

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x),$$

namely the set of all n times continuously differentiable functions $Y = \{y : y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x)\}$, is in one-to-one correspondence with the solution set of a certain **system** *of n* **first order** *ordinary differential equations having the form*

$$x_1'(t) = \phi_1(x_1, \ldots, x_n, t)$$
$$x_2'(t) = \phi_2(x_1, \ldots, x_n, t)$$
$$\vdots$$
$$x_n'(t) = \phi_n(x_1, \ldots, x_n, t).$$

Such a system may be understood to have solution a single **vector valued** *function $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{x} = (x_1, \ldots, x_n)$. Thus, the system may be written as*

$$\mathbf{x}' = \Phi(\mathbf{x}, t)$$

where $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$ is a given vector valued function of n variables, and the solution set is as set of **once** *continuously differentiable vector valued functions $\Sigma = \{\mathbf{x} : \mathbf{x}' = \Phi(\mathbf{x}, t)\}$.*

Exercise 8 *What is the first order system equivalent to the ODE*

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x)?$$

Fully justify your answer.

In these terms, we can state the general existence and uniqueness theorem for ODEs:

Theorem 3 *If Φ is continuous on some open set containing the point*

$$(x_1^0, x_2^0, \ldots, x_n^0, t_0) \in \mathbb{R}^{n+1}$$

and each of the first partial derivatives of the coordinate functions $\phi_1, \phi_2, \ldots, \phi_n$ of Φ are likewise continuous on the same open set, then there exists some number $\epsilon > 0$ and **there exists a unique solution** $\mathbf{x} \in \Sigma$ *defined on the interval $(t_0 - \epsilon, t_0 + \epsilon)$ and satisfying $\mathbf{x}(t_0) = (x_1^0, x_2^0, \ldots, x_n^0)$.*

Exercise 9 *In view of the correspondence from the previous exercise, what is required of the function F in the ODE $y'' = F(y', y, y, x)$ in order to apply the general existence and uniqueness theorem to the equivalent system? What conclusion can be drawn about existence and uniqueness in $Y = \{y : y'' = F(y', y, y, x)\}$?*

Exercise 10 *Notice that the size of ϵ is not easily related to the set on which Φ and the original equations are well-defined (and regular). Consider the equation $y' = y^2$ with initial condition $y(t_0) = y_0$. What is the function Φ and on what set is Φ continuous? What can you say about the value of ϵ given by the thorem?*

Exercise 11 *Consider the equation $y' = y^{1/3}$. Does the general existence and uniqueness theorem apply to this ODE? What can you say about the existence and uniqueness of solutions?*

Of course, there are many other aspects of ODE which may be studied, but these three,

1. existence and uniqueness for linear ODE,

2. equivalence of n-th order ODE with first order systems, and

3. general existence and uniqueness

are generally representative of how one should think about ODE when starting to study PDE. The overall conclusion is that if one writes down a "reasonable" ODE, especially a linear one, then it should have solutions—solutions should exist (at least locally), and under some natural conditions solutions should be unique.

Exercise 12 *What can you say about the existence and uniqueness of solutions for the **singular first order ODE** $xy' = y$?*

The point of Exercise 11 and Exercise 12 is to suggest that the hypotheses in the general existence and uniqueness theorem are mostly needed for *uniqueness*. It is much easier to get existence, and it is really true that most ODEs you (can) write down and/or have solutions.

Before we make an attempt to say something about PDEs, let me point out two final related aspects of the existence and uniqueness theorem for ODEs. We didn't necessarily emphasize it, but the vector field $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$ is required to be (only) continuous, and the solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ will be **continuously differentiable**. This means each of the functions $x_1, x_2, \ldots x_n$ will be differentiable

and the derivative will be continuous. In fact, we would want the solution components x_1, x_2, ... x_n to be differentiable so that it makes sense to even substitute $\mathbf{x}$ into the ODE. On the other hand, we do not actually need to assume the functions x_1, x_2, ... x_n have derivatives that are continuous.

Exercise 13 *Find a function which is differentiable, say at every point on the interval* $(-1, 1)$, *but the derivative is not a continuous function.*

In the case of an ODE $y^n = F(y^{(n-1)}, \ldots, y, t)$, there is only one "top order," i.e., highest order, derivative of the function y, so it is natural to write the equation in a form where that derivative is given in terms of lower order derivatives. Similarly, for a system, it is natural to assume that each derivative x_j' is given in terms of the other unknown components (without derivatives):

$$x_j' = \phi_j(x_1, x_2, \ldots, x_n, t).$$

In these cases, because the existence of a derivative does imply continuity of the function itself (though not immediately the continuity of the derivative) we can conclude the continuity of the derivative, i.e., the continuous differentiability, of the solution from the equation itself. This is called an issue of **regularity**. Let me repeat how regularity works for ODEs:

In the case of a single equation, $y^n = F(y^{(n-1)}, \ldots, y, t)$, if you have a solution y with n well-defined derivatives, then the lower order derivatives $y^{(n-1)}$, $y^{(n-2)}$, ... y', and y are all continuous. Furthermore, the n-th derivative of y is also continuous because the equation says

$$y^n = F(y^{(n-1)}, \ldots, y, t),$$

and the right side is known to be continuous (because it's a composition of continuous functions).

In the case of a system $\mathbf{x}' = \Phi(\mathbf{x}, t)$, if you have a differentiable solution $\mathbf{x}$, then $\mathbf{x}$ must be contiuous and, therefore,

$$x_j' = \phi_j(\mathbf{x}, t)$$

must be continous as well for each $j = 1, 2, \ldots, n$. Again, the equation says x_j' is a composition of a continuous function ϕ with the continuous functions in $\mathbf{x}$. Thus, with ODEs you get regularity for free.

Exercise 14 *In a previous exercise you found a differentiable function whose derivative was not continuous. Notice that such a function y can't be a solution of an ode*

$y' = f(x)$ *with* f *a continuous function. If you have a twice differentiable function* y *for which* $y'' = 0$, *what can you say about the continuity of* y''?

A **weak formulation** *of the equation* $y'' = 0$ *is as follows: A continuous real valued function* f *defined on the interval* $(0, 1)$ *is said to be a* **weak solution** *if*

$$\int_0^1 f(x)g''(x)\,dx = 0 \tag{6}$$

for every twice differentiable real valued function g *defined on* $(0, 1)$ *and vanishing near* $x = 0$ *and* $x = 1$. *In other words, for each twice differentiable* $g : (0, 1) \to \mathbb{R}$ *for which there is some* $\epsilon > 0$ *such that* $g(x) = 0$ *when* $x < \epsilon$ *or* $x > 1 - \epsilon$, *we know that the integral condition (6) holds. What can you say (and prove) about such a continuous function* f?

In summary, **regularity** is not emphasized in the study of ODEs, but it's there and can be added to the list of basic topics in ODE given above. Regularity is much more important and central in the study of PDEs for reasons we'll touch on below.

PDEs

For PDE, I would like to write down some well-known functions from calculus (just as we did for ODEs) and take some partial derivatives of them. Let's start with

$$u(x, y) = x^2 + y^2 \qquad \text{and} \qquad v(x, y) = x^2 - y^2. \tag{7}$$

The homogeneous second partials of the first function are

$$u_{xx} = 2 \qquad \text{and} \qquad u_{yy} = 2.$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}, \tag{8}$$

and this is a PDE. So, it's natural to ask: Can you start with the PDE in (8) and, forgetting about (7), get back to the function u given in (7)? Likewise,

$$v_{xx} + v_{yy} = 0,$$

and this is Laplace's equation in two dimensions. What can we say about it and its solutions?

Exercise 15 *Show that*

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}.$$

Exercise 16 *The previous exercise suggests we look at the first order PDE*

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0. \tag{9}$$

Do you see why? Imagine we restrict a function $w = w(x, y)$ of two variables to a line $y = x - x_0$ of slope 1 and passing through the point $(x_0, 0)$. Compute

$$\frac{d}{dx}w(x, x - x_0).$$

What does this computation tell you if (9) holds for w?

Exercise 17 *If $w(x, 0) = w_0(x)$ is a given (continuously differentiable) function of one variable, and w satisfies (9), then find w in terms of w_0.*

Exercise 18 *Let's, for a moment, remember $u(x, y) = x^2 + y^2$ from which we got the PDE $u_{xx} - u_{yy} = (u_x - u_y)_x + (u_x - u_y)_y = 0$. Which choice of the function w_0 gives the solution $w(x, y) = 2(x - y)$ of (9)?*

The last two exercises bring out an interesting point. If you have a linear first order homogeneous ODE $y' + ay = 0$, then you expect a one-parameter family of solutions, namely,

$$Y_h = \{ce^{-\alpha} : c \in \mathbb{R}\} \qquad \text{where} \qquad \alpha(x) = \int_{x_0}^x a(\xi)\, d\xi. \tag{10}$$

At least for the first order homogeneous linear constant coefficient PDE $w_x + w_y = 0$ instead of a one-parameter family of solutions determined by a single constant c, we get a solution corresponding to every function of one variable:

$$W = \{w_0(x - y) : w_0 \in C^1(\mathbb{R})\}.$$

Here $C^1(\mathbb{R})$ denotes the set of all continuously differentiable functions of one variable defined on the real line $\mathbb{R}$.

Exercise 19 *Show that the set of functions Y_h given in (10) is a vector space in the sense that*

10

1. *There is an element $Z \in Y_h$ such that $y + Z = y$ for every $y \in Y_h$.*

2. *Given any two elements y_1 and y_2 in Y_h, there is a sum $y_1 + y_2$, and this sum is also in Y_h.*

3. *Given any element $y \in Y_h$ and any scalar $\lambda \in \mathbb{R}$, there is a scaling λy, and this scaling is also in Y_h.*

There are some other properties of a vector space, but since we are dealing with a set of real valued functions, we really only need the second two listed above. For example, we need to know that for each $y \in Y_h$, there is an element $-y \in Y_h$ such that $y + (-y) = Z$ (the existence of additive inverses), but this follows from closure under addition and scaling when one is talking about a set of real valued functions. The other properties follow similarly simply from the vector space properties of $\mathbb{R}$.
*   What is the dimension of Y_h as a vector space?*

Exercise 20 *Show that the solution set W for our PDE is also a vector space. What is the dimension of this vector space?*

Exercise 21 *Solve the PDE $u_{xx} - u_{yy} = 0$. Hint: Fix a point $(x_0, 0)$ and consider the line $y = x_0 - x$ of slope -1 passing through $(x_0, 0)$. Solve an ODE for $y = f(x) = u(x, x_0 - x)$.*

Exercise 22 *Assuming you have found all solutions of $u_{xx} - u_{yy} = 0$ defined on all of $\mathbb{R}^2$, explain how to obtain $u(x, y) = x^2 + y^2$ as an element of your solution set.*

Unfortunately, the approach we have used involving factoring the operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)$$

Doesn't really work for the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

but as we shall see later, the Laplace operator and the solutions of Laplace's equation satisfy some nice properties.

Exercise 23 *Show*

$$\Delta = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

In fact, the Laplace operator and Laplace's equation plays a very important—even central—role in complex analysis.

Now that we have a little experience with how PDEs work, let me get to the point of this introduction, which is to try to compare PDEs to ODEs. One would like to know about existence and uniqueness (for PDEs). As with ODEs, it's not hard to reduce any PDE to some kind of first order system of PDEs. For example, setting $w_1 = v_x$ and $w_2 = v_y$, Laplace's equation for $v = v(x, y)$ is equivalent to the system

$$v_x = w_1$$
$$v_y = w_2$$
$$\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0.$$

Exercise 24 *Find a system of first order equations equivalent to the hyperbolic PDE*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

Notice that there is very often more than one "top order" derivative appearing in a system of PDEs. Because we have derivatives (partial derivatives) with respect to more than one variable, this is very natural and to be expected. This is a fundamental difference between PDE and ODE, and it is one of the reasons regularity plays a much more important role in PDE. Before we give an example where the regularity story works out nicely, let me formulate what we might expect as a **general existence theorem** for PDEs.

In the general case, we can expect to have a number of unknown functions w_1, w_2, ... w_k each of which is a function of several variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in some (open) set in $\mathbb{R}^n$. Recall that the **gradient** of such a function is just the vector containing the partial derivatives:

$$Dw_i = \left(\frac{\partial w_i}{\partial x_1}, \frac{\partial w_i}{\partial x_2}, \ldots, \frac{\partial w_i}{\partial x_n} \right).$$

Sometimes this gradient vector is denoted ∇w_i, but we will use Dw_i here. We would expect to have at least one PDE for each unknown function w_i, but we can't expect to be able to solve for any particular element of the gradient of any one of the unknown functions. Therefore, a first try at what to expect might be something like this:

Conjecture 1 (General Existence for PDEs) *If ϕ_1, ϕ_2, ..., ϕ_ℓ are continuous functions of $nk + k + n$ variables on an open region $\mathbb{R}^{nk} \times \mathbb{R}^k \times U$ where U is an open*

set in $\mathbb{R}^n$ containing a point $\mathbf{x}_0$ and $\ell \geq k$, then there exists some $\delta > 0$ such that the system of PDEs

$$\phi_1(Dw_1, Dw_2, \ldots, Dw_k, w_1, w_2, \ldots, w_k, x_1, x_2, \ldots, x_n) = 0,$$
$$\phi_2(Dw_1, Dw_2, \ldots, Dw_k, w_1, w_2, \ldots, w_k, x_1, x_2, \ldots, x_n) = 0,$$
$$\vdots$$
$$\phi_\ell(Dw_1, Dw_2, \ldots, Dw_k, w_1, w_2, \ldots, w_k, x_1, x_2, \ldots, x_n) = 0$$

has a solution $(w_1, w_2, \ldots, w_k)$ defined on $B_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < \epsilon\}$ consisting of continuously differentiable functions w_1, w_2, $\ldots w_k$.

The restriction $\ell \geq k$ may be ignored because we could just add extra equations reading $0 = 0$. Put another way, having fewer equations can only make our task of finding a solution easier, but based on the examples of first order systems of PDEs above, the inequality $\ell \geq k$ seems reasonable. On the other hand, as Fritz John points out in his book on PDE, no such theorem can be proven or expected simply because we have said nothing about the dependence on the top order derivatives $Dw_1, \ldots, Dw_k$. To be very explicit, we could include an impossible condition/equation like

$$e^{\frac{\partial w}{\partial x}} = 0.$$

We have said nothing to rule this out.[2] Nevertheless, the conclusion of this theorem does hold for many PDEs. The important thing to note is that those PDEs have some reasonable structure, i.e., the functions $\phi_1, \ldots, \phi_\ell$ are not just anything. As an example, let's consider briefly the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Here the functions u and v are functions of two variables x and y. We can, and should, think of the pair (u, v) as determining a **map** from some region U in the x, y-plane into the u, v-plane. The identity map works. That is, $u(x, y) = x$ and $v(x, y) = y$ give two functions satisfying these equations. Also, a constant, i.e., mapping every single point in the x, y-plane to a single point (u_0, v_0) gives a solution. These are not such interesting solutions, but there are more interesting ones. Here is a way to construct some of them:

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[2]Of course, there are completely unreasonable ODEs too. There is no solution of $e^{y'} = 0$.

Think of any function for which you know a formula. Consider that function on the **complex** variable $x + iy$. Then take the real part to be u and the imaginary part to be v.

For example $f(t) = t^2$ gives $f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy$. This means $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ should solve the Cauchy-Riemann equations. (And they do.) You can imagine that you get some relatively impressive looking solutions by taking higher and higher powers. Here are a couple more nice examples:

$$f(t) = e^t \quad \text{leads to the complex exponential} \quad e^{x+iy} = e^x \, e^{iy} = e^x(\cos y + i \sin y).$$

Here we used Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ when θ is real. This means $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ should satisfy the Cauchy-Riemann equations.

Exercise 25 *Consider the complex sine function determined by $f(t) = \sin t$, that is*

$$\sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}.$$

What solutions of the Cauchy-Riemann equations do you get out of this? Hint: $\cosh y = (e^y + e^{-y})/2$ *and* $\sinh y = (e^y - e^{-y})/2$.

The Cauchy-Riemann equations capture the geometric condition that the mapping from the x, y-plane to the u, v-plane is **conformal**. This means that the linearization of the mapping at each point has the form

$$L\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

That is, $(x, y) \mapsto (u, v)$ behaves like a rotation (angles are preserved) with a scaling. The angle of rotation θ depends on the point (x, y) and so does the scaling factor λ, but at each point this is the infinitesimal behavior of such a map. In particular, angles are always preserved. And the converse is true too. If a map of the plane looks locally like a rotation with a homogeneous scaling at each point, then the mapping is conformal and the component functions will satisfy the Cauchy-Riemann equations.

Exercise 26 *I mentioned earlier that Laplace's equation was related to complex analysis. Let u and v be solutions of the Cauchy-Riemann equations. Show u and v are solutions of Laplace's equation. Solutions of Laplace's equation are called **harmonic functions**, and pairs of solutions of Laplace's equation satisfying the Cauchy-Riemann equations are called **conjugate harmonic functions**.*

14

We have seen there are a lot of solutions of the Cauchy-Riemann equations. Here is a nice result which everyone should probably know. It gives a lot of solutions of the Cauchy-Riemann equations, and they are **global solutions**; note there is no "ϵ neighborhood of a point" involved in the statement. It's hard to prove.

Theorem 4 (Riemann mapping theorem) *If U is a simply connected open set[3] in the x, y-plane and $\mathcal{U}$ is a simply connected open set in the u, v-plane, then there is a solution of the Cauchy-Riemann equations (u, v) defined on U such that the mapping $(x, y) \mapsto (u, v)$ is one-to-one and onto the region $\mathcal{U}$.*

Exercise 27 *Find the image of the unit disk $U = \{(x, y) : x^2 + y^2 < 1\}$ under the complex sine function considered in Exercise 25.*

Exercise 28 *Given a conformal mapping (u, v) of a simply connected open set U in the plane to a second set $\mathcal{U}$, a **boundary limit** is determined as follows: Given a sequence $p_1, p_2, p_2, \ldots$ in U converging to a boundary point p of U, we say a point q in the boundary of $\mathcal{U}$ is the boundary limit of the sequence if*

$$\lim_{j \to \infty} (u(p_j), v(p_j)) = q.$$

Assuming the Riemann mapping theorem, and given boundary limits q_1 and q_2 on the boundary of $\mathcal{U}$ with q_1 determined with respect to a conformal mapping (u_1, v_1) so that

$$\lim_{j \to \infty} (u_1(p_j), v_1(p_j)) \to q_1,$$

Prove there is a different conformal mapping (u_2, v_2) of U onto $\mathcal{U}$ with

$$\lim_{j \to \infty} (u_2(p_j), v_2(p_j)) \to q_2.$$

We will end this discussion by noting something about the **regularity** of the Cauchy-Riemann equations. As mentioned above, an equation like

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{11}$$

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[3]We also need the sets U and $\mathcal{U}$ to be non-empty and "proper," i.e., not the whole plane. The condition **simply connected** means that any loop/closed curve in the set can be contracted to a single point while always remaining in the set. The region between two concentric circles is not simply connected because you can take a loop going around the excluded region inside the inner circle; such a loop cannot be contracted to a point without entering that excluded region.

does not express either first partial derivative as a function of lower order derivatives or, in this case since we only have first order derivatives, in terms of the functions u and v. Thus, we might imagine the possibility that u and v are both differentiable but not continuously differentiable, so that the discontinuity in the derivatives cancels out and the linear combination vanishes as required by the PDE (11). It turns out that this can't happen.

Theorem 5 (Regularity of solutions for the Cauchy-Riemann equations) *If u and v are differentiable functions satisfying the Cauchy-Riemann equations, then not only are u and v continuously differentiable, but they have (partial) derivatives of* **all orders***; they are infinitely differentiable.*

So, the Cauchy-Riemann equations are a system of PDEs where the existence and regularity work out quite happily.

Let's return to our proposed existence and regularity theorem above (see the conjecture and recall the existence and uniqueness properties of ODEs) and introduce some kind of structural condition giving something that we really might expect can be true and proved. We say a system of first order partial differential equations for the unknown functions w_1, w_2, $\ldots w_k$ of n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is **linear** if it has the form

$$
A \begin{pmatrix} Dw_1 \\ Dw_2 \\ \vdots \\ Dw_k \\ w_1 \\ w_2 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\ell \end{pmatrix} \tag{12}
$$

where $A = (a_{ij})$ is a $\ell \times (nk + k)$ matrix with components $a_{ij} = a_{ij}(\mathbf{x})$ and the functions $f_j = f_j(\mathbf{x})$ are defined for $\mathbf{x}$ in some open set of $\mathbb{R}^n$.

Conjecture 2 (Existence and Regularity for linear PDEs) *If the coefficients a_{ij} and f_j are continuously differentiable on all of $\mathbb{R}^n$, then given $\mathbf{x}_0 \in \mathbb{R}^n$, there is some $\epsilon > 0$ such that the system (12) has a continuously differentiable solution on $B_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < \epsilon\}$.*

In 1957 Hans Lewy published a paper in which he considered a very specific example of a first order linear system of PDEs having the form (12). To be very

precise, Lewy's equation was for two functions of three variables,

$$u = u(x, y, z) \qquad \text{and} \qquad v = v(x, y, z)$$

and had the form

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + 2y\frac{\partial u}{\partial z} + 2x\frac{\partial v}{\partial z} = f_1(x, y, z)$$

$$\tag{13}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + 2y\frac{\partial v}{\partial z} - 2x\frac{\partial u}{\partial z} = f_2(x, y, z).$$

Exercise 29 *Lewy's system above may be written in a form that is simpler than (12), namely*

$$A \begin{pmatrix} Du \\ Dv \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{14}$$

where

$$A = \begin{pmatrix} 1 & 0 & 2y & 0 & -1 & 2x \\ 0 & 1 & -2x & 1 & 0 & 2y \end{pmatrix}. \tag{15}$$

Exercise 30 *Let $w = u + iv$ and $f = f_1 + if_2$. Show Lewy's system is equivalent to the single PDE*

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} - 2i(x + iy)\frac{\partial w}{\partial z} = f$$

for the single complex valued function $w = w(x, y, z)$.

Exercise 31 *Show that the coefficients $a_{ij} = a_{ij}(x, y, z)$ given in (15) have partial derivatives of all orders (which are all continuous). Such functions are said to be infinitely differentiable on all of $\mathbb{R}^3$, and we write $a_{ij} \in C^\infty(\mathbb{R}^3)$.*

This last exercise should suggest to you that Lewy's equation is a very reasonable equation. If the conjecture above is correct, then it should definitely apply to Lewy's equation. Before I state what Lewy proved about this equation, I need to discuss a little more about the regularity of functions. The last exercise was about some functions which were very regular $C^\infty(\mathbb{R}^3)$. Let's start at the other end with some notation for concepts we've already considered. If a function u is simply continuous on an open set U, then we write $u \in C^0(U)$. In other words, the set $C^0(U)$ is the set of all continuous functions defined on the set U. For this discussion, you can think

of U simply as an interval (a, b) in $\mathbb{R}^1$ and the derivatives we're going to talk about as ordinary derivatives. Of course, if $U \subset \mathbb{R}^n$, then the derivatives will be partial derivatives.

If u is continuously differentiable, then we say $u \in C^1(U)$. That is, $C^1(U)$ is the set of functions with first order derivatives that are continuous. Now, let's think for a moment about the functions that are in $C^0(U)$ but not in $C^1(U)$. Examples of such functions are given by differentiable functions with derivatives which are not continuous as considered in Exercise 13. There are other functions that are continuous but not even differentiable. For example, $f(x) = |x|$ or $g(x) = x^{1/3}$ which are both in $C^0(\mathbb{R})\backslash C^1(\mathbb{R})$. There are special names to distinguish the level of continuity for functions like these. In fact, if you plot both of them, you'll see that f is somehow "more regular" than g in the sense that any secant line connecting two points on the graph of f has slope no more than 1. In symbols

$$|f(x_2) - f(x_1)| \leq |x_2 - x_1| \qquad \text{for every } x_1, x_2 \in \mathbb{R}.$$

More generally, a real valued function u for which there is a constant λ such that

$$|u(x_2) - u(x_1)| \leq \lambda |x_2 - x_1| \qquad \text{for all } x_1, x_2 \in U$$

is said to be **Lipschitz continuous**. The constant λ is called the Lipschitz constant for such a function.

Exercise 32 *Show that $g(x) = x^{1/3}$ has $g \in C^0(\mathbb{R})$, but g is not Lipschitz.*

Nevertheless, g is better (more regular) than some other functions.

Exercise 33 *Show there is a constant c such that $g(x) = x^{1/3}$ satisfies*

$$|g(x_2) - g(x_1)| \leq c|x_2 - x_1|^{1/3} \qquad \text{for every } x_1, x_2 \in \mathbb{R}.$$

Show that $h(x) = x^{1/5}$ satisfies a similar inequality (with $\alpha = 1/5$), and that g satisfies that same inequality on any closed and bounded interval.

A function $u \in C^0(U)$ is said to be **Hölder continuous** with exponent $\alpha < 1$ if there is some constant c for which

$$|u(x_2) - u(x_1)| \leq c|x_2 - x_1|^\alpha.$$

In this case we write $u \in C^{0,\alpha}(U)$. Technically, we need to be a bit more careful when we define the set $C^{0,\alpha}(U)$ for various reasons, but we only need a very simple

understanding of Hölder continuous functions here. Everything we have said is okay, as long as U is a subset of a closed and bounded set like $B_R(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$. In this case, to say a function u is Hölder continuous is to say there is some α with $0 < \alpha < 1$ for which $u \in C^{0,\alpha}(U)$.

Just like the Hölder exponent "grades" the continuous functions between $C^0(U)$ and the Lipschitz functions $C^{0,1}(U)$, we can consider the first derivatives of functions in $C^1(U)$ and require a little bit more regularity of them by requiring those derivatives to be Hölder continuous. Thus, we have the classes $C^{1,\alpha}(U)$ consisting of continuously differentiable functions with derivatives in the Hölder class $C^{0,\alpha}$. Now we are ready for Lewy's result.

Theorem 6 (Hans Lewy) *There are functions f_1 and f_2 which are infinitely differentiable on all of $\mathbb{R}^3$ ($f_1, f_2 \in C^\infty(\mathbb{R}^3)$) and for which the system of equations (13)* **has no** *solution on* **any open set** *$U \subset \mathbb{R}^3$ consisting of functions u and v with Hölder continuous partial derivatives.*

Notice that the line between Lewy's equation being a decisive counterexample to our conjecture is very small. Our conjecture for linear PDE only asserts *continuously differentiable* solutions. But if we require just a tiny bit more—namely continuity of the partial derivatives and any (Hölder) continuity estimate whatsoever—then there is a (very definitive) counterexample.

Lewy's theorem is broadly interpreted to mean that **there is no general existence theory for partial differential equations**. This is the big difference between PDE and ODE. There is a general theory of ODE (even nonlinear ODE). The important (or at least most interesting) PDE are all nonlinear, and even for linear PDE there is no known comprehensive existence theory. There probably can't be one. Heuristically speaking, if $\mathbf{x} = \mathbf{x}(t)$ is a vector valued function of one variable $t \in \mathbb{R}$, then one can specify the derivative $\mathbf{x}' = \mathbf{x}'(t)$ in any manner that makes sense, and there will be a function $\mathbf{x}$ (at least on some small interval $(t_0 - \delta, t_0 + \delta)$, where $t_0 \in \mathbb{R}$ and $\delta > 0$, which solves the associated ODE. If, on the other hand, one writes down seemingly reasonable relations between the first order partial derivatives of even a pair of functions $u = u(x, y, z)$ and $v = v(x, y, z)$ like the Hans Lewy sytstem, then because of the way the partial derivatives fit together in space, it can be the space that there is simply no pair of functions u and v for which the condition can hold on any open set (assuming some nominal extra regularity).

Of course, a basic question at this point might be: Does there exist a solution $(u, v) \in C^1(B_\delta) \times C^1(B_\delta)$ of Hans Lewy's system of partial differential equations for any ball $B = B_\delta(\mathbf{x}_0) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \epsilon\}$ where $\mathbf{x}_0 \in \mathbb{R}^3$ and $\epsilon > 0$. In other

words, is there a solution without the extra little bit of regularity. I don't know the answer to this question, and I don't know if anyone really understands fundamentally what is happening with the Hans Lewy system. I would say this: The Navier-Stokes equations seem much more complicated than the Hans Lewy system, and if I wanted to understand what could go wrong with the existence and uniqueness for the Navier-Stokes equations, I would start by understanding what is going wrong with the Hans Lewy system.

In any case, we are reduced to considering various special cases. In this course, we will consider the second order linear PDE in the list above: Laplace's equation, the heat equation, and the wave equation. Quite a bit is known about each of these.

As a final note for this introductory material, when it comes to understanding PDE it is usually quite important to have some kind of interpretation of what the PDE "means." A good example of this is with the Cauchy-Riemann equations. Those equations encapsulate the geometric condition of having a **conformal mapping**. This kind of interpretation can give, on the one hand, some intuition about what to expect. Riemann stated the Riemann mapping theorem long before anyone could prove it.[4] So these interpretations can put you on the right track. Also, on the other hand, a good interpretation can actually produce a proof sometimes. We'll see this in regard to energy considerations with both the heat equation and the wave equation.

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[4]Statement in 1851. First correct proof in 1912 by Constantin Carathéodory.

1   Comments and Solutions

In Exercise 4 it is mentioned that the value of

$$\int_0^\infty \sin t^2 \, dt$$

can be computed explicitly. Here is a way[5] to do that using a little complex analysis. We consider the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = e^{iz^2}$$

which is holomorphic on all of $\mathbb{C}$, i.e., entire. It follows from Cauchy's theorem that the integral of f around any closed contour vanishes. In particular, if we take a segment from $z = 0$ to $z = R \in \mathbb{R}$ and then traverse the circular arc

$$A = \{Re^{it} : 0 \le t \le \pi/4\}$$

counterclockwise to $w = (1 + i)R/\sqrt{2}$ in the first quadrant, and then the segment returning to the origin, we obtain by Cauchy's theorem:

$$\int_0^R (\cos t^2 + i \sin t^2) \, dt + \int_A f - \frac{1+i}{\sqrt{2}} \int_0^R e^{-t^2} \, dt = 0.$$

The middle integral has the form

$$\int_0^{\pi/4} iR \exp[iR\cos t - R\sin t] \, e^{it} \, dt.$$

This can be estimated as

$$\left| \int_A f \right| \le R \int_0^{\pi/4} e^{-R\sin t} \, dt.$$

We can write the interval $[0, \pi/4] = [0, \epsilon] \cup [\epsilon, \pi/4]$ where $\epsilon = \epsilon(R) \searrow 0$ as $R \nearrow \infty$ and consider two integrals:

$$\int_0^\epsilon e^{-R\sin t} \, dt \le \epsilon$$

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[5]An alternative approach is given by Boas in Problem 41 of Section 7 in Chapter 14 of her book *Mathematical Methods in the Physical Sciences.*

and
$$\int_{\epsilon}^{\pi/4} e^{-R\sin t}\, dt \le \frac{\pi}{4} e^{-R\sin \epsilon}.$$

For an appropriate choice of $\epsilon = \epsilon(R)$ both of these integrals tend to zero faster than R. That is,
$$\lim_{R\to\infty} \int_A f = 0.$$

This means
$$\int_0^\infty \cos t^2\, dt + i \int_0^\infty \sin t^2\, dt = \frac{1+i}{\sqrt{2}} \int_0^\infty e^{-t^2}\, dt.$$

The integral on the right has a well-known[6] value obtained as follows:

$$\left(\int_0^\infty e^{-t^2}\, dt\right) = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right)$$
$$= \int_0^\infty \left(\int_0^\infty e^{-y^2}\, dy\right) e^{-x^2}\, dx$$
$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$$
$$= \int_{[0,\infty)\times[0,\infty)} e^{-(x^2+y^2)}$$
$$= \int_0^\infty \int_0^{\pi/2} r e^{-r^2}\, d\theta\, dr$$
$$= \frac{\pi}{4} \int_0^\infty e^{-u}\, du$$
$$= \frac{\pi}{4}.$$

Therefore,
$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

Finally then
$$\int_0^\infty \cos t^2\, dt + i \int_0^\infty \sin t^2\, dt = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4},$$

[6]See Gaussian distribution function

and
$$\int_0^\infty \cos t^2 \, dt = \int_0^\infty \sin t^2 \, dt = \frac{\sqrt{2\pi}}{4}$$
as claimed.