

The Heat Equation

The Fundamental Solution and Mean Value Property

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April 19, 2021

We have considered the heat equation

$$u_t = \Delta u$$

as the gradient flow in $C^\infty(\bar{U})$ of the Dirichlet energy

$$\mathcal{D}[u] = \int_U |Du|^2.$$

We recall also that the Euler-Lagrange equation for the Dirichlet energy is Laplace's equation $\Delta u = 0$. The solutions of Laplace's equation enjoyed many nice properties mainly following (for us) from the mean value property. We now briefly attempt to extend this discussion (without proofs) to the heat equation.¹

Fundamental Solution

Perhaps the easiest starting point is with the **fundamental solution**. Recall that the fundamental solution $\Phi(\mathbf{x} - \mathbf{p})$ for Laplace's equation had a singularity at the point \mathbf{p} and gave, by convolution $\Phi * f$, solutions of Poisson's equation on all of \mathbb{R}^n . The fundamental solution for the heat equation (centered at $\mathbf{0} \in \mathbb{R}^n$) is $\Phi : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ given by

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}.$$

Exercise 1 *It should be clear to you how to move the center of the fundamental solution to another point $\mathbf{p} \in \mathbb{R}^n$. Use mathematical software to plot the fundamental solution in one space dimension ($n = 1$).*

You should see that the limit as $t \searrow 0$ of the fundamental solution develops a singularity at the center in the limit. In fact, the singularity is quite severe, and you can verify the following:

- (i) $\lim_{t \searrow 0} \Phi(\mathbf{x} - \mathbf{p}, t) \equiv 0$ if $\mathbf{x} \neq \mathbf{p}$.
- (ii) $\lim_{t \searrow 0} \Phi(\mathbf{x} - \mathbf{p}, t) \equiv +\infty$ if $\mathbf{x} = \mathbf{p}$.
- (iii) Given $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the **spatial convolution** $\Phi * u_0 : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ given by

$$(\Phi * u_0)(\mathbf{p}, t) = \int_{\mathbf{x} \in \mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{p}, t) u_0(\mathbf{x})$$

satisfies $\Phi * u_0 \in C^\infty(\mathbb{R}^n \times (0, \infty))$ with $(\Phi * u_0)_t = \Delta(\Phi * u_0)$ and

$$\lim_{t \searrow 0} \Phi * u_0(\mathbf{p}, t) = u_0(\mathbf{p}) \quad \text{for all } \mathbf{p} \in \mathbb{R}^n.$$

¹A good reference for this material is Chapter 2 of Craig Evans' book *Partial Differential Equations*.

Mean Value Property; Heat Balls

There is a **mean value property** for the heat equation, though it is a little complicated:

Theorem 1 *If $u \in C^2(\mathcal{U} \times (0, T))$ with $u_t = \Delta u$, then*

$$u(\mathbf{p}, t) = \frac{1}{4r^n} \int_{(\mathbf{x}, \tau) \in B_r^{\text{heat}}(\mathbf{p}, t)} u(\mathbf{x}, \tau) \frac{|\mathbf{x} - \mathbf{p}|^2}{(t - \tau)^2}$$

for every **heat ball**

$$B_r^{\text{heat}}(\mathbf{p}, t) \subset \mathcal{U} \times (0, T)$$

where

$$B_r^{\text{heat}}(\mathbf{p}, t) = \left\{ (\mathbf{x}, \tau) \in \mathbb{R}^{n+1} : \Phi(\mathbf{p} - \mathbf{x}, t - \tau) \geq \frac{1}{r^n} \right\}.$$

As for solutions of Laplace's equation, the mean value property has various striking consequences.

Exercise 2 *The heat ball is not a spherical ball in \mathbb{R}^{n+1} . Use numerical software to draw a picture of heat balls in one and two spatial dimensions. Find the "equator time," i.e., the time corresponding to maximum spatial radius of the heat ball. How does the maximum radius at the equator compare to the "height" (in the time dimension) of the heat ball? Determine the $(n + 1)$ -dimensional measure of the heat ball. Note carefully the scaling factor in front of the integral in the mean value property.*

Strong Maximum Principle

Theorem 2 *(strong maximum principle for the heat equation) If*

$$u \in C^2(\mathcal{U} \times (0, T)) \cap C^0(\bar{\mathcal{U}} \times [0, T])$$

solves $u_t = \Delta u$, then

$$\max_{\bar{\mathcal{U}} \times [0, T]} u = \max_{(\partial\mathcal{U} \times [0, T]) \cap (\bar{\mathcal{U}} \times [0, T])} u,$$

and if \mathcal{U} is connected,

$$u(\mathbf{x}, t) < M = \max_{\bar{\mathcal{U}} \times [0, T]} u$$

unless $u(\mathbf{x}, t) \equiv M$ is constant.

Note, the sets $\bar{\mathcal{U}} \times [0, T]$ and $(\partial\mathcal{U} \times [0, T]) \cap (\bar{\mathcal{U}} \times [0, T])$ are not closed as subsets of \mathbb{R}^{n+1} and the fact that a maximum value of a solution u is attained on these sets is part of the assertion of the theorem.

Corollary 1 *(uniqueness) Say $u, v \in C^2(\mathcal{U} \times (0, T)) \cap C^0(\bar{\mathcal{U}} \times [0, T])$ are two solutions of*

$$\begin{cases} u_t = \Delta u & \text{on } \mathcal{U} \times (0, T) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \\ u|_{\mathbf{x} \in \partial\mathcal{U}} = g(\mathbf{x}). \end{cases}$$

Then $u \equiv v$.

Corollary 2 *(infinite propagation speed) Say $u \in C^2(\mathcal{U} \times (0, T)) \cap C^0(\bar{\mathcal{U}} \times [0, T])$ is a solution of*

$$\begin{cases} u_t = \Delta u & \text{on } \mathcal{U} \times (0, T) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \\ u|_{\partial\mathcal{U}} \equiv 0 \end{cases}$$

with $u_0 \geq 0$ and $u(\mathbf{p}) > 0$ (at one point), then $u > 0$ on $\mathcal{U} \times (0, \infty)$.

Regularity

If $u \in C^2(\mathcal{U} \times (0, T))$ satisfies $u_t = \Delta u$, then $u \in C^\infty(\mathcal{U} \times (0, T))$. Note that this does not require anything at all about the limit

$$\lim_{t \searrow 0} u(\mathbf{x}, t).$$

In fact, this limit may be discontinuous or worse.

The results/properties listed here are, as for solutions of Laplace's equation, all consequences of the mean value property.