47. $w = \sqrt{x^2 + y^2}$; $x = \ln(rs + tu)$,
$y = \dfrac{t}{u}\cosh rs$; $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial u}$

48. $s = p^2 + q^2 - r^2 + 4t$; $p = \phi e^{3\theta}$, $q = \cos(\phi + \theta)$, $r = \phi\theta^2$,
$t = 2\phi + 8\theta$; $\dfrac{\partial s}{\partial \phi}, \dfrac{\partial s}{\partial \theta}$

In Problems 49–52, use (8) to find the indicated derivative.

49. $z = \ln(u^2 + v^2)$; $u = t^2$, $v = t^{-2}$; $\dfrac{dz}{dt}$

50. $z = u^3 v - uv^4$; $u = e^{-5t}$, $v = \sec 5t$; $\dfrac{dz}{dt}$

51. $w = \cos(3u + 4v)$; $u = 2t + \dfrac{\pi}{2}$, $v = -t - \dfrac{\pi}{4}$; $\left.\dfrac{dw}{dt}\right|_{t=\pi}$

52. $w = e^{xy}$; $x = \dfrac{4}{2t + 1}$, $y = 3t + 5$; $\left.\dfrac{dw}{dt}\right|_{t=0}$

53. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that **Laplace's equation** $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

54. Van der Waals' equation of state for the real gas CO_2 is

$$P = \frac{0.08T}{V - 0.0427} - \frac{3.6}{V^2}.$$

If dT/dt and dV/dt are rates at which the temperature and volume change, respectively, use the Chain Rule to find dP/dt.

55. The equation of state for a thermodynamic system is $F(P, V, T) = 0$, where P, V, and T are pressure, volume, and temperature, respectively. If the equation defines V as a function of P and T, and also defines T as a function of V and P, show that

$$\frac{\partial V}{\partial T} = -\frac{\dfrac{\partial F}{\partial T}}{\dfrac{\partial F}{\partial V}} = -\frac{1}{\dfrac{\partial T}{\partial V}}.$$

56. The voltage across a conductor is increasing at a rate of 2 volts/min and the resistance is decreasing at a rate of 1 ohm/min. Use $I = E/R$ and the Chain Rule to find the rate at which the current passing through the conductor is changing when $R = 50$ ohms and $E = 60$ volts.

57. The length of the side labeled x of the triangle in **FIGURE 9.4.6** increases at a rate of 0.3 cm/s, the side labeled y increases at a rate of 0.5 cm/s, and the included angle θ increases at a rate of 0.1 rad/s. Use the Chain Rule to find the rate at which the area of the triangle is changing at the instant $x = 10$ cm, $y = 8$ cm, and $\theta = \pi/6$.

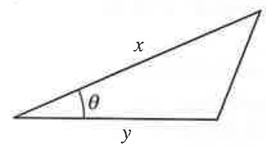

FIGURE 9.4.6 Triangle in Problem 57

58. A particle moves in 3-space so that its coordinates at any time are $x = 4\cos t$, $y = 4\sin t$, $z = 5t$, $t \geq 0$. Use the Chain Rule to find the rate at which its distance

$$w = \sqrt{x^2 + y^2 + z^2}$$

from the origin is changing at $t = 5\pi/2$ seconds.

9.5 Directional Derivative

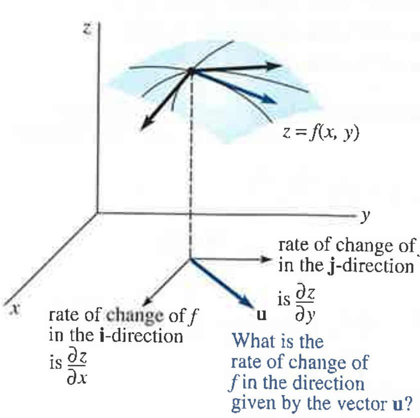

FIGURE 9.5.1 An arbitrary direction is denoted by the vector **u**

Introduction We saw in the last section that for a function f of two variables x and y, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ give the slope of the tangent to the trace, or curve of intersection of the surface defined by $z = f(x, y)$ and vertical planes that are, respectively, parallel to the x- and y-coordinates axes. Equivalently, we can think of the partial derivative $\partial z/\partial x$ as the rate of change of the function f in the direction given by the vector **i**, and $\partial z/\partial y$ as the rate of change of the function f in the **j**-direction. There is no reason to confine our attention to just two directions. In this section we shall see how to find the rate of change of a differentiable function in *any* direction. See **FIGURE 9.5.1**.

The Gradient of a Function In this and the next sections it is convenient to introduce a new vector based on partial differentiation. When the **vector differential operator**

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \quad \text{or} \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is applied to a differentiable function $z = f(x, y)$ or $w = F(x, y, z)$, we say that the vectors

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{1}$$

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \tag{2}$$

Remarks

If $w = F(x, y, z)$ has continuous partial derivatives of any order, then analogous to (3), the mixed partial derivatives are equal:

$$F_{xyz} = F_{yzx} = F_{zyx}, \qquad F_{xxy} = F_{yxx} = F_{xyx},$$

and so on.

9.4 Exercises Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–6, sketch some of the level curves associated with the given function.

1. $f(x, y) = x + 2y$
2. $f(x, y) = y^2 - x$
3. $f(x, y) = \sqrt{x^2 - y^2 - 1}$
4. $f(x, y) = \sqrt{36 - 4x^2 - 9y^2}$
5. $f(x, y) = e^{y - x^2}$
6. $f(x, y) = \tan^{-1}(y - x)$

In Problems 7–10, describe the level surfaces but do not graph.

7. $F(x, y, z) = \dfrac{x^2}{9} + \dfrac{z^2}{4}$
8. $F(x, y, z) = x^2 + y^2 + z^2$
9. $F(x, y, z) = x^2 + 3y^2 + 6z^2$
10. $F(x, y, z) = 4y - 2z + 1$
11. Graph some of the level surfaces associated with $F(x, y, z) = x^2 + y^2 - z^2$ for $c = 0$, $c > 0$, and $c < 0$.
12. Given that

$$F(x, y, z) = \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{9},$$

find the x-, y-, and z-intercepts of the level surface that passes through $(-4, 2, -3)$.

In Problems 13–32, find the first partial derivatives of the given function.

13. $z = x^2 - xy^2 + 4y^5$
14. $z = -x^3 + 6x^2y^3 + 5y^2$
15. $z = 5x^4y^3 - x^2y^6 + 6x^5 - 4y$
16. $z = \tan(x^3y^2)$
17. $z = \dfrac{4\sqrt{x}}{3y^2 + 1}$
18. $z = 4x^3 - 5x^2 + 8x$
19. $z = (x^3 - y^2)^{-1}$
20. $z = (-x^4 + 7y^2 + 3y)^6$
21. $z = \cos^2 5x + \sin^2 5y$
22. $z = e^{x^2 \tan^{-1} y^2}$
23. $f(x, y) = xe^{x^3 y}$
24. $f(\theta, \phi) = \phi^2 \sin \dfrac{\theta}{\phi}$
25. $f(x, y) = \dfrac{3x - y}{x + 2y}$
26. $f(x, y) = \dfrac{xy}{(x^2 - y^2)^2}$
27. $g(u, v) = \ln(4u^2 + 5v^3)$
28. $h(r, s) = \dfrac{\sqrt{r}}{s} - \dfrac{\sqrt{s}}{r}$
29. $w = 2\sqrt{xy} - ye^{y/z}$
30. $w = xy \ln xz$
31. $F(u, v, x, t) = u^2w^2 - uv^3 + vw\cos(ut^2) + (2x^2t)^4$
32. $G(p, q, r, s) = (p^2q^3)^{r^4s^5}$

In Problems 33 and 34, verify that the given function satisfies **Laplace's equation**:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

33. $z = \ln(x^2 + y^2)$
34. $z = e^{x^2 - y^2} \cos 2xy$

In Problems 35 and 36 verify that the given function satisfies the **wave equation**:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

35. $u = \cos at \sin x$
36. $u = \cos(x + at) + \sin(x - at)$
37. The molecular concentration $C(x, t)$ of a liquid is given by $C(x, t) = t^{-1/2} e^{-x^2/kt}$. Verify that this function satisfies the **diffusion equation**:

$$\frac{k}{4} \frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t}.$$

38. The pressure P exerted by an enclosed ideal gas is given by $P = k(T/V)$, where k is a constant, T is temperature, and V is volume. Find:
    (a) the rate of change of P with respect to V,
    (b) the rate of change of V with respect to T, and
    (c) the rate of change of T with respect to P.

In Problems 39–48, use the Chain Rule to find the indicated partial derivatives.

39. $z = e^{uv^2}$; $u = x^3$, $v = x - y^2$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
40. $z = u^2 \cos 4v$; $u = x^2y^3$, $v = x^3 + y^3$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
41. $z = 4x - 5y^2$; $x = u^4 - 8v^3$, $y = (2u - v)^2$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$
42. $z = \dfrac{x - y}{x + y}$; $x = \dfrac{u}{v}$, $y = \dfrac{v^2}{u}$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$
43. $w = (u^2 + v^2)^{3/2}$; $u = e^{-t} \sin \theta$, $v = e^{-t} \cos \theta$; $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial \theta}$
44. $w = \tan^{-1}\sqrt{uv}$; $u = r^2 - s^2$, $v = r^2s^2$; $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$
45. $R = rs^2t^4$; $r = ue^{v^2}$, $s = ve^{-u^2}$, $t = e^{u^2v^2}$; $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial v}$
46. $Q = \ln(pqr)$; $p = t^2 \sin^{-1} x$, $q = \dfrac{x}{t^2}$, $r = \tan^{-1} \dfrac{x}{t}$; $\dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial t}$

Remarks

By writing (6) as

$$\mathbf{a}(t) = \frac{ds}{dt}\frac{d\mathbf{T}}{dt} + \frac{d^2s}{dt^2}\mathbf{T},$$

we note that the so-called scalar acceleration d^2s/dt^2, referred to in the last remark, is now seen to be the tangential component of the acceleration a_T.

9.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1 and 2, for the given position function, find the unit tangent.

1. $\mathbf{r}(t) = (t\cos t - \sin t)\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + t^2\mathbf{k};\ t > 0$
2. $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
3. Use the procedure outlined in Example 2 to find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for motion on a general circular helix that is described by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$.
4. Use the procedure outlined in Example 2 to show on the twisted cubic of Example 3 that at $t = 1$:

$$\mathbf{T} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}),\quad \mathbf{N} = -\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}),$$

$$\mathbf{B} = -\frac{1}{\sqrt{6}}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}),\quad \kappa = \frac{\sqrt{2}}{3}.$$

In Problems 5 and 6, find an equation of the osculating plane to the given space curve at the point that corresponds to the indicated value of t.

5. The circular helix of Example 2; $t = \pi/4$
6. The twisted cubic of Example 3; $t = 1$

In Problems 7–16, $\mathbf{r}(t)$ is the position vector of a moving particle. Find the tangential and normal components of the acceleration at any t.

7. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$
8. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$
9. $\mathbf{r}(t) = t^2\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t^2\mathbf{k}$
10. $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j} + t^4\mathbf{k}$
11. $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\mathbf{j}$
12. $\mathbf{r}(t) = \tan^{-1}t\,\mathbf{i} + \frac{1}{2}\ln(1 + t^2)\mathbf{j}$
13. $\mathbf{r}(t) = 5\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j}$
14. $\mathbf{r}(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j}$
15. $\mathbf{r}(t) = e^{-t}(\mathbf{i} + \mathbf{j} + \mathbf{k})$
16. $\mathbf{r}(t) = t\mathbf{i} + (2t - 1)\mathbf{j} + (4t + 2)\mathbf{k}$
17. Find the curvature of an elliptical helix that is described by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\mathbf{k}$, $a > 0$, $b > 0$, $c > 0$.
18. (a) Find the curvature of an elliptical orbit that is described by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + c\mathbf{k}$, $a > 0$, $b > 0$, $c > 0$.
    (b) Show that when $a = b$, the curvature of a circular orbit is the constant $\kappa = 1/a$.
19. Show that the curvature of a straight line is the constant $\kappa = 0$. [*Hint*: Use (2) in Section 7.5.]
20. Find the curvature at $t = \pi$ of the cycloid that is described by

$$\mathbf{r}(t) = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ a > 0.$$

21. Let C be a plane curve traced by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g have second derivatives. Show that the curvature at a point is given by

$$\kappa = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}.$$

22. Show that if $y = F(x)$, the formula for κ in Problem 21 reduces to

$$\kappa = \frac{|F''(x)|}{[1 + (F'(x))^2]^{3/2}}.$$

In Problems 23 and 24, use the result of Problem 22 to find the curvature and radius of curvature of the curve at the indicated points. Decide at which point the curve is "sharper."

23. $y = x^2$; $(0, 0)$, $(1, 1)$
24. $y = x^3$; $(-1, -1)$, $(\frac{1}{2}, \frac{1}{8})$
25. Discuss the curvature near a point of inflection of $y = F(x)$.
26. Show that $\|\mathbf{a}(t)\|^2 = a_N^2 + a_T^2$.

9.4 Partial Derivatives

Introduction In this section we consider functions of two or more variables and how to find the instantaneous rate of change—that is, the derivative—of such functions with respect to each variable.

(b) Explain why the angular momentum $\mathbf{L}$ of a planet is constant.

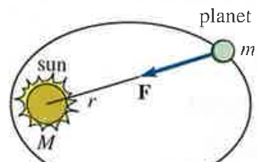

FIGURE 9.2.9 Force $\mathbf{F}$ in Problem 27

28. (*This problem could present a challenge.*) In this problem the student will use the properties in Sections 7.4 and 9.1 to prove **Kepler's first law of planetary motion**: The orbit of a planet is an ellipse with the Sun at one focus. We assume that the Sun has mass M and is located at the origin, $\mathbf{r}$ is the position vector of a body of mass m moving under the gravitational attraction of the Sun, and $\mathbf{u} = \mathbf{r}/r$ is a unit vector in the direction of $\mathbf{r}$.

(a) Use Problem 27 and Newton's second law of motion $\mathbf{F} = m\mathbf{a}$ to show that

$$\frac{d^2\mathbf{r}}{dt^2} = -kM\frac{\mathbf{u}}{r^2}.$$

(b) Use part (a) to show that $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$.

(c) Use part (b) to show that $\dfrac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}$.

(d) It follows from part (c) that $\mathbf{r} \times \mathbf{v} = \mathbf{c}$, where $\mathbf{c}$ is a constant vector. Show that $\mathbf{c} = r^2(\mathbf{u} \times \mathbf{u}')$.

(e) Show that $\dfrac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = 0$ and consequently $\mathbf{u} \cdot \mathbf{u}' = 0$.

(f) Use parts (a), (e), and (d) to show that

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{c}) = kM\frac{d\mathbf{u}}{dt}.$$

(g) By integrating the result in part (f) with respect to t, we get $\mathbf{v} \times \mathbf{c} = kM\mathbf{u} + \mathbf{d}$, where $\mathbf{d}$ is another constant vector. Dot both sides of this last expression by the vector $\mathbf{r} = r\mathbf{u}$ and use Problem 61 in Exercises 7.4 to show that

$$r = \frac{c^2/kM}{1 + (d/kM)\cos\theta},$$

where $c = \|\mathbf{c}\|$, $d = \|\mathbf{d}\|$, and θ is the angle between $\mathbf{d}$ and $\mathbf{r}$.

(h) Explain why the result in part (g) proves Kepler's first law.

(i) At perihelion* the vectors $\mathbf{r}$ and $\mathbf{v}$ are perpendicular and have magnitudes r_0 and v_0, respectively. Use this information and parts (d) and (g) to show that $c = r_0v_0$ and $d = r_0v_0^2 - kM$.

*This is the point in the orbit where the body is closest to the Sun.

9.3 Curvature and Components of Acceleration

■ Introduction Let C be a smooth curve in either 2- or 3-space traced out by a vector function $\mathbf{r}(t)$. In this section we are going to consider in greater detail the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t)$ introduced in the last section. But before doing this, we need to examine a scalar quantity called the **curvature** of a curve.

■ A Definition We know that $\mathbf{r}'(t)$ is a tangent vector to the curve C, and consequently

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{1}$$

is a **unit tangent**. But recall from the end of Section 9.1 that if C is parameterized by arc length s, then a unit tangent to the curve is also given by $d\mathbf{r}/ds$. The quantity $\|\mathbf{r}'(t)\|$ in (1) is related to arc length s by $ds/dt = \|\mathbf{r}'(t)\|$. Since the curve C is smooth, we know from page 442 that $ds/dt > 0$. Hence by the Chain Rule,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \quad \text{and so} \quad \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t). \tag{2}$$

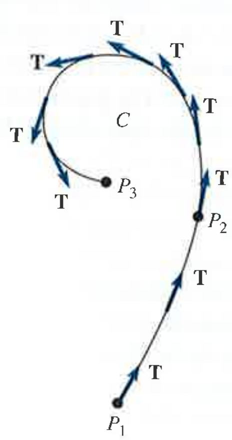

FIGURE 9.3.1 Unit tangents

Now suppose C is as shown in **FIGURE 9.3.1**. As s increases, $\mathbf{T}$ moves along C, changing direction but not length (it is always of unit length). Along the portion of the curve between P_1 and P_2 the vector $\mathbf{T}$ varies little in direction; along the curve between P_2 and P_3, where C obviously bends more sharply, the change in the direction of the tangent $\mathbf{T}$ is more pronounced. We use the *rate* at which the unit vector $\mathbf{T}$ changes direction with respect to arc length as an indicator of the curvature of a smooth curve C.

Show that the projectile will strike the target in midair. See **FIGURE 9.2.6**. [*Hint*: Assume that the origin is at the muzzle of the cannon and that the angle of elevation is θ. If $\mathbf{r}_p$ and $\mathbf{r}_t$ are position vectors of the projectile and target, respectively, is there a time at which $\mathbf{r}_p = \mathbf{r}_t$?]

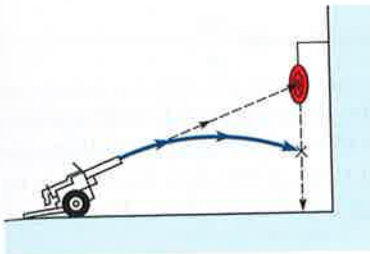

FIGURE 9.2.6 Cannon in Problem 17

18. In army field maneuvers sturdy equipment and supply packs are simply dropped from planes that fly horizontally at a slow speed and a low altitude. A supply plane flies horizontally over a target at an altitude of 1024 ft at a constant speed of 180 mi/h. Use (1) to determine the horizontal distance a supply pack travels relative to the point from which it was dropped. At what line-of-sight angle α should the supply pack be released in order to hit the target indicated in **FIGURE 9.2.7**?

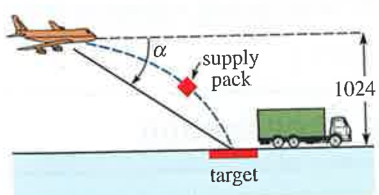

FIGURE 9.2.7 Supply plane in Problem 18

19. Suppose that $\mathbf{r}(t) = r_0 \cos \omega t\mathbf{i} + r_0 \sin \omega t\mathbf{j}$ is the position vector of an object that is moving in a circle of radius r_0 in the xy-plane. If $\|\mathbf{v}(t)\| = v$, show that the magnitude of the centripetal acceleration is $a = \|\mathbf{a}(t)\| = v^2/r_0$.

20. The motion of a particle in space is described by

$$\mathbf{r}(t) = b \cos t\mathbf{i} + b \sin t\mathbf{j} + ct\mathbf{k}, \quad t \geq 0.$$

(a) Compute $\|\mathbf{v}(t)\|$.

(b) Compute $s = \int_0^t \|\mathbf{v}(t)\|\, dt$ and verify that ds/dt is the same as the result of part (a).

(c) Verify that $d^2s/dt^2 \neq \|\mathbf{a}(t)\|$.

21. The **effective weight** w_e of mass m at the equator of the Earth is defined by $w_e = mg - ma$, where a is the magnitude of the centripetal acceleration given in Problem 19. Determine the effective weight of a 192-lb person if the radius of the Earth is 4000 mi, $g = 32$ ft/s^2, and $v = 1530$ ft/s.

22. Consider a bicyclist riding on a flat circular track of radius r_0. If m is the combined mass of the rider and bicycle, fill in the blanks in **FIGURE 9.2.8**. [*Hint*: Use Problem 19 and force = mass × acceleration. Assume that the directions are upward and to the left.] The resultant vector **U** gives the direction the bicyclist must be tipped to avoid falling. Find the angle ϕ from the vertical at which the bicyclist must be tipped if her speed is 44 ft/s and the radius of the track is 60 ft.

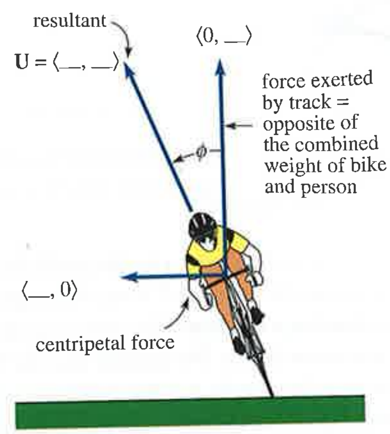

FIGURE 9.2.8 Bicyclist in Problem 22

23. Use the results given in (1) to show that the trajectory of a ballistic projectile is parabolic.

24. A projectile is launched with an initial speed v_0 from ground level at an angle of elevation θ. Use (1) to show that the maximum height and range of the projectile are, respectively,

$$H = \frac{v_0^2 \sin^2\theta}{2g} \quad \text{and} \quad R = \frac{v_0^2 \sin 2\theta}{g}.$$

25. The velocity of a particle moving in a fluid is described by means of a **velocity field** $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, where the components v_1, v_2, and v_3 are functions of x, y, z, and time t. If the velocity of the particle is $\mathbf{v}(t) = 6t^2x\mathbf{i} - 4ty^2\mathbf{j} + 2t(z+1)\mathbf{k}$, find $\mathbf{r}(t)$. [*Hint*: Use separation of variables.]

26. Suppose m is the mass of a moving particle. Newton's second law of motion can be written in vector form as

$$\mathbf{F} = m\mathbf{a} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt},$$

where $\mathbf{p} = m\mathbf{v}$ is called **linear momentum**. The angular momentum of the particle with respect to the origin is defined to be $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is its position vector. If the torque of the particle about the origin is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times d\mathbf{p}/dt$, show that $\boldsymbol{\tau}$ is the time rate of change of angular momentum.

27. Suppose the Sun is located at the origin. The gravitational force **F** exerted on a planet of mass m by the Sun of mass M is

$$\mathbf{F} = -k\frac{Mm}{r^2}\mathbf{u}.$$

F is a **central force**—that is, a force directed along the position vector **r** of the planet. Here k is the gravitational constant, $r = \|\mathbf{r}\|$, $\mathbf{u} = \mathbf{r}/r$ is a unit vector in the direction of **r**, and the minus sign indicates that **F** is an attractive force—that is, a force directed toward the Sun. See **FIGURE 9.2.9**.

(a) Use Problem 26 to show that the torque acting on the planet due to this central force is **0**.

Integrating again gives

$$\mathbf{r}(t) = (384\sqrt{3}t)\mathbf{i} + (-16t^2 + 384t)\mathbf{j}.$$

Hence, the parametric equations of the shell's trajectory are

$$x(t) = 384\sqrt{3}t, \qquad y(t) = -16t^2 + 384t. \tag{4}$$

(b) From (4) we see that $dy/dt = 0$ when $-32t + 384 = 0$ or $t = 12$. Thus, the maximum height H attained by the shell is

$$H = y(12) = -16(12)^2 + 384(12) = 2304 \text{ ft}.$$

(c) From (4) we see that $y(t) = 0$ when $-16t(t - 24) = 0$ or $t = 0$, $t = 24$. The range R is then

$$R = x(24) = 384\sqrt{3}(24) \approx 15{,}963 \text{ ft}.$$

(d) From (3) we obtain the impact speed of the shell:

$$\|\mathbf{v}(24)\| = \sqrt{(-384)^2 + (384\sqrt{3})^2} = 768 \text{ ft/s}.$$ ≡

Remarks

We have seen that the rate of change of arc length ds/dt is the same as the speed $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$. However, as we shall see in the next section, it does *not* follow that the *scalar acceleration* d^2s/dt^2 is the same as $\|\mathbf{a}(t)\| = \|\mathbf{r}''(t)\|$. See Problem 20 in Exercises 9.2.

9.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–8, $\mathbf{r}(t)$ is the position vector of a moving particle. Graph the curve and the velocity and acceleration vectors at the indicated time. Find the speed at that time.

1. $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{4}t^4\mathbf{j};\ t = 1$
2. $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t^2}\mathbf{j};\ t = 1$
3. $\mathbf{r}(t) = -\cosh 2t\mathbf{i} + \sinh 2t\mathbf{j};\ t = 0$
4. $\mathbf{r}(t) = 2\cos t\mathbf{i} + (1 + \sin t)\mathbf{j};\ t = \pi/3$
5. $\mathbf{r}(t) = 2\mathbf{i} + (t - 1)^2\mathbf{j} + t\mathbf{k};\ t = 2$
6. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t^3\mathbf{k};\ t = 2$
7. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k};\ t = 1$
8. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + t\mathbf{k};\ t = 1$
9. Suppose $\mathbf{r}(t) = t^2\mathbf{i} + (t^3 - 2t)\mathbf{j} + (t^2 - 5t)\mathbf{k}$ is the position vector of a moving particle. At what points does the particle pass through the xy-plane? What are its velocity and acceleration at these points?
10. Suppose a particle moves in space so that $\mathbf{a}(t) = \mathbf{0}$ for all time t. Describe its path.
11. A shell is fired from ground level with an initial speed of 480 ft/s at an angle of elevation of 30°. Find:
    (a) a vector function and parametric equations of the shell's trajectory,
    (b) the maximum altitude attained,
    (c) the range of the shell, and
    (d) the speed at impact.
12. Rework Problem 11 if the shell is fired with the same initial speed and the same angle of elevation but from a cliff 1600 ft high.
13. A used car is pushed off an 81-ft-high sheer seaside cliff with a speed of 4 ft/s. Find the speed at which the car hits the water.
14. A small projectile is launched from ground level with an initial speed of 98 m/s. Find the possible angles of elevation so that its range is 490 m.
15. A football quarterback throws a 100-yd "bomb" at an angle of 45° from the horizontal. What is the initial speed of the football at the point of release?
16. A quarterback throws a football with the same initial speed at an angle of 60° from the horizontal and then at an angle of 30° from the horizontal. Show that the range of the football is the same in each case. Generalize this result for any release angle $0 < \theta < \pi/2$.
17. A projectile is fired from a cannon directly at a target that is dropped from rest simultaneously as the cannon is fired.

9.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–10, graph the curve traced by the given vector function.

1. $\mathbf{r}(t) = 2 \sin t\mathbf{i} + 4 \cos t\mathbf{j} + t\mathbf{k};\ t \geq 0$
2. $\mathbf{r}(t) = \cos t\mathbf{i} + t\mathbf{j} + \sin t\mathbf{k};\ t \geq 0$
3. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \cos t\mathbf{k};\ t \geq 0$
4. $\mathbf{r}(t) = 4\mathbf{i} + 2 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$
5. $\mathbf{r}(t) = \langle e^t, e^{2t} \rangle$
6. $\mathbf{r}(t) = \cosh t\mathbf{i} + 3 \sinh t\mathbf{j}$
7. $\mathbf{r}(t) = \langle \sqrt{2} \sin t, \sqrt{2} \sin t, 2 \cos t \rangle;\ 0 \leq t \leq \pi/2$
8. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$
9. $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}$
10. $\mathbf{r}(t) = \langle t \cos t, t \sin t, t^2 \rangle$

In Problems 11–14, find the vector function that describes the curve C of intersection between the given surfaces. Sketch the curve C. Use the indicated parameter.

11. $z = x^2 + y^2,\ y = x;\ x = t$
12. $x^2 + y^2 - z^2 = 1,\ y = 2x;\ x = t$
13. $x^2 + y^2 = 9,\ z = 9 - x^2;\ x = 3 \cos t$
14. $z = x^2 + y^2,\ z = 1;\ x = \sin t$
15. Given that $\mathbf{r}(t) = \dfrac{\sin 2t}{t}\mathbf{i} + (t-2)^5\mathbf{j} + t \ln t\mathbf{k}$, find $\lim_{t\to 0^+} \mathbf{r}(t)$.
16. Given that $\lim_{t\to a} r_1(t) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\lim_{t\to a} \mathbf{r}_2(t) = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, find:
    (a) $\lim\limits_{t\to a} [-4\mathbf{r}_1(t) + 3\mathbf{r}_2(t)]$
    (b) $\lim\limits_{t\to a} \mathbf{r}_1(t) \cdot \mathbf{r}_2(t)$.

In Problems 17–20, find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for the given vector function.

17. $\mathbf{r}(t) = \ln t\mathbf{i} + \mathbf{j},\ t > 0$
18. $\mathbf{r}(t) = \langle t \cos t - \sin t, t + \cos t \rangle$
19. $\mathbf{r}(t) = \langle te^{2t}, t^3, 4t^2 - t \rangle$
20. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + \tan^{-1} t\mathbf{k}$

In Problems 21–24, graph the curve C that is described by $\mathbf{r}$ and graph $\mathbf{r}'$ at the indicated value of t.

21. $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 6 \sin t\mathbf{j};\ t = \pi/6$
22. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j};\ t = -1$
23. $\mathbf{r}(t) = 2\mathbf{i} + t\mathbf{j} + \dfrac{4}{1+t^2}\mathbf{k};\ t = 1$
24. $\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + 2t\mathbf{k};\ t = \pi/4$

In Problems 25 and 26, find parametric equations of the tangent line to the given curve at the indicated value of t.

25. $x = t,\ y = \frac{1}{2}t^2,\ z = \frac{1}{3}t^3;\ t = 2$
26. $x = t^3 - t,\ y = \dfrac{6t}{t+1},\ z = (2t+1)^2;\ t = 1$

In Problems 27–32, find the indicated derivative. Assume that all vector functions are differentiable.

27. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)]$
28. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot (t\mathbf{r}(t))]$
29. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))]$
30. $\dfrac{d}{dt}[\mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))]$
31. $\dfrac{d}{dt}\left[\mathbf{r}_1(2t) + \mathbf{r}_2\left(\dfrac{1}{t}\right)\right]$
32. $\dfrac{d}{dt}[t^3\mathbf{r}(t^2)]$

In Problems 33–36, evaluate the given integral.

33. $\displaystyle\int_{-1}^{2} (t\mathbf{i} + 3t^2\mathbf{j} + 4t^3\mathbf{k})\, dt$
34. $\displaystyle\int_{0}^{4} (\sqrt{2t+1}\mathbf{i} - \sqrt{t}\mathbf{j} + \sin \pi t\mathbf{k})\, dt$
35. $\displaystyle\int (te^t\mathbf{i} - e^{-2t}\mathbf{j} + te^{t^2}\mathbf{k})\, dt$
36. $\displaystyle\int \frac{1}{1+t^2}(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\, dt$

In Problems 37–40, find a vector function $\mathbf{r}$ that satisfies the indicated conditions.

37. $\mathbf{r}'(t) = 6\mathbf{i} + 6t\mathbf{j} + 3t^2\mathbf{k};\ r(0) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$
38. $\mathbf{r}'(t) = t \sin t^2\mathbf{i} - \cos 2t\mathbf{j};\ \mathbf{r}(0) = \frac{3}{2}\mathbf{i}$
39. $\mathbf{r}''(t) = 12t\mathbf{i} - 3t^{-1/2}\mathbf{j} + 2\mathbf{k};\ \mathbf{r}'(1) = \mathbf{j},\ \mathbf{r}(1) = 2\mathbf{i} - \mathbf{k}$
40. $\mathbf{r}''(t) = \sec^2 t\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k};$
    $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \mathbf{r}(0) = -\mathbf{j} + 5\mathbf{k}$

In Problems 41–44, find the length of the curve traced by the given vector function on the indicated interval.

41. $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k};\ 0 \leq t \leq 2\pi$
42. $\mathbf{r}(t) = t\mathbf{i} + t \cos t\mathbf{j} + t \sin t\mathbf{k};\ 0 \leq t \leq \pi$
43. $\mathbf{r}(t) = e^t \cos 2t\mathbf{i} + e^t \sin 2t\mathbf{j} + e^t\mathbf{k};\ 0 \leq t \leq 3\pi$
44. $\mathbf{r}(t) = 3t\mathbf{i} + \sqrt{3}t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k};\ 0 \leq t \leq 1$
45. Express the vector equation of a circle $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j}$ as a function of arc length s. Verify that $\mathbf{r}'(s)$ is a unit vector.
46. If $\mathbf{r}(s)$ is the vector function given in (4), verify that $\mathbf{r}'(s)$ is a unit vector.
47. Suppose $\mathbf{r}$ is a differentiable vector function for which $\|\mathbf{r}(t)\| = c$ for all t. Show that the tangent vector $\mathbf{r}'(t)$ is perpendicular to the position vector $\mathbf{r}(t)$ for all t.
48. In Problem 47, describe geometrically the kind of curve C for which $\|\mathbf{r}(t)\| = c$.

Miscellaneous Problems

49. Prove Theorem 9.1.4(*ii*).
50. Prove Theorem 9.1.4(*iii*).
51. Prove Theorem 9.1.4(*iv*).
52. If $\mathbf{v}$ is a constant vector and $\mathbf{r}$ is integrable on $[a, b]$, prove that $\int_a^b \mathbf{v} \cdot \mathbf{r}(t)\, dt = \mathbf{v} \cdot \int_a^b \mathbf{r}(t)\, dt$.