On the inclination of a parameterized curve

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Abstract

Given a planar curve Γ smoothly parameterized by arclength s on an open interval $I \subset \mathbb{R}$ by a function $\gamma : I \to \mathbb{R}^2$ with twice continuously differentiable component functions and an initial inclination angle $\theta_0 \in \mathbb{R}$ satisfying $\dot\gamma(s_0) = (\cos\theta_0, \sin\theta_0)$ for some $s_0 \in I$, we show there exists a unique function $\psi \in C^1(I)$ with $\dot\gamma(s) = (\cos\psi(s), \sin\psi(s))$ for all $s \in I$ and $\psi(s_0) = \theta_0$. Topologically, the result follows with $\psi \in C^0(I)$ from a familiar result concerning the universal covering of the circle by the real line. From this point of view, our construction is primarily of interest with regard to the regularity of the inclination, its relation to a singular system of ordinary differential equations, and its derivation from that system of differential equations in particular. We give one other related example of a similar singular system of ordinary differential equations, and we strongly suspect the development of a general axiomatic theory of such singular systems should be possible, though we are unaware of such a development.

Let I be an open interval in $\mathbb{R}$ with $s_0 \in I$. Let $\gamma : I \to \mathbb{R}^2$ have coordinate functions $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_j \in C^2(I)$ for $j = 1, 2$ satisfying $\dot\gamma_1^2 + \dot\gamma_2^2 = 1$ (parameterization by arclength) where

$$\dot\gamma_j = \frac{d\gamma_j}{ds} \qquad \text{for } j = 1, 2.$$

We will prove the following:

Theorem 1 *If $\theta_0 \in \mathbb{R}$ satisfies*

$$\begin{cases} \cos\theta_0 = \dot\gamma_1(s_0) \\ \sin\theta_0 = \dot\gamma_2(s_0), \end{cases} \tag{1}$$

there exists a unique function $\psi \in C^1(I)$ *satisfying*

$$\begin{cases} \cos\psi = \dot\gamma_1, & s \in I \\ \sin\psi = \dot\gamma_2, & s \in I \\ \psi(s_0) = \theta_0. \end{cases} \tag{2}$$

Moreover, the relations (2) *are equivalent to the singular system of ordinary differential equations*

$$\begin{cases} -\sin\psi\ \dot\psi &=& \ddot\gamma_1, \quad s \in I \\ \cos\psi\ \dot\psi &=& \ddot\gamma_2, \quad s \in I \\ \psi(s_0) &=& \theta_0. \end{cases} \tag{3}$$

Having assumed γ is an arclength parameterization, we have for each $s \in I$ that $\dot\gamma(s) \in \mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Each such point $\dot\gamma(s)$ and each $a \in \mathbb{R}$ determine a unique angle θ in the interval $[a, a+2\pi) \subset \mathbb{R}$ for which

$$\begin{cases} \cos\theta = \dot\gamma_1(s) \\ \sin\theta = \dot\gamma_2(s), \end{cases} \tag{4}$$

and a unique discrete collection of arguments $\{\theta + 2\pi k : k \in \{0, \pm1, \pm2, \pm3, \ldots\}\}$. On the one hand, this restricts the possible choices of the initial angle θ_0 appearing in (1).

On the other hand, the projection $p : [a, a+2\pi) \to \mathbb{S}^1$ by $p(\theta) = (\cos\theta, \sin\theta)$ is one-to-one and onto, and the inverse $p^{-1} : \mathbb{S}^1 \to [a, a+2\pi)$ is well-defined and continuous except at $p(a) = (\cos(a), \sin(a)) \in \mathbb{S}^2$. Thus, given any θ_0 satisfying (1) we can take $a = \theta_0 - \pi$ to obtain some $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_0 < s_2$ and a function $\psi \in C^0(s_1, s_2)$ given by

$$\psi(s) = p^{-1}(\dot\gamma(s))$$

satisfying $p \circ \psi(s) = (\cos\psi, \sin\psi) = \dot\gamma(s)$ and $\psi(s_0) = \theta_0$. The relation $p \circ \psi(s) = \dot\gamma(s)$ may be assumed to hold and determine ψ uniquely for $s_1 < s_0 < s_2$ as long as $\dot\gamma(s) \in \{(\cos\theta, \sin\theta) \in \mathbb{S}^1 : |\theta - \theta_0| < \pi\}$ for each arclength s in the same interval.

More generally, a continuous function $\mathbf{v} : I \to \mathbb{S}^1$ with $\mathbf{v}(s_0) = (\cos\theta_0, \sin\theta_0)$ gives rise to a unique continuous lifting $\alpha : I \to \mathbb{R}$ for which

$$p \circ \alpha = \mathbf{v} \qquad \text{and} \qquad \alpha(s_0) = \theta_0. \tag{5}$$

Taking $\mathbf{v} = \dot\gamma$, we obtain a unique $\psi_c \in C^0(I)$ for which (2) holds. See Lemma 4.1 of [2]. Though $\dot\gamma_1, \dot\gamma_2 \in C^1(I)$ in (2), it does not immediately follow that $\psi_c \in C^1(I)$, so we cannot immediately differentiate to obtain the singular ordinary differential equations in (3).

Remarks on the construction of plane curves

Associated with an arclength parameterization $\gamma : I \to \mathbb{R}^2$ as introduced above, the function $\dot\gamma : I \to \mathbb{S}^1$ illustrated in Figure 1 is familiar from differential geometry. In this context, the inclination angle is usually defined as the angle between the tangent

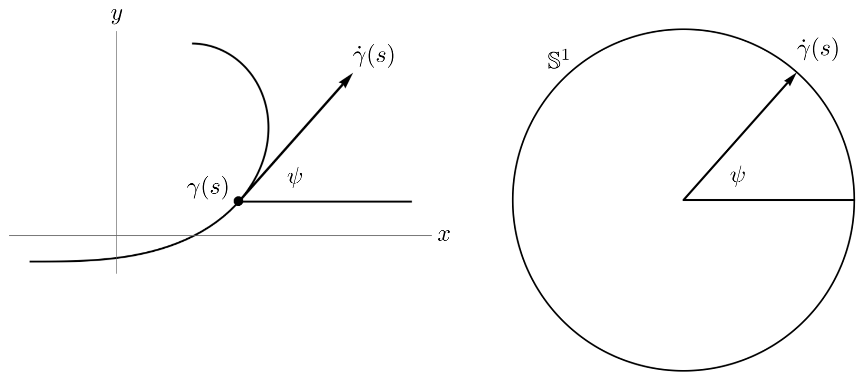

Figure 1: The inclination angle of a planar curve.

vector $\dot\gamma$ and the positive horizontal direction. It is pointed out in most elementary differential geometry texts, e.g., [1], that the **signed curvature** k of such a curve is given by the formula

$$k = \frac{d\psi}{ds}.$$

This assumes, of course, that the inclination angle is a well-defined differentiable function of arclength.

This construction is very often turned around to obtain arclength parameterizations of particular planar curves using a system of the form

$$\begin{cases} \dot\gamma_1 = \cos\psi, & \gamma_1(0) = x_0 \\ \dot\gamma_2 = \sin\psi, & \gamma_2(0) = y_0 \\ \dot\psi = f, & \psi(0) = \theta_0 \end{cases}$$

where the function $f = f(\gamma, \psi, s)$ prescribes the signed curvature of the curve. All such constructions again assume the curve one wishes to parameterize admits/determines a differentiable inclination angle. Many examples can be given. Of note are the Euler elastica for which the signed curvature is proportional to the height $y = \gamma_2$ and the meridians of axially symmetric surfaces of prescribed mean curvature, e.g., constant mean curvature or capillary surfaces with mean curvature an affine function of height. In fact, in view of the structure theorem for planar curves which states that every such curve is essentially determined by the value $k(s) = f$ of the signed curvature as a function of arclength along the curve, it can be said that every planar curve is an example. The curve in Figure 1, incidentally, was numerically computed with the signed curvature equal to the arclength along the curve.

I have used this approach for constructing various special curves numerous times without reflecting either on the fact that the existence of a smooth inclination angle was being assumed or the fact that I did not know a reference where that existence was justified.

1 Topological lifting

The existence of the topological lifting asserted above is fairly standard. Technically, however, we need a slight variant of a special case of Lemma 4.1 in [2] to justify our assertion. Specifically it is important for us to allow I to be an open interval instead of a closed (and compact) one. Taking this into account, as well as the facts that we can (and will) prove a somewhat more general result and that the details of the proof are relevant to the discussion to follow, we state and prove a topological lifting result first. Let $p : \mathbb{R} \to \mathbb{S}^1$ now denote the (universal) covering map of the circle given by $p(\theta) = (\cos\theta, \sin\theta)$.

Lemma 1 *If* $\mathbf{v} = (v_1, v_2) : I \to \mathbb{S}^1$ *is continuous and there are values* $s_0 \in I$ *and* $\theta_0 \in \mathbb{R}$ *with*

$$\begin{cases} \cos\theta_0 = v_1(s_0) \\ \sin\theta_0 = v_2(s_0), \end{cases} \tag{6}$$

then there exists a unique function $\alpha \in C^0(I)$ *such that*

$$p \circ \alpha = \mathbf{v} \qquad \text{and} \qquad \alpha(s_0) = \theta_0. \tag{7}$$

The function α *is called the continuous* **lifting** *of* $\mathbf{v}$ *to* $\mathbb{R}$.

Proof: By continuity, there exist (finite) arclengths s_1 and s_2 with $s_1 < s_0 < s_2$ for which $I_0 = (s_1, s_2) \subset I$ and

$$\mathbf{v}(I_0) \subset B_1(\mathbf{v}(s_0)) = \{\mathbf{x} \in \mathbb{S}^1 : \|\mathbf{x} - \mathbf{v}(s_0)\| < 1\}.$$

Note that $\mathbf{v}(s_0) = (\cos\theta_0, \sin\theta_0)$ and

$$p_0 = p\Big|_{(\theta_0-\pi,\theta_0+\pi)} : (\theta_0 - \pi, \theta_0 + \pi) \to \mathbb{S}^1 \backslash \{(-\cos\theta_0, -\sin\theta_0)\}$$

is a homeomorphism, i.e., continuous bijection with continuous inverse. We claim, furthermore, that

$$\{\mathbf{x} \in \mathbb{S}^1 : \|\mathbf{x} - \mathbf{v}(s_0)\| < 1\} \subset \mathbb{S}^1 \backslash \{(-\cos\theta_0, -\sin\theta_0)\}. \tag{8}$$

This is immediate since

$$\|(-\cos\theta_0, -\sin\theta_0) - \mathbf{v}(s_0)\| = \|(-\cos\theta_0, -\sin\theta_0) - (\cos\theta_0, \sin\theta_0)\| = 2.$$

We thus begin by defining $\alpha_0 : I_0 = (s_1, s_2) \to \mathbb{R}$ by

$$\alpha_0(s) = p_0^{-1} \circ \mathbf{v}(s). \tag{9}$$

This clearly gives $\alpha_0 \in C^0(s_1, s_2)$ with

$$p \circ \alpha_0 = \mathbf{v}\Big|_{(s_0-\delta,s_0+\delta)} \qquad \text{and} \qquad \alpha_0(s_0) = p_0^{-1} \circ p(\theta_0) = \theta_0.$$

Note that the "endpoints" of the open interval I

$$\inf I \in [-\infty, s_0) \qquad \text{and} \qquad \sup I \in (s_1, +\infty]$$

exist as extended real numbers with $I = (\inf I, \sup I)$. As a preliminary assertion let us distinguish

Case 0.0 $s_1 = \inf I$ and $s_2 = \sup I$.

In this case $I_0 = I$ and the unique lifting $\alpha = \alpha_0$ of $\mathbf{v}$ satisfying (7) is defined in (9), and the proof of the lemma is complete. Let us go further by elevating the reasoning behind **Case 0.0** to a formally stated result:

Corollary 1 *If $\mathbf{v} : I \to \mathbb{S}^1$ is continuous and there are values $s \in I$ and $\theta \in \mathbb{R}$ with $\mathbf{v}(s) = p(\theta)$ and*

$$\mathbf{v}(I) \subset \{(\cos(\theta + t), \sin(\theta + t)) \in \mathbb{S}^1 : -\pi < t < \pi\}.$$

then there exists a unique continuous lifting $\alpha \in C^0(I)$ such that

$$p \circ \alpha = \mathbf{v} \qquad \text{and} \qquad \alpha(s) = \theta. \tag{10}$$

In order to see what happens in other cases in more detail, we begin by sharpening the inclusion (8). There in fact holds

$$\{\mathbf{x} \in \mathbb{S}^1 : \|\mathbf{x} - \mathbf{v}(s_0)\| < 1\} \subset \{(\cos(\theta_0 + t), \sin(\theta_0 + t)) \in \mathbb{S}^1 : -\pi/2 < t < \pi/2\}. \quad (11)$$

To see this consider $(\cos(\theta_0 + t), \sin(\theta_0 + t))$ with $-\pi \le t \le -\pi/2$ or $\pi/2 \le t \le \pi$. For such points there holds

$$\|(\cos(\theta_0 + t), \sin(\theta_0 + t)) - \mathbf{v}(s_0)\|^2 = 2(1 - \cos t) \ge 2.$$

Thus,

$$\|(\cos(\theta_0 + t), \sin(\theta_0 + t)) - \mathbf{v}(s_0)\| \ge \sqrt{2} > 1.$$

If $\inf I < s_1$, then $s_1 \in I$ and

$$\theta_1 = \lim_{s \searrow s_1} \alpha_0(s) = \lim_{s \searrow s_1} p_0^{-1} \circ \mathbf{v}(s) = p_0^{-1} \circ \mathbf{v}(s_1).$$

Notice the evaluation on the right is possible because

$$\mathbf{v}(s_1) \in \overline{\mathbf{v}(I_0)} \subset\subset \mathbb{S}^1 \backslash \{(-\cos\theta_0, -\sin\theta_0)\}.$$

We have then $\mathbf{v}(s_1) = (\cos\theta_1, \sin\theta_1)$, and we can repeat the argument starting at the beginning of this proof with s_1 and θ_1 in place of s_0 and θ_0. In this way we obtain some $s_3 < s_1$ for which $I_1 = (s_3, 2s_1 - s_3) \subset I$ and

$$\mathbf{v}(I_1) \subset B_1(\mathbf{v}(s_1)) = \{\mathbf{x} \in \mathbb{S}^1 : \|\mathbf{x} - \mathbf{v}(s_1)\| < 1\}.$$

The argument continues by noting

$$p_1 = p\big|_{(\theta_1 - \pi, \theta_1 + \pi)} : (\theta_1 - \pi, \theta_1 + \pi) \to \mathbb{S}^1 \backslash \{(-\cos\theta_1, -\sin\theta_1)\}$$

is a homeomorphism, and

$$\begin{aligned} \mathbf{v}(I_1) \subset\subset & \{\mathbf{x} \in \mathbb{S}^1 : \|\mathbf{x} - \mathbf{v}(s_1)\| < \sqrt{2}\} \\ & \subset \{(\cos(\theta_1 + t), \sin(\theta_1 + t)) \in \mathbb{S}^1 : -\pi/2 < t < \pi/2\}. \end{aligned}$$

It may be that $s_3 = \inf I$. If this is the case, $\alpha_1 : I_1 \cup I_0 = (s_3, s_2) \to \mathbb{R}$ by

$$\alpha_1(s) = \begin{cases} \alpha_0(s) = p_0^{-1} \circ \mathbf{v}(s), & s \in I_0 \\ p_1^{-1} \circ \mathbf{v}(s), & s \in I_1 \end{cases}$$

defines a continuous lifting of

$$\mathbf{w}_1 = \mathbf{v}\Big|_{(s_3,s_2)}.$$

If $\tilde{\alpha} : (s_3, s_2) \to \mathbb{R}$ were another lifting of $\mathbf{w}_1$, then $p \circ \tilde{\alpha} \equiv p \circ \alpha_1$, and clearly for $s \in I_0$ we have $\tilde{\alpha}(s) = p_0^{-1} \circ \mathbf{v}(s) = \alpha_0(s) = \alpha_1(s)$. The same identity holds for $s \in I_1 \cap I_0 = (s_1, 2s_1 - s_3)$. Applying Corollary 1 to

$$\mathbf{v}\Big|_{I_1}$$

at $s = (3s_1 - s_3)/2$ with $\theta = \alpha_1(s)$, we conclude $\tilde{\alpha}(s) \equiv \alpha_1(s)$ for $s \in I_1$ as well. Thus, there is a unique lifting $\alpha_1 : I_1 \cup I_0 = (s_3, s_2) \to \mathbb{R}$ of

$$\mathbf{v}\Big|_{(s_3,s_2)}.$$

The argument just given can be generalized in several ways. First of all, we consider a second special case.

Case 0.1 $\inf I < s_1$ and $s_2 = \sup I$.

We let $s_1^* = \inf U$ where

$$U = \left\{ s_3 \in (\inf I, s_0) : \mathbf{v}\Big|_{(s_3,s_2)} \text{ has a unique lifting } \alpha \text{ with } \alpha(s_0) = \theta_0 \right\}.$$

In this case, we claim first that $s_1^* = \inf I$ and second that the unique lifting of $\mathbf{v}$ is defined by

$$\alpha_*(s) = \alpha(s)$$

for any lifting $\alpha \in C^0(s_3, s_2)$ of

$$\mathbf{v}\Big|_{(s_3,s_2)} \qquad \text{with} \qquad \alpha(s_0) = \theta_0$$

for some $s_3 \in U$ with $s_3 < s$.

Were we to assume $\inf I < s_1^*$, then we would have $s_1^* \in I$, and a version of the argument above may be repeated: There is some $s_3 > s_1^*$ such that $I_1^* = (2s_1^* - s_3, s_3) \subset I$ and

$$\|\mathbf{v}(s) - \mathbf{v}(s_1^*)\| < 1. \tag{12}$$

Since $s_1^* = \inf U$, there are two more arclengths s_1^{**} and s_* such that

$$s_1^* < s_1^{**} < s_* < s_3 < s_0 < s_2$$

and there is a unique lifting α_0^* of

$$\mathbf{w} = \mathbf{v}\Big|_{I_0^*} \qquad \text{where} \qquad I_0^* = (s_1^{**}, s_2)$$

with $\alpha_0^*(s_0) = \theta_0$. Setting

$$\theta_* = \alpha_0^*(s_*)$$

Corollary 1 may be applied to

$$\mathbf{w}_* = \mathbf{v}\Big|_{I_*} : I_1^* \to \mathbb{S}^1$$

taking $s_* \in I_1^*$ and $\theta = \theta_*$. Specifically,

$$\mathbf{v}(s_*) = p \circ \alpha_0^*(s_*) = p(\theta_*),$$

and if $s \in I_1^*$, then

$$\|\mathbf{v}(s) - \mathbf{v}(s_*)\| \le \|\mathbf{v}(s) - \mathbf{v}(s_1^*)\| + \|\mathbf{v}(s_1^*) - \mathbf{v}(s_*)\| < 2.$$

This last inequality means $\mathbf{v}(s) \neq -\mathbf{v}(s_*) = -(\cos\theta_*, \sin\theta_*)$ because $\|-2\mathbf{v}(s_*)\| = 2$, and

$$\mathbf{v}(s) \in \{(\cos(\theta_* + t), \sin(\theta_* + t)) \in \mathbb{S}^1 : -\pi < t < \pi\}$$

as required by Corollary 1. We thus obtain a unique lifting $\alpha_1^* \in C^0(I_1^*)$ of $\mathbf{w}_*$ with $\alpha_1^*(s_*) = \theta_*$. Since

$$\alpha_1^*\Big|_{I_*} \qquad \text{and} \qquad \alpha_0^*\Big|_{I_*} \quad \text{are both liftings of} \quad \mathbf{w}_* = \mathbf{v}\Big|_{I_*}$$

on

$$I_* = I_1^* \cap I_0^* = (s_1^{**}, s_3)$$

with $s_* \in I_*$ and

$$\alpha_1^*\Big|_{I_*}(s_*) = \theta_* = \alpha_0^*\Big|_{I_*}(s_*),$$

the hypotheses of Corollary 1 still hold on the smaller interval I_* and

$$\alpha_1^*(s) \equiv \alpha_0^*(s) \qquad \text{for} \qquad s \in I_*.$$

We may thus define $\alpha_1 \in C^0(2s_1^* - s_3, s_2)$ by

$$\alpha_1(s) = \begin{cases} \alpha_0^*(s), & s_1^{**} < s < s_2 \\ \alpha_1^*(s), & 2s_1^* - s_3 < s < s_3. \end{cases}$$

This function ψ_1 is a continuous lifting of

$$\mathbf{w}_1 = \mathbf{v}\Big|_{(2s_1^* - s_3, s_2)}$$

with $\alpha_1(s_*) = \theta_*$. It is also the case that $\alpha_1(s_0) = \alpha_0^*(s_0) = \theta_0$.

If $\tilde{\alpha} \in C^0(2s_1^* - s_3, s_2)$ were any other lifting of $\mathbf{w}_1$ with $\tilde{\alpha}(s_0) = \theta_0$, then on the one hand

$$\tilde{\alpha}_0^* = \tilde{\alpha}\Big|_{I_0^*} \qquad \text{is a lifting of} \qquad \mathbf{v}\Big|_{I_0^*}$$

with $\tilde{\alpha}_0^*(s_0) = \theta_0$ where we recall $I_0^* = (s_1^{**}, s_2)$. By the uniqueness of α_0^* we have $\tilde{\alpha}_0^* = \alpha_0^*$, and for each s satisfying $s_1^{**} < s < s_2$, i.e., $s \in I_0^*$, there holds

$$\tilde{\alpha}(s) = \tilde{\alpha}_0^*(s) = \alpha_0^*(s) = \alpha_1(s).$$

In particular, $\tilde{\alpha}(s_*) = \theta_*$.

On the other hand,

$$\tilde{\alpha}_1^* = \tilde{\alpha}\Big|_{I_1^*} \qquad \text{is a lifting of} \qquad \mathbf{v}\Big|_{I_1^*}$$

with $\tilde{\alpha}_1^*(s_*) = \theta_*$ where we recall $I_1^* = (2s_1^* - s_3, s_3)$. By the uniqueness of α_1^* we have $\tilde{\alpha}_1^* = \alpha_1^*$, and for each s satisfying $2s_1^* - s_3 < s < s_3$, i.e., $s \in I_1^*$, there holds

$$\tilde{\alpha}(s) = \tilde{\alpha}_1^*(s) = \alpha_1^*(s) = \alpha_1(s).$$

We have shown α_1 is the unique lifting of $\mathbf{w}_1$ satisfying $\alpha_1(s_0) = \theta_0$. Since $2s_1^* - s_3 < s_1^* = \inf U$, this is a contradiction.

We have established our first claim: $s_1^* = \inf I$.

Next, for any $s \in I$, let $s_3 \in U$ with $s_3 < s$ and let α be the unique lifting of

$$\mathbf{v}\Big|_{(s_3, s_2)} \qquad \text{with} \qquad \alpha(s_0) = \theta_0.$$

If $\tilde{s}_3$ is any other arclength in U with $\tilde{s}_3 < s$ and corresponding unique lifting $\tilde{\alpha}$ of

$$\mathbf{v}\Big|_{(\tilde{s}_3, s_2)} \qquad \text{with} \qquad \tilde{\alpha}(s_0) = \theta_0,$$

Then $s > \sigma = \min\{s_3, \tilde{s}_3\} \in U$ and the restrictions of α and $\tilde{\alpha}$ to (σ, s_2), at least one of which the same lifting, must agree on (σ, s_2). In particular, $\alpha(s) = \tilde{\alpha}(s)$, and our construction of α_* gives a well-defined function $\alpha_* \in C^0(I)$. The function α_* is also a lifting of $\mathbf{v}$ and satisfies $\alpha_*(s_0) = \theta_0$.

Essentially the same uniqueness argument just given applies to show α_* is the unique continuous lifting of $\mathbf{v}$ with $\alpha_*(s_0) = \theta_1$ in **Case 0.1** as claimed.

Even when $s_2 < \sup I$, we have established the following:

Corollary 2 *If* $\mathbf{v} : I \to \mathbb{S}^1$ *is continuous and there are values* $s \in I$ *and* $\theta \in \mathbb{R}$ *with* $\mathbf{v}(s) = p(\theta)$ *then there is some* $s_2 \in I$ *with* $s < s_2$ *so that setting* $I_1 = (\inf I, s_2)$, *there is a unique lifting* $\alpha \in C^0(I_1)$ *of*

$$\mathbf{w} = \mathbf{v}\Big|_{I_1}$$

satisfying

$$p \circ \alpha = \mathbf{w} \qquad \text{and} \qquad \alpha(s) = \theta. \tag{13}$$

Finally, it is straightforward to generalize the argument(s) above to extend the unique lifting of Corollary 2 obtained by taking $\theta = \theta_0$ and $s = s_0$ to get the unique continuous lifting α_* of $\mathbf{v}$ on $I = (\inf I, s_2^*) = (\inf I, \sup I)$ satisfying $\alpha_*(s_0) = \theta_0$ as asserted in Lemma 1. $\square$

Remark: It is also straightforward to allow certain more general possibilities in the argument(s) above. For example, if I is assumed to be a half-closed interval of the form $[\min I, \sup I)$ where $\min I \in \mathbb{R}$ with $s_0 \in I_* = (\min I, \sup I)$ and θ_0 given as in Lemma 1, then it can be shown that

$$\lim_{s \searrow s_1^*} \psi_c(s) = \theta_*$$

exists where α_* is the continuous lifting of

$$\mathbf{v}\Big|_{I_*}$$

and $p \circ (\alpha_*) = \mathbf{v}(\theta_*)$. The situation when $\mathbf{v}(s_0) = p(\theta_0)$ is specified at an endpoint $s_0 = \min I$ can also be considered separately in this case using a variant of the argument above. As a consequence, we can state the following general version of Lemma 1.

Lemma 2 *Let* $I \subset \mathbb{R}$ *be* **any interval**, *open, closed, or half-open/closed. If* $\mathbf{v} = (v_1, v_2) : I \to \mathbb{S}^1$ *is continuous and there are values* $s_0 \in I$ *and* $\theta_0 \in \mathbb{R}$ *with*

$$\begin{cases} \cos\theta_0 = v_1(s_0) \\ \sin\theta_0 = v_2(s_0), \end{cases} \tag{14}$$

then there exists a unique function $\alpha \in C^0(I)$ *such that*

$$p \circ \alpha = \mathbf{v} \qquad \text{and} \qquad \alpha(s_0) = \theta_0. \tag{15}$$

For convenience with regard to application in the next section, we state and prove a final topological result.

Corollary 3 *Under the hypotheses of* **Lemma 1** *according to which one obtains the lifting* $\alpha \in C^0(I)$, *if*

(i) I_1 *is an open subinterval of* I, *and*

(ii) *there is some* $s_1 \in I_1$ *for which* $\theta_1 = \psi_c(s_1)$ *satisfies*

$$\begin{cases} \cos\theta_1 = v_1(s_1) \\ \sin\theta_1 = v_2(s_1), \end{cases} \tag{16}$$

then there is a function $\alpha_1 \in C^0(I_1)$ *for which*

$$p \circ \alpha_1 = \mathbf{v}\Big|_{I_1} \qquad \text{and} \qquad \alpha_1(s_1) = \alpha(s_1)$$

and

$$\alpha_1 = \alpha\Big|_{I_1}.$$

Proof: We can simply apply Lemma 1 to the restriction

$$\mathbf{v}\Big|_{I_1} \in C^0(I_1)$$

at $s_1 \in I_1$ and $\theta_1 \in \mathbb{R}$. We thus obtain a unique lifting $\alpha_1 \in C^0(I_1)$. However, the restriction

$$\alpha\Big|_{I_1}$$

of the global lifting α, in view of (16) and the condition $\theta_1 = \alpha(s_1)$, is also the unique lifting of

$$\mathbf{v}\Big|_{I_1}. \qquad \square$$

In the proof of Theorem 1 below, we assume Lemma 1 has been applied with $\mathbf{v} = \dot{\gamma}$ to obtain a continuous lifting $\alpha = \psi_c \in C^0(I)$ with

$$\dot{\gamma} = p \circ \psi_c \qquad \text{and} \qquad \psi_c(s_0) = \theta_0.$$

2 Analytic approach

If we know (or assume) $\psi \in C^1(I)$, then it is clear the relations (2) imply all the ordinary differential equations of (3). More generally, if J is any open interval in I and

$$\begin{cases} \cos\psi = \dot\gamma_1, & s \in J \\ \sin\psi = \dot\gamma_2, & s \in J \end{cases} \tag{17}$$

holds for some $\psi \in C^1(J)$, then simply by differentiating we obtain the singular system of ordinary differential equations

$$\begin{cases} -\sin\psi\ \dot\psi &=& \ddot\gamma_1, \quad s \in J \\ \cos\psi\ \dot\psi &=& \ddot\gamma_2, \quad s \in J. \end{cases} \tag{18}$$

Roughly speaking, we will prove Theorem 1 by showing (3) has a unique solution $\psi \in C^1(I)$. We will then show this solution satisfies (2) as well, and therefore must be the same as the continuous solution ψ_c of (2) obtained via topological lifting.

We begin with a local version of Theorem 1 with arbitrary initial condition at the point $s_1 \in I$.

Lemma 3 *If $\theta_1 \in \mathbb{R}$ satisfies*

$$\begin{cases} \cos\theta_1 = \dot\gamma_1(s_1) \\ \sin\theta_1 = \dot\gamma_2(s_1), \end{cases} \tag{19}$$

there exists some $\epsilon > 0$ and a unique function $\psi \in C^1(J)$ where $J = (s_1 - \epsilon, s_1 + \epsilon)$ such that

$$\begin{cases} \cos\psi = \dot\gamma_1, & s \in J \\ \sin\psi = \dot\gamma_2, & s \in J \\ \psi(s_1) = \theta_1. \end{cases} \tag{20}$$

Moreover, the relations (20) *are equivalent to the singular system of ordinary differential equations*

$$\begin{cases} -\sin\psi\ \dot\psi &=& \ddot\gamma_1, \quad s \in J \\ \cos\psi\ \dot\psi &=& \ddot\gamma_2, \quad s \in J \\ \psi(s_1) &=& \theta_1. \end{cases} \tag{21}$$

Proof: As suggested above, we begin with the system of ordinary differential equations (21). We know $\sin^2\theta_1 + \cos^2\theta_1 = 1$, so either $\sin\theta_1 \neq 0$ or $\cos\theta_1 \neq 0$. Let us consider the case where $\sin\theta_1 \neq 0$. In this case there is some $\epsilon_1 > 0$ for which

$$\sin\theta \neq 0 \qquad \text{for} \qquad \theta_1 - \epsilon_1 \leq \theta \leq \theta_1 + \epsilon_1 \tag{22}$$

and the initial value problem

$$\begin{cases} -\sin\psi\ \dot\psi &= \ddot\gamma_1 \\ \psi(s_1) &= \theta_1 \end{cases} \tag{23}$$

is nonsingular at $(s_1, \theta_1) \in (s_1 - \epsilon_1, s_1 + \epsilon_1) \times (\theta_1 - \epsilon_1, \theta_1 + \epsilon_1)$. By the existence and uniqueness theorem for ordinary differential equations, there is some $\epsilon > 0$ with $\epsilon < \epsilon_1$ for which (23) has a unique solution $\psi \in C^1(J)$ defined for $s \in J = (s_1 - \epsilon, s_1 + \epsilon) \subset I$.

This solution also satisfies the first relation of (20) because

$$\begin{aligned}\cos\psi(s) &= \cos\psi(s_1) + \int_{s_1}^{s} (-\sin\psi(\sigma)\ \dot\psi(\sigma))\, d\sigma \\ &= \cos\theta_1 + \int_{s_1}^{s} \ddot\gamma_1(\sigma)\, d\sigma \\ &= \cos\theta_1 + \dot\gamma_1(s) - \dot\gamma_1(s_1) \\ &= \dot\gamma_1(s).\end{aligned}$$

The second relation of (20) is somewhat more difficult to see. We know, however, that

$$\dot\gamma_2^2 = 1 - \dot\gamma_1^2 = 1 - \cos^2\psi = \sin^2\psi.$$

This means

$$\dot\gamma_2(s) \in \{-\sin\psi(s), \sin\psi(s)\} \qquad \text{for every } s \in (s_1 - \epsilon_1, s_1 + \epsilon_1). \tag{24}$$

We also have from (19) that $\sin\psi(s_1) = \dot\gamma_2(s_1)$ and from (22) that $-\sin\psi(s) \neq \sin\psi(s)$ on the same interval $(s_1 - \epsilon_1, s_1 + \epsilon_1)$. In particular, $\dot\gamma_2$ cannot vanish on this interval, and by continuity, e.g., the intermediate value theorem, there must hold

$$\dot\gamma_2(s) \equiv \sin\psi(s) \qquad \text{for every } s \in (s_1 - \epsilon_1, s_1 + \epsilon_1).$$

We have established (20) for a function $\psi \in C^1(J)$ and by differentiation (21) holds as well. Uniqueness now follows from the fact that $\psi : J \to \mathbb{R}$ is the unique topological lifting $\alpha \in C^0(J)$ of $\dot\gamma : J \to \mathbb{S}^1$, with $\alpha(s_1) = \theta_1$ since according to (20) $p \circ \psi = \dot\gamma$.

The case in which $\cos\theta_1 \neq 0$ may be treated much the same way. □

Proof of Theorem 1: We can apply Lemma 3 with $s_1 = s_0$ and $\theta_1 = \theta_0$ to obtain some interval $J_0 = (s_0 - \epsilon_1, s_0 + \epsilon_1)$ for which

$$\begin{cases} \cos\psi = \dot\gamma_1, & s \in J_0 \\ \sin\psi = \dot\gamma_2, & s \in J_0 \\ \psi(s_0) = \theta_0. \end{cases} \tag{25}$$

and

$$\begin{cases} -\sin\psi\ \dot{\psi} &= \ddot{\gamma}_1, \quad s \in J_0 \\ \cos\psi\ \dot{\psi} &= \ddot{\gamma}_2, \quad s \in J_0 \\ \psi(s_0) &= \theta_0 \end{cases} \tag{26}$$

are equivalent and have a unique solution $\psi \in C^1(J_0)$ which is also the topological lifting α_0 of $\dot{\gamma} : J_0 \to \mathbb{S}^1$, with $\alpha_0(s_0) = \theta_0$.

By the uniqueness of the topological lifting $\psi_c : I \to \mathbb{R}$ of $\dot{\gamma} : I \to \mathbb{S}^1$ we know any union

$$I_* = \bigcup_{\beta \in \Gamma} J_\beta$$

of open intervals J_β with $s_0 \in J_\beta \subset I$ for which

$$\begin{cases} \cos\psi = \dot{\gamma}_1, \quad s \in J_\beta \\ \sin\psi = \dot{\gamma}_2, \quad s \in J_\beta \\ \psi(s_0) = \theta_0. \end{cases} \tag{27}$$

and

$$\begin{cases} -\sin\psi\ \dot{\psi} &= \ddot{\gamma}_1, \quad s \in J_\beta \\ \cos\psi\ \dot{\psi} &= \ddot{\gamma}_2, \quad s \in J_\beta \\ \psi(s_0) &= \theta_0 \end{cases} \tag{28}$$

are equivalent and have a unique solution $\psi \in C^1(J_\beta)$ satisfies I_* is an open interval with $s_0 \in I_* \subset I$, as well as the condition that

$$\begin{cases} \cos\psi = \dot{\gamma}_1, \quad s \in I_* \\ \sin\psi = \dot{\gamma}_2, \quad s \in I_* \\ \psi(s_0) = \theta_0. \end{cases} \tag{29}$$

and

$$\begin{cases} -\sin\psi\ \dot{\psi} &= \ddot{\gamma}_1, \quad s \in I_* \\ \cos\psi\ \dot{\psi} &= \ddot{\gamma}_2, \quad s \in I_* \\ \psi(s_0) &= \theta_0 \end{cases} \tag{30}$$

are equivalent and have a unique solution $\psi = \psi_* \in C^1(I_*)$. Consequently, we may assume I_* is a maximal open interval in I with this property. If we assume $\inf I_* > \inf I$, then $a = \inf I_* \in I \subset \mathbb{R}$. Notice then that $\dot{\gamma}(a)$ is well-defined. Furthermore, considering $\alpha_* \in C^0(I_*)$ as the unique topological lifting of $\dot{\gamma} : I_* \to \mathbb{S}^1$ with , we know α_* must be the restriction of $\psi_c \in C^0(I)$ to the interval I_*. By the continuity of ψ_c, we know

$$\theta_a = \lim_{s \searrow a} \alpha_*(s) = \psi_c(a) \qquad \text{is well-defined}$$

and satisfies

$$\begin{cases} \cos\theta_a = \dot{\gamma}_1(a) \\ \sin\theta_a = \dot{\gamma}_2(a). \end{cases} \tag{31}$$

Thus, we can apply Lemma 3 to obtain some $\epsilon > 0$ and some $\psi_a \in C^1(a-\epsilon, a+\epsilon)$ which also agrees with ψ_c on the interval $(a-\epsilon, a+\epsilon)$. We may also assume $\epsilon < \sup I_* - a$ so that I_* is a proper subinterval of $J_* = (a-\epsilon, \sup I_*)$. It follows that $\psi_2 : J_* \to \mathbb{R}$ by

$$\psi_2(s) = \begin{cases} \alpha_*(s), & s \in I_* \\ \psi_a(s), & s \in (a-\epsilon, a+\epsilon) \end{cases}$$

is well-defined with $\psi_2 \in C^1(J_*)$. Furthermore, $s_0 \subset J_* \subset I$ and the problems

$$\begin{cases} \cos\psi = \dot{\gamma}_1, & s \in J_* \\ \sin\psi = \dot{\gamma}_2, & s \in J_* \\ \psi(s_0) = \theta_0. \end{cases} \tag{32}$$

and

$$\begin{cases} -\sin\psi\ \dot{\psi} &= \ddot{\gamma}_1, \quad s \in J_* \\ \cos\psi\ \dot{\psi} &= \ddot{\gamma}_2, \quad s \in J_* \\ \psi(s_0) &= \theta_0 \end{cases} \tag{33}$$

are equivalent with unique solution $\psi = \psi_2 \in C^1(J_*)$. This contradicts the maximality of I_*, and we conclude $\inf I_* = \inf I$.

The assumption $\sup I_* < \sup I$ leads to a similar contradiction, so $I = I_*$ and the assertion of Theorem 1 holds. $\square$

3 Another singular system

Given $\gamma \in C^2(I \to \mathbb{R}^2)$ as above, a technically different singular system of ordinary differential equations sharing the same singular/nonsingular character displayed by (3) and indeed an alternative for analytically defining the inclination $\psi \in C^1(I)$ determined by γ is

$$\begin{cases} -\dot{\gamma}_2\ \dot{\psi} &= \ddot{\gamma}_1, \quad s \in I \\ \dot{\gamma}_1\ \dot{\psi} &= \ddot{\gamma}_2, \quad s \in I \\ \psi(s_0) &= \theta_0. \end{cases} \tag{34}$$

We make two simple observations about the system (34).

First, in view of the condition

$$\dot{\gamma}_1^2 + \dot{\gamma}_2^2 = 1 \tag{35}$$

at least one of the ordinary differential equations in (34) is nonsingular at each $s \in I$. It will be recalled that this is a feature shared with the singular system (3). Proceeding as with the system (3) we may consider the case $\dot\gamma_2(s_0) \neq 0$ so that on some interval the first equation in (34) determines a unique function ψ locally.

Letting ψ_0 denote the solution of (3) given by Theorem 1, we can then write locally

$$\frac{d}{ds}(\psi - \psi_0) = -\frac{\ddot\gamma_1}{\dot\gamma_2} + \frac{\ddot\gamma_1}{\sin\psi_0} \equiv 0,$$

since it was established that the second equation in (2) namely $\dot\gamma_2 = \sin\psi_0$ holds for ψ_0. This implies the solution of (34) is locally identical to the solution of (3) as expected, and this reasoning can clearly be extended to the global assertion $\psi = \psi_0$. As implied, the global existence of the solution $\psi \in C^1(I)$ and the fact that this solution satisfies (2) may also be established along these lines.

Finally, we note the question of "consistency" for the system (34), that is for example showing the second relation $\dot\gamma_1\ \dot\psi = \ddot\gamma_2$ of (34) holds on an interval where the first relation $-\dot\gamma_2\ \dot\psi = \ddot\gamma_1$ holds and is nonsingular, is straightforward. This is in contrast to the slightly delicate argument arising in the proof of Lemma 3 in connection with (24) for the system (3). To see this, for example, we can differentiate the relation (35) and use $-\dot\gamma_2\ \dot\psi = \ddot\gamma_1$ to obtain directly

$$0 = \dot\gamma_1\ \ddot\gamma_1 + \dot\gamma_2\ \ddot\gamma_2 = -\dot\gamma_1\ \dot\gamma_2\ \dot\psi + \dot\gamma_2\ \ddot\gamma_2$$

which implies $\ddot\gamma_2 = \dot\gamma_1\ \dot\psi$.

References

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