Green's Function(s)
Under Construction

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1 Singular solutions of Laplace's equation; fundamental solution

Consider Laplace's equation $\Delta u = 0$ for a function $u \in C^\infty(\mathbb{R}^n)$ with $n \geq 3$. One can look for radial solutions having the form $u(\mathbf{x}) = \phi(|\mathbf{x}|)$ for some function $\phi : [0, \infty) \to \mathbb{R}$, and one finds and ODE for ϕ leading to the unique solutions

$$u(\mathbf{x}) = \text{constant}.$$

These are not very interesting solutions, but in the process one does see that the ODE for $\phi = \phi(r)$ has a (nonconstant) solution defined for $r > 0$. That solution, up to additive and multiplicative constants is

$$\phi(r) = \frac{1}{r^{n-2}}.$$

I'm going to start the discussion of Green's function(s) here with a discussion of the interesting solution $\Phi : \mathbb{R}^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ of Laplace's equation given by

$$\Phi(\mathbf{x}) = \frac{C}{|\mathbf{x}|^{n-2}}$$

where C is a multiplicative constant. First of all, let's take the constant C to be positive, but otherwise arbitrary, and see what we can find. Notice Φ is a solution on punctured Euclidean space $\mathbb{R}^n \backslash \{\mathbf{0}\}$ instead of the entire space. There is a singularity at the puncture $\mathbf{x} = \mathbf{0}$ and assuming $C > 0$, we can say

$$\lim_{\mathbf{x} \to \mathbf{0}} \Phi(\mathbf{x}) = +\infty.$$

In two dimensions, one can draw the graph of such a function, but you'll note that we're only considering this solution for $n \geq 3$ and the formula just gives a constant for $n = 2$, so perhaps something different happens when $n = 2$. In fact, I suggest it might be instructive for you to follow along with the discussion I give here and try to see what actually happens when $n = 2$ at each step of the way.

Exercise 1 Look for a radial solution Φ of Laplace's equation on the punctured plane $\mathbb{R}^2 \backslash \{\mathbf{0}\}$. You should find essentially one interesting solution $\Phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ up to a multiplicative constant. Restrict the sign of the constant so that

$$\lim_{\mathbf{x} \to \mathbf{0}} \Phi(\mathbf{x}) = +\infty.$$

Returning to the singularity of my function with

$$\Phi(\mathbf{x}) = \frac{C}{|\mathbf{x}|^{n-2}}$$

when $n \geq 3$, I'd like to know about the **integrability** at the singularity. The function Φ is everywhere positive, so the integral

$$\int_{\mathbb{R}^n} \Phi$$

should make sense, though the value may be $+\infty$. I'm going to introduce a kind of change of variable to evaluate this integral, and it may be one with which you are not familiar...and even of a kind with which you are not familiar. Don't worry if what I am about to present is a little confusing. I will go back later and discuss integration and the techniques I'm using in more detail. I am hopeful you will find what happens here so compelling that you'll be excited to figure out what is going on.

Okay, I'm going to think of $\mathbb{R}^n$ as the image of a mapping on an n-dimensional curved space, but a relatively simple one. That space is the product of an interval with a sphere: $(0, \infty) \times \mathbb{S}^{n-1}$. Here

$$\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\},$$

and this is a hypersurface in $\mathbb{R}^n$. This sphere has many interesting properties and can be parameterized in various different ways, but I'll try to avoid many of those interesting details. I'm interested in the map $\psi : (0, \infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n$ by

$$\psi(r, \mathbf{p}) = r\mathbf{p}.$$

Notice that ψ gives a one-to-one correspondence between $(0, \infty) \times \mathbb{S}^{n-1}$ and the punctured Euclidean space $\mathbb{R}^n \backslash \{\mathbf{0}\}$. Since only one point is missing, this is appropriate for integration. That is to say, in calculating for example

$$\int_{\mathbb{R}^n} \Phi$$

we can integrate via change of variables over $(0, \infty) \times \mathbb{S}^{n-1}$. Here is how that goes:

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi &= \int_{\mathbf{x} \in \mathbb{R}^n} \Phi(\mathbf{x}) \\
&= \int_{(r,\mathbf{p}) \in (0,\infty) \times \mathbb{S}^{n-1}} \Phi(\psi(r, \mathbf{p}))\ J \qquad (1) \\
&= \int_0^\infty \left(\int_{\mathbf{p} \in \mathbb{S}^{n-1}} \Phi \circ \psi(r, \mathbf{p})\ J \right) dr \\
&= \int_0^\infty \left(\int_{\mathbf{p} \in \mathbb{S}^{n-1}} \phi(r)\ J \right) dr \\
&= \int_0^\infty \phi(r)\ J \left(\int_{\mathbf{p} \in \mathbb{S}^{n-1}} 1 \right) dr \\
&= \left(\int_{\mathbf{p} \in \mathbb{S}^{n-1}} 1 \right) \int_0^\infty \frac{CJ}{r^{n-2}}\, dr. \qquad (2)
\end{aligned}$$

Okay, let's pause for a moment and look at some of what we have here. There is an interesting constant

$$\int_{\mathbf{p} \in \mathbb{S}^{n-1}} 1 = \int_{\mathbb{S}^{n-1}} 1$$

in the last line (2). This constant, it will be noted, has nothing to do with the function Φ. This is some interesting geometric constant. Were we in $n = 2$ dimensions, then $\mathbb{S}^{n-1}$ is a unit circle, and integrating the function 1 on the unit circle[1] simply gives the length of the unit circle:

$$\int_{\mathbb{S}^1} 1 = 2\pi.$$

You probably also know

$$\int_{\mathbb{S}^2} 1 = 4\pi,$$

[1] If this seems mysterious, you might want to have a look at the appendix on integration below.

and this is the first case of interest when $n \geq 3$ because $\mathbb{S}^2 \subset \mathbb{R}^3$. The numbers

$$\int_{\mathbb{S}^{n-1}} 1,$$

the $(n-1)$ dimensional measures of the unit spheres in $\mathbb{R}^n$ when $n > 3$ (and also for $n = 1$), are numbers you may not know. But these numbers might be interesting to know. Let's set the question of computing these numbers aside for the moment, and just note that they are some numbers. Specifically,

$$\int_{\mathbb{S}^{n-1}} 1$$

in (2) is just some constant that can be computed. Perhaps it is convenient to give this constant a short(er) or snappy symbolic name. One possibility that is technically rather nice is $\mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ which stands for the $(n-1)$ *dimensional Hausdorff measure of the unit sphere in* $\mathbb{R}^n$. This name, however, is not much shorter than just writing out the integral. There is another traditional name that is rather shorter. That name is $n\omega_n$. Thus, we can write

$$\int_{\mathbb{S}^{n-1}} 1 = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n.$$

I'll explain a bit more about each of these two names below, but needless to say ω_n is some constant with a snappy name and

$$\omega_n = \frac{1}{n} \int_{\mathbb{S}^{n-1}} 1.$$

The fact that $\mathbb{S}^{n-1}$ has associated with it this measure is an interesting property we are not going to avoid, but we will set aside the precise computation of that measure for the moment.

Returning to (1) notice the function J which appeared here. This is sometimes called a **Jacobian scaling factor**. We would get the wrong answer without it, and we need to know its precise value. Let me try to draw inspiration from something with which you are familiar: Say you want to integrate a function $f : B_R(\mathbf{0}) \to \mathbb{R}$ on a disk of radius R in the plane $\mathbb{R}^2$. For this, you might use polar coordinates writing

$$\int_{B_R(\mathbf{0})} f = \int_0^R \left(\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, d\theta \right) r\, dr. \tag{3}$$

In the background here is the polar coordinates map $\psi_0 : (0, R) \times [0, 2\pi) \to \mathbb{R}^2$ by

$$\psi_0(r, \theta) = (r\cos\theta, r\sin\theta).$$

We can certainly extend ψ_0 to the slightly larger domain $[0, R) \times [0, 2\pi)$, but then we lose injectivity, that is, the map is no longer one-to-one. On the other hand, the one-to-one correspondence we have is with the punctured disk. That is okay for integration because the single point at the center doesn't effect the integral. In fact, this entire mapping is a bit funky when it comes to the intersection of the positive x-axis with the disk. Notice the inverse mapping is not continuous along this line segment. Of course, that doesn't matter too much either because the two-dimensional measure, i.e., the area with which we are doing the integration, gives zero value to that segment, so omitting the positive x-axis from the disk doesn't effect the integral either. In any case, the mapping ψ_0 has associated with it the Jacobian scaling factor

$$J = |\det D\psi_0|$$

where $D\psi_0$ is the 2×2 matrix of the first partials of ψ_0 or what is called the **total derivative** in this case. Specifically, we have

$$J = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| = r.$$

The point here is that the composition $f(r\cos\theta, r\sin\theta)$ gives the same values of f at each point of the (punctured) disk, but assigned to points in the rectangle $(0, R) \times (0, 2\pi)$. If we were to just integrate over the rectangle without the Jacobian scaling factor r we would in effect be summing terms that look like

$$f \circ \psi_0(r_j, \theta_j) \ \text{area}(A_j)$$

where A_j is a small rectangle containing the point (r_j, θ_j) in big rectangle $(0, R) \times (0, 2\pi)$. That is what integration means. This rectangle A_j has a rather different area from the corresponding region $\psi_0(A_j)$ in the disk, so you would get the wrong answer. The Jacobian scaling factor needs to be there due to the approximate relation

$$\text{area}(\psi_0(A_j)) \approx r \ \text{area}(A_j).$$

To expand on what is happening when you write (3) there are really two steps: First the integral over $B_R(\mathbf{0})$ is written as an integral over the rectangle $(0, R) \times (0, 2\pi)$,

$$\int_{B_R(\mathbf{0})} f = \int_{(r,\theta)\in(0,R)\times(0,2\pi)} f \circ \psi_0(r, \theta) \ r.$$

This is the change of variables. Then the relation between an integral over a rectangle and iterated integrals is used:

$$\int_{(r,\theta)\in(0,R)\times(0,2\pi)} f\circ\psi_0(r,\theta)\ r = \int_0^R \left(\int_0^{2\pi} f\circ\psi_0(r\theta)\ r\right) dr.$$

This relation is called Fubini's theorem.

For application in higher dimensions, I want to approach this a little differently. Instead of the rectangle $(0,R)\times[0,2\pi)$, in this lower dimensional case I want to use the cross product

$$(0,R)\times\mathbb{S}^1$$

which is a cylinder. I want to integrate on this cylinder. For the change of variables I have $\psi : (0,R)\times\mathbb{S}^1 \to B_R(\mathbf{0})$ by $\psi(r,\mathbf{p}) = r\mathbf{p}$. In this way I can think of the scaling factor in terms of the cross product directly with the unit circle $\mathbb{S}^1$ mapping to the circle $\partial B_r(\mathbf{0})$ for each r:

$$\int_{B_R(\mathbf{0})} f = \int_{(r,\mathbf{p})\in(0,R)\times\mathbb{S}^1} f\circ\psi(r,\mathbf{p})\ r$$

where we now have the precise relation length$(\psi(\Gamma)) = r$ length(Γ) where Γ is a subset of the circle $\mathbb{S}^1$ which is dilated by r onto the circle $\partial B_r(\mathbf{0})$. This is the key point: In higher dimensions also, you can see the Jacobian scaling factor simply by considering dilation of the spheres.

Continuing with the two-dimensional integration we can apply a generalization of Fubini's theorem to write the iterated integral

$$\begin{aligned}\int_{(r,\mathbf{p})\in(0,R)\times\mathbb{S}^1} f\circ\psi(r,\mathbf{p})\ r &= \int_{r\in(0,R)}\left(\int_{\mathbf{p}\in\mathbb{S}^1} f\circ\psi(r,\mathbf{p})\ r\right)\\ &= \int_0^R\left(\int_{\mathbf{p}\in\mathbb{S}^1} f\circ\psi(r,\mathbf{p})\ r\right) dr.\end{aligned}$$

Together, we obtain the alternative expression

$$\int_{B_R(\mathbf{0})} f = \int_0^R\left(\int_{\mathbf{p}\in\mathbb{S}^1} f\circ\psi(r,\mathbf{p})\ \right) r\,dr$$

which is a little different from the formula you know, but you can then use any technique of integration for integrating on the circle to get back the familiar formula (3).

Hopefully this discussion makes more or less clear what happens in (1) when we write

$$\int_{\mathbb{R}^n} \Phi = \int_{(r,\mathbf{p})\in(0,\infty)\times\mathbb{S}^{n-1}} \Phi(\psi(r,\mathbf{p}))\ J.$$

Here, the Jacobian scaling factor should be the scaling factor for $(n-1)$ dimensional measure under scaling by r from the unit sphere $\mathbb{S}^{n-1}$. In general, as you might guess, that scaling factor is r^{n-1}. This has some nice consequences. For one thing, the value of

$$\mathcal{H}^{n-1}(\partial B_r(\mathbf{0})) = n\omega_n\ r^{n-1}.$$

More immediately, we can continue from (2) and write

$$\int_{\mathbb{R}^n} \Phi = n\omega_n \int_0^\infty \frac{Cr^{n-1}}{r^{n-2}}\, dr = Cn\omega_n \int_0^\infty r\, dr.$$

You may note right away that

$$\int_0^\infty r\, dr = +\infty$$

so Φ is not integrable globally. On the other hand, the source of the non-integrability is from $r = \infty$ not from the singularity at $r = 0$. Very specifically, if we redo the calculation integrating only on $B_R(\mathbf{0}) \subset \mathbb{R}^n$ we find

$$\int_{B_R(\mathbf{0})} \Phi = Cn\omega_n \int_0^R r\, dr = \frac{Cn\omega_n}{2}\ R^2 < \infty.$$

Indeed the singularity is an integrable singularity. This opens up an interesting possibility.

2 Poisson's equation

Having established the integrability of Φ at the singularity, the second thing I want to do is consider the **convolution** of my radial solution(s) $\Phi : \mathbb{R}^n\backslash\{\mathbf{0}\} \to \mathbb{R}$ with $n \geq 3$ with some function $f : \mathbb{R}^n \to \mathbb{R}$. Such a convolution integral defines a new function $u : \mathbb{R}^n \to \mathbb{R}$ given by

$$u(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} \Phi(\mathbf{x}-\xi) f(\xi). \tag{4}$$

For each fixed $\mathbf{x}$, the integrand is constructed by translating the singularity in Φ to $\mathbf{x}$ and then integrating against f. In some sense, the purpose of this section is to investigate the consequences of that construction: What is the nature of this function u constructed from f?

Before we get to that investigation, there are some preliminary considerations of which we should take account. Obviously, we can't take just any function $f : \mathbb{R}^n \to \mathbb{R}$. If we take f to be the constant function with $f \equiv 1$ for example, then $u \equiv \infty$ because we know Φ is not integrable at $\infty \in \mathbb{R}^n$. Thus, we had better restrict f in some way. We will make two restrictions both of which can be relaxed to some extent, but they will make the discussion below simpler.

1. We assume f is three times continuously differentiable, that is $f \in C^3(\mathbb{R}^n)$.

2. We assume f has **compact support**.

For functions defined on all of $\mathbb{R}^n$, the property of compact support is very simple. This just means there is some $R > 0$ for which

$$f(\mathbf{x}) \equiv 0 \qquad \text{for} \qquad |\mathbf{x}| > R.$$

If we have also $f \in C^0(\mathbb{R}^n)$, and remember $C^3(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$ so we we will have that in this case, then $f(\mathbf{x}) = 0$ for $|\mathbf{x}| = R$ as well. Okay then, under these assumptions the values of u given in (4) will all be finite, so we have $u : \mathbb{R}^n \to \mathbb{R}$. Incidentally, there is a nice notation for the set of all functions satisfying the conditions **1** and **2** above. The collection of all three times continuously differentiable functions with compact support in $\mathbb{R}^n$ is denoted $C^3_c(\mathbb{R}^n)$. The subscript "c" indicates the restriction to functions of compact support.

2.1 Continuity of the convolution with Φ

If we attempt to consider $|u(\mathbf{x}) - u(\mathbf{p})|$ with a view to showing $u = \Phi * f$ is continuous at $\mathbf{p} \in \mathbb{R}^n$, we might write down something like this:

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{p})| &= \left| \int_{\xi \in \mathbb{R}^n} \Phi(\mathbf{x} - \xi) f(\xi) - \int_{\xi \in \mathbb{R}^n} \Phi(\mathbf{p} - \xi) f(\xi) \right| \\ &= \left| \int_{\xi \in \mathbb{R}^n} [\Phi(\mathbf{x} - \xi) - \Phi(\mathbf{p} - \xi)] f(\xi) \right| \\ &\le \int_{\xi \in \mathbb{R}^n} |\Phi(\mathbf{x} - \xi) - \Phi(\mathbf{p} - \xi)| \, |f(\xi)|. \end{aligned}$$

Here, it turns out $|f(\xi)|$ can be bounded uniformly by a constant. This assertion follows from a result in calculus called the **extreme value theorem**:

Theorem 1 (extreme value theorem) If Ω is a bounded[2] open subset of $\mathbb{R}^n$ and $f \in C^0(\overline{\Omega})$, then there exist points $\mathbf{p}, \mathbf{q} \in \overline{\Omega}$ such that

$$f(\mathbf{p}) = \min_{\mathbf{x}\in\overline{\Omega}} f(\mathbf{x}) \qquad \text{and} \qquad f(\mathbf{q}) = \max_{\mathbf{x}\in\overline{\Omega}} f(\mathbf{x}).$$

Under our assumptions the function $|f|$ is continuous on all of $\mathbb{R}^n$ and we know $f(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \geq R$. Thus, by the extreme value theorem the value

$$\max_{\mathbf{x}\in\overline{B_R(\mathbf{0})}} |f(\mathbf{x})| < \infty$$

providing the claimed bound. In this way, we can obtain an estimate

$$|u(\mathbf{x}) - u(\mathbf{p})| \leq \max_{\eta\in\overline{B_R(\mathbf{0})}} |f(\eta)| \int_{\xi\in\mathbb{R}^n} |\Phi(\mathbf{x}-\xi) - \Phi(\mathbf{p}-\xi)|.$$

Unfortunately, at this point the situation for our efforts at estimation becomes difficult. The problem is that Φ is singular, so it is impossible to get a uniform bound on the integrand $|\Phi(\mathbf{x}-\xi) - \Phi(\mathbf{p}-\xi)|$ much less say the entire integral is small to show continuity.

Fortunately, there is another important alternative. We change variables $\eta = \mathbf{x}-\xi$:

$$u(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} \Phi(\mathbf{x}-\xi)\ f(\xi) = \int_{\eta\in\mathbb{R}^n} \Phi(\eta)\ f(\mathbf{x}-\eta).$$

It will be noticed that the new expression for u has the same form as the original convolution integral with the roles of Φ and f reversed. Thus, if we write

$$u(\mathbf{x}) = \Phi * f(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} \Phi(\mathbf{x}-\xi)\ f(\xi),$$

then we can also write

$$\Phi * f(\mathbf{x}) = f * \Phi(\mathbf{x}).$$

[2]Starting here, I will use some notation and terminology from Appendix C.

Thus, the convolution procedure is in this case *commutative.* Proceeding with the continuity estimate using the alternative form $f * \Phi$, we have

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{p})| &= \left| \int_{\eta\in\mathbb{R}^n} \Phi(\eta) f(\mathbf{x}-\eta) - \int_{\eta\in\mathbb{R}^n} \Phi(\eta) f(\mathbf{x}-\eta) \right| \\ &= \left| \int_{\eta\in\mathbb{R}^n} \Phi(\eta)[f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)] \right| \\ &\le \int_{\eta\in\mathbb{R}^n} \Phi(\eta)|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)|. \end{aligned} \tag{5}$$

Since f is continuous at $\mathbf{p}-\eta$, we know that for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|\xi - (\mathbf{p}-\eta)| < \delta \qquad \text{implies} \qquad |f(\xi) - f(\mathbf{p}-\eta)| < \epsilon.$$

In particular, if $|\mathbf{x}-\mathbf{p}| < \delta$, then $|\mathbf{x}-\eta-(\mathbf{p}-\eta)| < \delta$, and

$$|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| < \epsilon. \tag{6}$$

Exercise 2 We know there is some $R > 0$ for which $f(\mathbf{x}) = 0$ whenever $|\mathbf{x}| \ge R$. Show that given $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{x}$ with $|\mathbf{x}-\mathbf{p}| < a$, there is some $M > 0$ so that

$$f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta) \equiv 0 \qquad \text{whenever} \qquad |\eta - \mathbf{p}| > M.$$

In fact, obtain the stronger assertion

$$f(\mathbf{x}-\eta) = f(\mathbf{p}-\eta) \equiv 0 \qquad \text{whenever} \qquad |\eta - \mathbf{p}| \ge M.$$

Let us say we can use the estimate (6) in the integrand appearing in (5). Then we conclude that for $|\mathbf{x}-\mathbf{p}| < \delta$ there holds

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{p})| &\le \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| \\ &= \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| \\ &\le \epsilon \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta). \end{aligned} \tag{7}$$

Since

$$\int_{\eta\in B_M(\mathbf{p})} \Phi(\eta) < \infty,$$

this looks very promising: If we can take $\epsilon > 0$ as small as we like here, we can seemingly make $|u(\mathbf{x}) - u(\mathbf{p})|$ as small as we like, and that gives the continuity of u.

You may have felt uneasy at some point in the argument above and if so, you had good reason to be uneasy. Though it comes to the essentially correct conclusion that the convolution integral $u = \Phi * f$ is continuous, the argument above is seriously flawed and basically incorrect. Do you see why?

The function f is indeed continuous at the point $\mathbf{p} - \eta$, but the tolerance δ from the definition of continuity used to get the estimate (6) must be assumed to depend on the point $\mathbf{p} - \eta$ and the point η in particular. As a consequence, we do not really have a single tolerance δ here, but rather an **uncountably infinite collection of tolerances** $\delta = \delta_\eta$ one for each $\eta \in \mathbb{R}^n$. In order to make the argument above valid as it stands, we would have to ensure there is one fixed positive number $\delta_* > 0$ satisfying

$$\delta_* \le \delta_\eta \qquad \text{for all } \eta \in \mathbb{R}^n.$$

Fortunately, there is a relatively easy way around this difficulty because $f \in C_c^3(\mathbb{R}^n)$ has such nice regularity, and some continuous derivatives in particular. The conclusion that the function $u = \Phi * f$ is continuous is still true if f is only continuous, but I've relegated the details of the proof in that case to Appendix A. There are quite a few other potentially interesting and useful details in that appendix, so you might want to at least scan it even if you are not interested in the subtle and sometimes important topics of compact sets and uniform continuity.

The easy approach is along the following lines:

$$\begin{aligned}
|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| &= \left| \int_0^1 \frac{d}{dt} f((1-t)(\mathbf{p} - \eta) + t(\mathbf{x} - \eta))\, dt \right| \\
&= \left| \int_0^1 Df((1-t)(\mathbf{p} - \eta) + t(\mathbf{x} - \eta)) \cdot (\mathbf{x} - \mathbf{p})\, dt \right| \\
&\le \int_0^1 |Df((1-t)(\mathbf{p} - \eta) + t(\mathbf{x} - \eta))|\, dt\ |\mathbf{x} - \mathbf{p}| \\
&\le \max_{\xi \in \mathbb{R}^n} |Df(\xi)|\ |\mathbf{x} - \mathbf{p}|.
\end{aligned}$$

Here we have used the Cauchy-Schwarz inequality

$$|Df((1-t)(\mathbf{p} - \eta) + t(\mathbf{x} - \eta)) \cdot (\mathbf{x} - \mathbf{p})| \le |Df((1-t)(\mathbf{p} - \eta) + t(\mathbf{x} - \eta))|\ |\mathbf{x} - \mathbf{p}|$$

and the extreme value theorem applied to the (continuous) partial derivatives of f so that

$$\max_{\xi\in\mathbb{R}^n} |Df(\xi)| = \max_{\xi\in\overline{B_R(\mathbf{0})}} \sqrt{\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)^2} < \infty$$

where $f(\xi) \equiv 0$ for $|\xi| \geq R$ so that $Df(\xi) \equiv \mathbf{0}$ for $|\xi| > R$ as well.

In view of this estimate, we can take for any $\epsilon > 0$ the value

$$\delta = \frac{\epsilon}{(1 + \max_{\xi\in\mathbb{R}^n} |Df(\xi)|) \left(1 + \int_{B_M(\mathbf{p})} \Phi\right)} > 0$$

and if $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \leq \max_{\xi\in\mathbb{R}^n} |Df(\xi)|\ |\mathbf{x} - \mathbf{p}| < \frac{\epsilon}{1 + \int_{B_M(\mathbf{p})} \Phi}$$

and

$$|\Phi * f(\mathbf{x}) - \Phi * f(\mathbf{p})| \leq \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| < \epsilon.$$

2.2 Differentiability of $\Phi * f$

Consider a difference quotient

$$\frac{\Phi * f(\mathbf{p} + v\mathbf{e}_j) - \Phi * f(\mathbf{p})}{v} = \frac{1}{v} \int_{\eta\in\mathbb{R}^n} \Phi(\eta)[f(\mathbf{p} + v\mathbf{e}_j - \eta) - f(\mathbf{p} - \eta)].$$

Writing

$$\begin{aligned} &\frac{f(\mathbf{p} + v\mathbf{e}_j - \eta) - f(\mathbf{p} - \eta)}{v} \\ &\qquad = \frac{1}{v} \int_0^1 \frac{d}{dt} f((1-t)(\mathbf{p} - \eta) + t(\mathbf{p} + v\mathbf{e}_j - \eta))\, dt \\ &\qquad = \int_0^1 Df((1-t)(\mathbf{p} - \eta) + t(\mathbf{p} + v\mathbf{e}_j - \eta)) \cdot \mathbf{e}_j\, dt \end{aligned}$$

we have

$$\begin{aligned}
&\left|\frac{\Phi * f(\mathbf{p}+v\mathbf{e}_j)-\Phi(\mathbf{p})}{v} - \Phi * \frac{\partial f}{\partial x_j}(\mathbf{p})\right| \\
&\quad = \left|\int_{\eta\in\mathbb{R}^n} \Phi(\eta)\left(\int_0^1 Df((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta))\cdot\mathbf{e}_j\,dt - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right)\right| \\
&\quad = \left|\int_{\eta\in\mathbb{R}^n} \Phi(\eta)\int_0^1 \left(Df((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta))\cdot\mathbf{e}_j - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right)\,dt\right| \\
&\quad = \left|\int_{\eta\in\mathbb{R}^n} \Phi(\eta)\int_0^1 \left(\frac{\partial f}{\partial x_j}((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta)) - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right)\,dt\right| \\
&\quad \leq \int_{\eta\in\mathbb{R}^n} \Phi(\eta)\int_0^1 \left|\frac{\partial f}{\partial x_j}((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta)) - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right|\,dt. \qquad (8)
\end{aligned}$$

Now, let us consider $g \in C_c^2(\mathbb{R}^n)$ with

$$g = \frac{\partial f}{\partial x_j}.$$

The integrand of the inner integral becomes

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x_j}((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta)) - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right| \\
&\qquad |g((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta)) - g(\mathbf{p}-\eta)| \\
&\qquad = |g(\mathbf{p}-\eta+tv\mathbf{e}_j) - g(\mathbf{p}-\eta)| \\
&\qquad = \left|\int_0^1 \frac{d}{ds} g((1-s)(\mathbf{p}-\eta)+s(\mathbf{p}-\eta+tv\mathbf{e}_j))\,ds\right| \\
&\qquad = \left|tv\int_0^1 Dg((1-s)(\mathbf{p}-\eta)+s(\mathbf{p}-\eta+tv\mathbf{e}_j))\cdot\mathbf{e}_j\,ds\right| \\
&\qquad = t|v|\left|\int_0^1 \frac{\partial^2 f}{\partial x_j^2}((1-s)(\mathbf{p}-\eta)+s(\mathbf{p}-\eta+tv\mathbf{e}_j))\,ds\right| \\
&\qquad \leq t|v|\int_0^1 \max_{\xi\in\mathbb{R}^n}\left|\frac{\partial^2 f}{\partial x_j^2}(\xi)\right|\,ds \\
&\qquad = t|v|\max_{\xi\in\mathbb{R}^n}\left|\frac{\partial^2 f}{\partial x_j^2}(\xi)\right|.
\end{aligned}$$

Continuing from (8)

$$\begin{aligned}
&\left|\frac{\Phi * f(\mathbf{p}+v\mathbf{e}_j)-\Phi(\mathbf{p})}{v} - \Phi * \frac{\partial f}{\partial x_j}(\mathbf{p})\right| \\
&\qquad \le \int_{\eta\in\mathbb{R}^n} \Phi(\eta) \int_0^1 \left|\frac{\partial f}{\partial x_j}((1-t)(\mathbf{p}-\eta)+t(\mathbf{p}+v\mathbf{e}_j-\eta)) - \frac{\partial f}{\partial x_j}(\mathbf{p}-\eta)\right| dt \\
&\qquad \le \int_{\eta\in\mathbb{R}^n} \Phi(\eta) \int_0^1 t|v| \max_{\xi\in\mathbb{R}^n} \left|\frac{\partial^2 f}{\partial x_j^2}(\xi)\right| dt \\
&\qquad = \frac{|v|}{2} \max_{\xi\in\mathbb{R}^n} \left|\frac{\partial^2 f}{\partial x_j^2}(\xi)\right| \int_{\eta\in\mathbb{R}^n} \Phi(\eta).
\end{aligned}$$

Notice that if for any $\epsilon > 0$, we take

$$|v| < \frac{\epsilon}{\left(1+\max_{\xi\in\mathbb{R}^n}\left|\frac{\partial^2 f}{\partial x_j^2}(\xi)\right|\right)\left(1+\int_{\eta\in\mathbb{R}^n}\Phi(\eta)\right)}$$

then

$$\left|\frac{\Phi * f(\mathbf{p}+v\mathbf{e}_j)-\Phi(\mathbf{p})}{v} - \Phi * \frac{\partial f}{\partial x_j}(\mathbf{p})\right| < \epsilon.$$

This means $\Phi * f$ has partial derivatives given by

$$\frac{\partial}{\partial x_j}\Phi * f = \Phi * \frac{\partial f}{\partial x_j}$$

for $j = 1, 2, \ldots, n$. Notice that since $\partial f/\partial x_j$ is continuous, the convolution $\Phi * (\partial f/\partial x_j)$ is also continuous (just as we showed for $\Phi * f$ above) so $\Phi * f \in C^1(\mathbb{R}^n)$. This formula also illustrates that convolution with Φ commutes with taking a derivative (of the function in the convolution which has a derivative if there is one).

Applying this last principle to the first derivatives of $\Phi * f$ we see $\Phi * f \in C_c^2(\mathbb{R}^n)$ and

$$\frac{\partial^2}{\partial x_i \partial x_j}\Phi * f = \Phi * \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

In particular, this means the Laplacian of $\Phi * f$ is well-defined and

$$\Delta\Phi * f = \Phi * \Delta f.$$

This seems like it may not tell us very much, since f was simply an arbitrary function in $C_c^3(\mathbb{R}^n)$. The Laplacian of f could be, more or less, anything.

But there is more to say. We have not yet used in any serious way the properties/form of the function Φ. We have only used that Φ has an integrable singularity.

2.3 The Laplacian of $\Phi * f$

We have shown

$$\Delta\Phi * f(\mathbf{x}) = \Phi * \Delta f(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} \Phi(\mathbf{x}-\xi)\Delta f(\xi).$$

We have not used the particular form of Φ and the form of the singularity in Φ in particular. We have only used the fact that the singularity in Φ at the origin is integrable. Now, we will need to use the detailed properties of Φ.

Before we begin to examine the value of $\Phi * \Delta f$ carefully, let me note that we used the regularity of f to get the derivatives "inside" the convolution integral onto f. One might imagine one could accomplish the same kind of differentiation putting the derivatives on the function Φ and arrive at

$$(\Delta\Phi) * f(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} \Delta\Phi(\mathbf{x}-\xi) f(\xi) \equiv 0$$

so that we obtain by the formula $u = \Phi * f$ some new solutions of Laplace's equation on all of $\mathbb{R}^n$. This cannot be accomplished, and this is not a correct conclusion.

In order to obtain a correct conclusion, we observe first that the Laplace operator is a **divergence form operator** in the sense the

$$\Delta f = \operatorname{div} Df \tag{9}$$

where the divergence of the gradient field may be understood in the simple sense of the formula for the divergnece of a vector field $\mathbf{v} = (v_1, v_2, \dots, v_n) \in C^1(\Omega \to \mathbb{R}^n)$ in rectangular coordinates:

$$\operatorname{div}\mathbf{v} = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}.$$

Here Ω is an open subset of $\mathbb{R}^n$, so that (9) is valid whenever $f \in C^2(\Omega)$.

We will use also the **product formula** for the divergence of a scaled vector field

$$\operatorname{div}(g\mathbf{v}) = Dg \cdot \mathbf{v} + g \operatorname{div}\mathbf{v} \tag{10}$$

where $\mathbf{v} \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$.

Finally, we will use the **divergence theorem** in the following form:

Theorem 2 (divergence theorem) If Ω is a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega$ a smooth hypersurface in $\mathbb{R}^n$ admitting a continuous outward unit normal field $\mathbf{n} \in C^0(\partial\Omega \to \mathbb{R}^n)$ and $\mathbf{v} \in C^1(\overline{\Omega})$, then

$$\int_\Omega \operatorname{div}\mathbf{v} = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}. \tag{11}$$

A discussion of the divergence operator in a more general context along with a sketch of the proof of the divergence theorem is given in Appendix D.

With this handful of tools in hand, we start by isolating the singularity in Φ: Recall that $\mathbf{x}$ is fixed in this argument, and we are looking to determine $\Delta\Phi * f(\mathbf{x})$. First take $M > 0$ so that

$$f(\mathbf{x}-\eta) \equiv 0 \qquad \text{for} \qquad \eta \in \mathbb{R}^n \backslash B_M(\mathbf{0}).$$

Consider $r > 0$ with $r < M$. Then

$$\begin{aligned} \Delta\Phi * f(\mathbf{x}) &= \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) + \int_{\eta\in\mathbb{R}^n\backslash B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) \\ &= \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) + \int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta). \end{aligned}$$

Recall that by the extreme value theorem

$$\max_{\xi\in\mathbb{R}^n} |\Delta f(\xi)| = \max_{\xi\in B_R(\mathbf{0})} |\Delta f(\xi)| < \infty.$$

Therefore,

$$\left| \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) \right| \leq \max_{\xi\in\mathbb{R}^n} |\Delta f(\xi)| \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta).$$

Note that

$$\begin{aligned} \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta) &= C\int_0^r \left(\int \partial B_t(\mathbf{0}) \frac{1}{t^{n-2}} \right) dt \\ &= C\int_0^r \frac{1}{t^{n-2}} \left(\int \partial B_t(\mathbf{0}) 1 \right) dt \\ &= C\int_0^r \frac{1}{t^{n-2}}\, n\omega_n t^{n-1}\, dt \\ &= Cn\omega_n \int_0^r t\, dt \\ &= \frac{Cn\omega_n}{2}\, r^2. \end{aligned}$$

This means

$$\lim_{r\searrow 0} \left| \int_{\eta\in B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) \right| = 0$$

and so

$$\lim_{r\searrow 0}\int_{\eta\in B_r(\mathbf{0})}\Phi(\eta)\Delta f(\mathbf{x}-\eta)=0.$$

Here we have used the specific form of the fundamental solution Φ, though really we have only used the basic integrability again. Consider however, the other integral

$$\int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})}\Phi(\eta)\Delta f(\mathbf{x}-\eta).$$

The Laplacian here is taken directly of f, but in order to apply the divergence theorem, we need a divergence with respect to the variable of integration. Thus we note

$$\text{div}^{\eta}Df(\mathbf{x}-\eta)=-\Delta f(\mathbf{x}-\eta).$$

Moreover if we consider the scaled field $\Phi Df(\mathbf{x}-\beta)$, then the product formula gives

$$\text{div}^{\eta}[\Phi(\eta)Df(\mathbf{x}-\eta)]=D\phi(\eta)\cdot Df(\mathbf{x}-\eta)-\Phi(\eta)\Delta f(\mathbf{x}-\eta).$$

It follows that

$$\begin{aligned}\int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})}\Phi(\eta)\Delta f(\mathbf{x}-\eta)=&\int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})}D\phi(\eta)\cdot Df(\mathbf{x}-\eta)\\&-\int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})}\text{div}^{\eta}[\Phi(\eta)Df(\mathbf{x}-\eta)].\end{aligned}$$

Applying the divergence theorem to the last integral, we have

$$\begin{aligned}\int_{\eta\in B_M(\mathbf{0})\backslash B_r(\mathbf{0})}\text{div}^{\eta}[\Phi(\eta)Df(\mathbf{x}-\eta)]=&\int_{\eta\in\partial B_M(\mathbf{0})}\Phi(\eta)Df(\mathbf{x}-\eta)\cdot\frac{\eta}{M}\\&-\int_{\eta\in\partial B_r(\mathbf{0})}\Phi(\eta)Df(\mathbf{x}-\eta)\cdot\frac{\eta}{r}\\=&-\frac{1}{r}\int_{\eta\in\partial B_r(\mathbf{0})}\Phi(\eta)Df(\mathbf{x}-\eta)\cdot\eta\end{aligned}$$

because $f(\mathbf{x}-\eta) \equiv 0$ for $\eta \in \partial B_M(\mathbf{0})$. We conclude

$$\begin{aligned}\left|\int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} \operatorname{div}^\eta[\Phi(\eta) Df(\mathbf{x}-\eta)]\right| &\leq \int_{\eta\in\partial B_r(\mathbf{0})} \Phi(\eta)|Df(\mathbf{x}-\eta)| \\ &\leq \max_{\xi\in\overline{B_R(\mathbf{0})}} |Df(\xi)| C \int_{\eta\in\partial B_r(\mathbf{0})} \frac{1}{|\eta|^{n-2}} \\ &= \frac{C}{r^{n-2}} \max_{\xi\in\overline{B_R(\mathbf{0})}} |Df(\xi)| \int_{\eta\in\partial B_r(\mathbf{0})} 1 \\ &= \frac{C}{r^{n-2}} \max_{\xi\in\overline{B_R(\mathbf{0})}} |Df(\xi)| n\omega_n\, r^{n-1} \\ &= Cn\omega_n \max_{\xi\in\overline{B_R(\mathbf{0})}} |Df(\xi)|\; r.\end{aligned}$$

Therefore,

$$\lim_{r\searrow 0}\left|\int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} \operatorname{div}^\eta[\Phi(\eta) Df(\mathbf{x}-\eta)]\right| = 0$$

and

$$\lim_{r\searrow 0}\int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} \operatorname{div}^\eta[\Phi(\eta) Df(\mathbf{x}-\eta)] = 0.$$

Let us summarize what we have found so far. We have written $\Delta\Phi * f(\mathbf{x})$ as a sum of three integrals:

$$\begin{aligned}\Delta\Phi * f(\mathbf{x}) = \int_{\eta\in B_r(\mathbf{0})} &\Phi(\eta)\Delta f(\mathbf{x}-\eta) \\ &+ \int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} D\Phi(\eta)\cdot Df(\mathbf{x}-\eta) \\ &\quad - \int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} \operatorname{div}^\eta[\Phi(\eta) Df(\mathbf{x}-\eta)].\end{aligned}$$

The first and last integrals satisfy

$$\lim_{r\searrow 0}\int_{\eta\in B_r(\mathbf{0})} \Phi(\eta)\Delta f(\mathbf{x}-\eta) = 0$$

and

$$\lim_{r\searrow 0}\int_{\eta \in B_M(\mathbf{0})\setminus B_r(\mathbf{0})} \operatorname{div}^\eta[\Phi(\eta) Df(\mathbf{x}-\eta)] = 0.$$

It remains to consider the middle integral. Notice that this middle integral

$$\int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} D\Phi(\eta) \cdot Df(\mathbf{x} - \eta)$$

is over a region where Φ is nonsingular and $\Delta\Phi \equiv 0$. We again apply the product formula for the divergence of a scaled field:

$$\operatorname{div}^\eta[f(\mathbf{x}-\eta)D\Phi(\eta)] = -Df(\mathbf{x}-\eta)\cdot D\Phi(\eta) + f(\mathbf{x}-\eta)\Delta\Phi(\eta) = -Df(\mathbf{x}-\eta)\cdot D\Phi(\eta).$$

This means

$$\begin{aligned}
\int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} D\Phi(\eta) \cdot Df(\mathbf{x} - \eta) &= -\int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} \operatorname{div}^\eta[f(\mathbf{x}-\eta)D\Phi(\eta)] \\
&= \int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\eta) D\Phi(\eta) \cdot \frac{\eta}{r} \\
&\quad - \int_{\eta \in \partial B_M(\mathbf{0})} f(\mathbf{x}-\eta) D\Phi(\eta) \cdot \frac{\eta}{M} \\
&= \frac{1}{r}\int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\eta) D\Phi(\eta) \cdot \eta. \qquad (12)
\end{aligned}$$

We need to compute the gradient of Φ. Remember

$$\Phi(\mathbf{x}) = \frac{C}{|\mathbf{x}|^{n-2}}$$

where C is some positive constant and $n \geq 3$. This means

$$\frac{\partial \Phi}{\partial x_j} = \frac{\partial}{\partial x_j} C(|\mathbf{x}|^2)^{-(n-2)/2} = -C(n-2)\ x_j\ (|\mathbf{x}|^2)^{-(n-2)/2-1} = -C(n-2)\frac{x_j}{|\mathbf{x}|^n}.$$

That is,

$$D\Phi(\mathbf{x}) = -C(n-2)\frac{\mathbf{x}}{|\mathbf{x}|^n}.$$

Substituting this value in (12) we have

$$\begin{aligned}
\int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} D\Phi(\eta) \cdot Df(\mathbf{x} - \eta) &= -\frac{C(n-2)}{r}\int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\eta)\frac{\eta}{r^n} \cdot \eta \\
&= -\frac{C(n-2)}{r^{n-1}}\int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\eta).
\end{aligned}$$

This is a rather interesting value. Recall in particular that $\mathcal{H}(\partial B_r(\mathbf{0})) = n\omega_n r^{n-1}$, and the power r^{n-1} matches the expression we have obtained. Specifically, we can write

$$\int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} D\Phi(\eta) \cdot Df(\mathbf{x} - \eta) = -Cn(n-2)\omega_n \, \frac{1}{n\omega_n r^{n-1}} \int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x} - \eta).$$

The constant $Cn(n-2)\omega_n$ is independent of r, and the expression

$$\frac{1}{n\omega_n r^{n-1}} \int_{\eta \in \partial B_r(\mathbf{0})} f(\mathbf{x} - \eta) = \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} f(\xi)$$

is precisely the average of the function f over $\partial B_r(\mathbf{x})$. By continuity we should expect

$$\lim_{r \searrow 0} \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} f(\xi) = f(\mathbf{x}).$$

Here is a proof of this fact: By continuity, for any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$|\xi - \mathbf{x}| < \delta \qquad \text{implies} \qquad |f(\xi) - f(\mathbf{x})| < \frac{\epsilon}{2}.$$

Thus, for $r < \delta$ we have

$$\begin{aligned}
\left| \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} f(\xi) - f(\mathbf{x}) \right| &= \left| \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} [f(\xi) - f(\mathbf{x})] \right| \\
&\le \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} |f(\xi) - f(\mathbf{x})| \\
&\le \frac{\epsilon}{2} \frac{1}{n\omega_n r^{n-1}} \int_{\xi \in \partial B_r(\mathbf{x})} 1 \\
&= \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Here is the limit result for the last/middle integral:

$$\lim_{r \searrow 0} \int_{\eta \in B_M(\mathbf{0}) \setminus B_r(\mathbf{0})} D\Phi(\eta) \cdot Df(\mathbf{x} - \eta) = -Cn(n-2)\omega_n \, f(\mathbf{x}).$$

That is,

$$\Delta\Phi * f(\mathbf{x}) = -Cn(n-2)\omega_n \, f(\mathbf{x}).$$

I've been calling $\Phi \in C^\infty(\mathbb{R}^n \backslash \{\mathbf{0}\}$ by

$$\Phi(\mathbf{x}) = \frac{C}{|\mathbf{x}|^{n-2}}$$

for $n \geq 3$ the "fundamental solution" of Laplace's equation. That is not quite accurate. The real **fundamental solution** for $n \geq 3$ is the one with the constant C taking the value

$$C = \frac{1}{n(n-2)\omega_n}.$$

Then we have the striking result

$$\Delta\Phi * f(\mathbf{x}) = -f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Thus, we have solved a somewhat different PDE than Laplace's equation $\Delta u = 0$, but we know how to find a solution of what is called **Poisson's equation**

$$\Delta u = -f$$

on all of $\mathbb{R}^n$ for any $f \in C_c^3(\mathbb{R}^n)$.

2.4 The boundary value problem and the boundary value swap

Remember the existence and uniqueness theorem for ODEs was based on the initial value problem (IVP). We could try to directly generalize this approach for PDE. The interval (a, b) initially considered as the domain on which the ODE is considered would be replaced by some open set Ω on which we consider the PDE. The internal point $t_0 \in (a, b)$ where the initial values are defined would "inflate" to a hypersurface in $\mathcal{N} \subset \Omega$. On the hypersurface $\mathcal{N}$ we can consider what is called **Cauchy data** which corresponds to the initial value $\mathbf{x}(t_0)$ in the system of ODEs. There is a theorem about such questions, but it applies only to a single PDE and not to every PDE. Still it's a nice theorem, especially for the time dependent PDEs like the heat equation and the wave equation, and one of the "big theorems" in PDE. We will not cover that theorem in this course, but it's called the **Cauchy-Kowalevski theorem**, so you'll know to look for it when you take a more advanced course in partial differential equations.

The usual approach, and what we will focus on for Laplace's equation, is to consider a **boundary value problem** (BVP)

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in \Omega \\ u\big|_{\partial\Omega} = g. \end{cases} \tag{13}$$

Typically, we seek a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If we try to specialize this back to a single ODE, we get what is called the **two point boundary value problem** in ODEs which might naturally look something like this:

$$\begin{cases} y'' = F(y', y, t), & t \in (a, b) \\ y(a) = c, \ y(b) = d. \end{cases} \tag{14}$$

Of course, you could try to rephrase this problem in terms of some kind of equivalent first order system, but you would be in some trouble because notice that we haven't said anything about boundary values for the function y'. In fact, the theory for such a problem is not so ideal in some ways. That is why there is an emphasis on the initial value problem instead. Nevertheless, in certain contexts one must face the two point boundary value problem in ODEs. This is something you can look forward to when you take an advanced course on ODEs.

Exercise 3 Give an example of a two point boundary value problem as posed in (14) with no solution. Give another example of a two point boundary value problem as posed in (14) with more than one solution. Hint(s):

$$\frac{d^2}{dt^2} \sin t = -\sin t \qquad \text{and} \qquad \sin(0) = \sin(\pi) = 0.$$

The reason things work out reasonably well when considering the boundary value problem for PDE is that we only consider some very special PDEs like Laplace's equation and Poisson's equation.

Aside from introducing the boundary value problem and suggesting that we'd like to solve it, which is what the Green's function for Laplacian can nominally do at least in some cases, the main point of this section is the following observation:

Imagine you have a solution of the BVP (13) and it happens that the boundary values $g : \partial\Omega \to \mathbb{R}$ are given by the restriction of some function $f : \overline{\Omega} \to \mathbb{R}$. In fact, let us assume $f \in C^2(\overline{\Omega})$ and

$$g(\mathbf{x}) = f(\mathbf{x}) \qquad \text{for} \qquad \mathbf{x} \in \partial\Omega.$$

Then we can consider the function $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ given by $v = u - f$ where u is a presumed solution of (13). Then notice that

$$\Delta v = \Delta u - \Delta f = -\Delta f \qquad \text{and} \qquad v\Big|_{\partial\Omega} \equiv 0.$$

Conversely, if we can solve the BVP

$$\begin{cases} \Delta v = -\Delta f, & \mathbf{x} \in \Omega \\ v\Big|_{\partial\Omega} \equiv 0, \end{cases} \tag{15}$$

then $u = v + f$ solves (13).

The simple consequence of this discussion is that certain boundary values for the Laplace equation can be solved in terms of (or swapped out with) certain boundary value problems for Poisson's equation **with a homogeneous boundary condition**

$$v\Big|_{\partial\Omega} \equiv 0. \tag{16}$$

We know how to find lots of solutions of Poisson's equation in the form $\Phi * f$ for various functions $f \in C_c^3(\mathbb{R}^n)$, and it is this homogeneous bondary condition (16) which plays a key role in "souping up" the fundamental solution of Laplace's equation to a Green's function which will allow us to solve some problems like (15) and consequently some problems like (13) for Laplace's equation.

Here are two nice problems to consider in this context:

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in \Omega = (0, L) \times (-M, M) \\ u\Big|_{\mathbf{x} \in \partial\Omega} = x_2^2. \end{cases} \tag{17}$$

In the BVP (17) Ω is a rectangular domain in $\mathbb{R}^2$. Here is another BVP:

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in \Omega = B_1(\mathbf{0}) \subset \mathbb{R}^n \\ u\Big|_{\mathbf{x} \in \partial\Omega} = f(\mathbf{x}) \end{cases} \tag{18}$$

where $f \in C^2(\mathbb{R}^n)$. Of course for the first problem one should (try to) use the fundamental solution and Green's function for $\mathbb{R}^2$. I will discuss the second problem for $n \geq 3$ in detail below.

2.5 The boundary value problem for Poisson's equation

In light of the discussion of the previous section, we start with consideration of the BVP

$$\begin{cases} \Delta v = -f, & \mathbf{x} \in \Omega \\ v\big|_{\partial\Omega} \equiv 0, & \end{cases} \tag{19}$$

for a function $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. In some abstract sense, we know how to deal with this problem when $\Omega = \mathbb{R}^n$, but now we focus on the case where Ω is some bounded domain with $\partial\Omega$ a smooth hypersurface like $\partial B_1(\mathbf{0})$.

The foundational idea of the Green's function for the Laplace equation is the following: We know what the singularity of the fundamental solution can do for us. Consequently, given $\xi \in \Omega$ we seek a solution of the particular problem

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in \Omega\backslash\{\xi\} \\ u\big|_{\partial\Omega} \equiv 0, & \end{cases} \tag{20}$$

but we also want the solution u to have a singularity like the fundamental solution $\Phi(\mathbf{x} - \xi)$ at the point ξ. Specifically, we use a boundary value swap and look to solve

$$\begin{cases} \Delta u_c = 0, & \mathbf{x} \in \Omega \\ u_c\big|_{\mathbf{x}\in\partial\Omega} \equiv \Phi(\mathbf{x} - \xi). & \end{cases} \tag{21}$$

This is called the *corrector problem,* and you can see immediately that given a solution u_c of (21) the function $u : \Omega\backslash\{\xi\} \to \mathbb{R}$ with

$$u(\mathbf{x}) = G(\mathbf{x}, \xi) = \Phi(\mathbf{x} - \xi) - u_c(\mathbf{x})$$

has all the properties we want. This is the Green's function. . . sort of. In reality, this is a good way to start thinking about the Green's function, but technically, the Green's function has an important depenence on twice as many variables and a number of interesting properties. Specifically, $G : \Omega \times \Omega\backslash\mathcal{D} \to \mathbb{R}$ where

$$\mathcal{D} = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \Omega\}$$

is the **diagonal** in $\Omega \times \Omega$. Also, you will note that the corrector function u_c depends also on ξ in a potentially complicated way. In any case, the corrector problem (21) has a unique solution $u \in C^2(\overline{\Omega})$ for each $\xi \in \Omega$, and thus a Green's function for a given domain Ω will always exist.

I'd like to do two things here. First, I'd like to derive some of the main properties of the Green's function. Chief amongst these, I'd like to calculate

$$\Delta^{\mathbf{x}} \int_{\xi \in \Omega} G(\mathbf{x}, \xi) f(\xi)$$

which should remind you of the calculation

$$\Delta \Phi * f(\mathbf{x}) = \Delta^{\mathbf{x}} \int_{\xi \in \Omega} \Phi(\mathbf{x} - \xi) f(\xi)$$

but with the inclusion of a corrector function. Second, I'd like to find the Green's function for the ball $B_1(\mathbf{0}) \subset \mathbb{R}^n$. This construction will apply to $n \geq 2$.

A If f is only continuous

As mentioned above, if we only know f is continuous, say $f \in C_c^0(\mathbb{R}^n)$ with $n \geq 3$ and we want to show $u = \Phi * f$ is continuous we have to work harder. In particular, given a tolerance $\epsilon > 0$, if we want to use the continuity of f at a point $\mathbf{p} - \eta$, then we can assert

$$|f(\xi) - f(\mathbf{p} - \eta)| < \epsilon \qquad \text{whenever} \qquad |\xi - (\mathbf{p} - \eta)| < \delta$$

but in this case $\delta = \delta(\eta)$ depens on the point η. We have to be more careful about how we choose or deal with the tolerances $\delta(\eta)$ arising from the continuity of f at $\mathbf{p} - \eta$. In particular, we need to get some kind of **uniformity** which we haven't been careful enough to get so far. Let's see if we can get some uniformity locally first.

We do know f is continuous at each point $\mathbf{q} \in \mathbb{R}^n$. Thus, taking a fixed point $\mathbf{q}$ we have for any given $\epsilon > 0$ a tolerance $\delta = \delta(\mathbf{q}) > 0$ for which

$$|\xi - \mathbf{q}| < \delta \qquad \text{implies} \qquad |f(\xi) - f(\mathbf{q})| < \frac{\epsilon}{2}.$$

Now say we have points ξ_1 and ξ_2 with $\xi_1, \xi_2 \in B_\delta(\mathbf{q})$. Then

$$|\xi_1 - \mathbf{q}|, |\xi_2 - \mathbf{q}| < \delta$$

so

$$|f(\xi_2) - f(\xi_1)| \leq |f(\xi_2) - f(\mathbf{q})| + |f(\mathbf{q}) - f(\xi_1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Notice that this gives a kind of uniformity on $B_{\delta/2}(\mathbf{q})$. Let's see what we can do with this idea.

Let $\epsilon > 0$. We wish to show there exists some $\delta > 0$ so that if $|\mathbf{x} - \mathbf{p}| < \delta$, then

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{p})| &\leq \int_{\eta \in \mathbb{R}^n} \Phi(\eta) |f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \\ &= \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta) |f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \\ &< \epsilon. \end{aligned} \tag{22}$$

Here M is some positive radius for which $f(\mathbf{x} - \eta) = 0$ for every $\eta \in \mathbb{R}^n \backslash B_M(\mathbf{p})$. Notice that $f(\mathbf{p} - \eta) = 0$ for $\eta \in \mathbb{R}^n \backslash B_M(\mathbf{p})$ in particular. The value M comes from

Exercise 2 above. Also, we should keep in mind that $\mathbf{p}$ is fixed. By the calculation above, if we can assert

$$|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| < \frac{\epsilon}{1 + \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)}$$

whenever $|\mathbf{x}-\mathbf{p}| < \delta$, then we can get (22) and we will know $u = \Phi * f$ is continuous (at any arbitrary point $\mathbf{p}$). Thus, let us take the new positive tolerance

$$\epsilon_1 = \frac{\epsilon}{1 + \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)}$$

and see what we can do.

For each $\eta \in \mathbb{R}^n$, it is indeed true that f is continuous at $\mathbf{p}-\eta$. Thus, there is some $\delta(\eta) > 0$ for which

$$\begin{aligned} |\mathbf{x}-\mathbf{p}| < \delta(\eta) \qquad &\text{implies} \qquad |\mathbf{x}-\eta-(\mathbf{p}-\eta)| < \delta(\eta) \\ &\text{implies} \qquad |f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| < \frac{\epsilon_1}{2}. \end{aligned}$$

Now if we take any $\xi_1 \in B_{\delta(\eta)/2}(\mathbf{p}-\eta)$ then for every ξ with $|\xi-\xi_1| < \delta(\eta)/2$ we have

$$|\xi - (\mathbf{p}-\eta)| \le |\xi-\xi_1| + |\xi_1 - (\mathbf{p}-\eta)| < \delta(\eta)$$

so

$$|f(\xi) - f(\xi_1)| \le |f(\xi) - f(\mathbf{p}-\eta)| + |f(\mathbf{p}-\eta) - f(\xi_1)| < \epsilon_1.$$

In this way, we have a uniform tolerance for continuity $\delta(\eta)/2$ that can be used for any point $\xi_1 \in B_{\delta(\eta)/2}(\mathbf{p}-\eta)$. Notice in particular that if $\eta_1 \in B_{\delta(\eta)/2}(\mathbf{p}-\eta)$, then

$$|f(\mathbf{x}-\eta_1) - f(\mathbf{p}-\eta_1)| < \epsilon_1 \qquad \text{whenever} \qquad |\mathbf{x}-\mathbf{p}| = |\mathbf{x}-\eta_1-(\mathbf{p}-\eta_1)| < \frac{\delta(\eta)}{2}. \tag{23}$$

This is because

$$|\mathbf{x}-\eta_1-(\mathbf{p}-\eta)| \le |\mathbf{x}-\mathbf{p}| + |\eta_1-\eta| < \delta(\eta) \qquad \text{and} \qquad |\mathbf{p}-\eta_1-(\mathbf{p}-\eta)| = |\eta_1-\eta| < \delta(\eta).$$

Now, notice that every point $\eta_1 \in \mathbb{R}^n$ falls into some ball $B_{\delta(\eta)/2}(\mathbf{p}-\eta)$ for some $\eta \in \mathbb{R}^n$. Thus (23) always holds, but notice this still does not give us a uniform tolerance for $|\mathbf{x}-\mathbf{p}|$.

In order to illustrate what I am saying here another way and change the notation around a bit, imagine we had finitely many points $\eta_1, \eta_2, \ldots, \eta_k \in \mathbb{R}^n$ and each one has associated with it a tolerance $\delta(\eta_j)$ according to which

$$|\xi - (\mathbf{p} - \eta_j)| < \delta(\eta_j) \qquad \text{implies} \qquad |f(\xi) - f(\mathbf{p} - \eta_j)| < \frac{\epsilon}{2}.$$

Consider any η with

$$\eta \in \bigcup_{j=1}^{k} B_\delta(\eta_j)$$

where δ is the fixed positive tolerance given by

$$\delta = \min\left\{\frac{\delta(\eta_k)}{2}, \frac{\delta(\eta_k)}{2}, \ldots, \frac{\delta(\eta_k)}{2}, \right\}.$$

The key here is that because there are only finitely many points and consequently **finitely many positive numbers** $\delta(\eta_j)/2$ we still get a positive number when we take the minimum. Now consider

$$|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)|.$$

Since

$$\eta \in \bigcup_{j=1}^{k} B_\delta(\eta_j)$$

there is some particular j with

$$\eta \in B_\delta(\eta_j) \subset B_{\delta(\eta_j)/2}(\eta_j).$$

Also, if $|\mathbf{x} - \mathbf{p}| < \delta$, then

$$|\mathbf{x} - \eta - (\mathbf{p} - \eta_j)| \le |\mathbf{x} - \mathbf{p}| + |\eta - \eta_j| < \delta(\eta_j).$$

Thus,

$$|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \le |f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta_j)| + |f(\mathbf{p} - \eta) - f(\mathbf{p} - \eta_j)| < \epsilon_1.$$

We conclude

$$\int_{\eta \in U} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \le \epsilon_1 \int_{\eta \in U} \Phi(\eta) = \frac{\int_{\eta \in U} \Phi(\eta)}{1 + \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)}\, \epsilon \qquad (24)$$

where

$$U = \bigcup_{j=1}^{k} B_\delta(\eta_j).$$

This looks very promising, but there are still a few pesky details that need to be ironed out.

Notice first of all that if we could somehow include enough points η_j so that

$$B_M(\mathbf{p}) \subset U = \bigcup_{j=1}^{k} B_\delta(\eta_j,)$$

then we could start with the estimate in (7) and conclude

$$\begin{aligned}|u(\mathbf{x}) - u(\mathbf{p})| &\le \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| \\ &\le \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)\epsilon_1 \\ &= \frac{\int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)}{1 + \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)}\,\epsilon \\ &< \epsilon.\end{aligned}$$

In view of this calculation, consider the viability of the following proposition:

> There are finitely many points $\eta_1, \eta_2, \ldots, \eta_k$ for which
>
> $$B_M(\mathbf{p}) \subset U = \bigcup_{j=1}^{k} B_\delta(\eta_j). \qquad (25)$$

Recall that for each $\xi \in \mathbb{R}^n$ there is some $\delta(\xi)$ for which

$$|\mathbf{x} - \mathbf{p}| < \delta(\xi) \qquad \text{implies} \qquad |f(\mathbf{x}-\xi) - f(\mathbf{p}-\xi)| < \frac{\epsilon_1}{2}.$$

This is a consequence of the continuity of f at the point $\mathbf{p} - \xi$. We can also observe that if we use infinitely many balls $B_{\delta(\xi)/2}(\xi)$, then we can have

$$B_M(\mathbf{p}) \subset V = \bigcup_{\xi \in B_M(\mathbf{p})} B_{\delta(\xi)/2}(\xi).$$

The problem, of course, is that we still do not have uniformity over the radii of the balls in the union. There are infinitely many points $\xi \in B_M(\mathbf{p})$, and it is very likely that

$$\min_{\xi \in B_M(\mathbf{p})} \frac{\delta(\xi)}{2} \tag{26}$$

does not exist. The numbers $\delta(\xi)/2$ are each positive, but they may be numbers that get arbitrarily small so that the **greatest lower bound** of the set

$$\left\{ \frac{\delta(\xi)}{2} \right\}_{\xi \in B_M(\mathbf{p})}$$

of positive numbers is 0. In this case, there is no single $\xi_{\min} \in B_M(\mathbf{p})$ for which $\delta(\xi_{\min}) = 0$. When we write a **minimum** as in (26) this is what we mean, so we have no reason to believe the least upper bound is achieved, and this is what I mean by saying the minimum may not exist.

We can still consider the greatest lower bound of this set of positive numbers and write for example

$$a = \text{g.l.b.} \left\{ \frac{\delta(\xi)}{2} \right\}_{\xi \in B_M(\mathbf{p})}.$$

If this number is positive, then we would probably be in pretty good shape. Incidentally, the more common notation for the greatest lower bound in situations like this is

$$a = \inf \left\{ \frac{\delta(\xi)}{2} \right\}_{\xi \in B_M(\mathbf{p})},$$

and the number a (the greatest lower bound), which as far as we know is zero, is called the **infemum** of the set. Please note the general principle(s): The infemum of a finite collection of numbers is always the minimum of those numbers. The infemum of any set of real numbers always exists. If the set $A \subset \mathbb{R}$ is bounded below in the sense that there is some number a with $a \leq x$ for every $x \in A$, then the infemum $\inf A$ is the greatest lower bound which is always well-defined and unique in this case. If on the other hand the set $A \subset \mathbb{R}$ is not bounded below, then $\inf A$ is still well-defined as an extended real number, and we write $\inf A = -\infty$. In our case with

$$A = \left\{ \frac{\delta(\xi)}{2} \right\}_{\xi \in B_M(\mathbf{p})}$$

we know the set is bounded below by zero, so we know $\inf A \geq 0$.

Returning to the proposition associated with (25) let us contemplate the possibility of throwing some of the points $\xi \in B_M(\mathbf{p})$ away. The set

$$\{B_{\delta(\xi)/2}(\xi)\}_{\xi \in A}$$

is an example of an **open cover** of A, that is a collection of open sets $\mathcal{C}$ for which

$$A \subset \bigcup_{U \in \mathcal{C}} U.$$

As mentioned in the appendix, there is a special name for sets having the property that every open cover may be reduced to a finite subcover. These sets are called **compact** sets. In $\mathbb{R}^n$ the compact sets are characterized by being closed and bounded. This means, in particular, that every closed and bounded set in $\mathbb{R}^n$ is compact. An outline of the proof of this fact is given at the end of Appendix C below.

Let us start again at the beginning and try to go through the details of successfully estimating

$$|u(\mathbf{x}) - u(\mathbf{p})| \leq \int_{\eta \in \mathbb{R}^n} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)|.$$

As usual, we let $\epsilon > 0$ be a given positive tolerance.

We have a positive number M so that $f(\mathbf{x} - \eta) = f(\mathbf{p} - \eta) = 0$ whenever $|\eta - \mathbf{p}| \geq M$ and $|\mathbf{x} - \mathbf{p}| < 1$. See Exercise 2. Thus we write

$$\int_{\eta \in \mathbb{R}^n} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| = \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)|.$$

For each η in the compact set $\overline{B_M(\mathbf{p})}$ there is some $\delta(\eta) > 0$ for which

$$|f(\xi) - f(\mathbf{p} - \eta)| < \frac{\epsilon}{2\left(1 + \int_{B_M(\mathbf{p})} \Phi\right)} \qquad \text{whenever} \qquad |\xi - (\mathbf{p} - \eta)| < \delta(\eta). \quad (27)$$

Therefore $\{B_{\delta(\eta)/2}(\eta)\}_{\eta \in \overline{B_M(\mathbf{p})}}$ is an open cover of the compact set $\overline{B_M(\mathbf{p})}$. Therefore, there exist finitely many points

$$\eta_1, \eta_2, \ldots, \eta_k \in \overline{B_M(\mathbf{p})}$$

for which

$$\overline{B_M(\mathbf{p})} \subset \bigcup_{j=1}^{k} B_{\delta(\eta_j)/2}(\eta_j).$$

Consider

$$\delta = \min\left\{1, \frac{\delta(\eta_1)}{2}, \frac{\delta(\eta_2)}{2}, \ldots, \frac{\delta(\eta_k)}{2}\right\}. \tag{28}$$

This is a positive number. The number $a = 1$ is included to allow application of Exercise 2 from which we obtain the important ball $B_M(\mathbf{p})$. Now if we take any $\eta \in B_M(\mathbf{p})$, there is some point η_j and some open ball $B_{\delta(\eta_j)/2}(\eta_j)$ so that

$$\eta \in B_{\delta(\eta_j)/2}(\eta_j).$$

Let us assume also that $|\mathbf{x} - \mathbf{p}| < \delta$ with δ given in (28). On the face of it, this has nothing to do with η, but remember if we write

$$|\mathbf{x} - \mathbf{p}| = |\mathbf{x} - \eta - (\mathbf{p} - \eta)|$$

then we have

$$|\mathbf{x} - \eta - (\mathbf{p} - \eta)| < \frac{\delta(\eta_j)}{2}.$$

This gives

$$|\mathbf{x} - \eta - (\mathbf{p} - \eta_j)| \le |\mathbf{x} - \eta - (\mathbf{p} - \eta)| + |\mathbf{p} - \eta - (\mathbf{p} - \eta_j)| < \delta(\eta_j).$$

Then by continuity taking $\xi = \mathbf{x} - \eta$ and $\xi = \mathbf{p} - \eta$ in we have

$$|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| \le |f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta_j)| + |f(\mathbf{p}-\eta) - f(\mathbf{p}-\eta_j)| < \frac{\epsilon}{1 + \int_{B_M(\mathbf{p})} \Phi}.$$

This is an estimate we can use uniformly in application to the integrand in $\Phi * f$: If $|\mathbf{x} - \mathbf{p}| < \delta$, then

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{p})| &\le \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| \\ &\le \frac{\epsilon}{1 + \int_{B_M(\mathbf{p})} \Phi} \int_{\eta \in B_M(\mathbf{p})} \Phi(\eta) \\ &< \epsilon. \end{aligned}$$

So showing continuity of $\Phi * f$ when $f \in C_c^0(\mathbb{R}^n)$ and Φ is the fundamental solution is a **simple as that**!

Actually, I'm being a little sarcastic. I have drawn out many of the complicated details in the discussion above to illustrate the subtle and difficult aspects of this assertion. There is a way to get this assertion more simply, and it is perhaps worth describing this phrasing. It starts with a definition:

Definition 1 A function $g \in C^0(A)$ where A is any subset of $\mathbb{R}^n$ is said to be **uniformly continuous** on A if for any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$|g(\mathbf{x}) - g(\mathbf{p})| < \epsilon \qquad \text{whenever} \qquad |\mathbf{x} - \mathbf{q}| < \delta.$$

Here it is understood of course that the points $\mathbf{x}$ and $\mathbf{p}$ satisfy $\mathbf{x}, \mathbf{p} \in A$.

Notice the difference between the definition of uniform continuity and what it means for g to be continuous at $\mathbf{p} \in A$.

With this definition one can state a nice theorem:

Theorem 3 If K is compact and $g \in C^0(K)$, then g is uniformly continuous on K.

The essential elements of the proof of this theorem are contained in the discussion above.

To apply the theorem to the continuity of $u = \Phi * f$ at $\mathbf{p} \in \mathbb{R}^n$ where Φ is the fundamental solution of Laplace's equation, $n \geq 3$ and $f \in C^0_c(\mathbb{R}^n)$, let $\epsilon > 0$ and observe that the restriction of f to K is uniformly continuous on the compact set

$$K = \overline{U} \qquad \text{where} \qquad U = B_{M+1}(\mathbf{p}).$$

Therefore, there exists some $\delta > 0$ for which

$$\begin{cases} \xi_1, \xi_2 \in K, \\ |\xi_1 - \xi_2| < \delta \end{cases} \qquad \text{implies} \qquad |f(\xi_1) - f(\xi_2)| < \frac{\epsilon}{1 + \int_K \Phi}.$$

We may also assume $\delta < 1$ and recall the condition

$$|\mathbf{x} - \mathbf{p}| < 1 \qquad \text{implies} \qquad f(\mathbf{x} - \eta) = f(\mathbf{p} - \eta) = 0 \quad \text{for} \quad \eta \in B_M(\mathbf{p})$$

from Exercise 2.

Now, if $|\mathbf{x} - \mathbf{p}| < \delta$, then for any $\eta \in B_M(\mathbf{p})$ we have

$$|\mathbf{x} - \eta| \leq |\mathbf{x} - \mathbf{p}| + |\mathbf{p} - \eta| < M + 1,$$

$$|\mathbf{p} - \eta| < M,$$

and

$$|\mathbf{x} - \eta - (\mathbf{p} - \eta)| \leq |\mathbf{x} - \mathbf{p}| < \delta.$$

Therefore, for $\eta \in B_M(\mathbf{p})$

$$|f(\mathbf{x} - \eta) - f(\mathbf{p} - \eta)| < \frac{\epsilon}{1 + \int_K \Phi}$$

and

$$
\begin{aligned}
|\Phi * f(\mathbf{x}) - \Phi * f(\mathbf{p})| &\le \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta)|f(\mathbf{x}-\eta) - f(\mathbf{p}-\eta)| \\
&\le \frac{\epsilon}{1+\int_K \Phi} \int_{\eta\in B_M(\mathbf{p})} \Phi(\eta) \\
&< \epsilon.
\end{aligned}
$$

The function $\Phi * f$ is continuous at $\mathbf{p}$.

B Integration

The integration used above may be somewhat unfamiliar both in concept and notation. The basic concept is really just a natural generalization of the integration you learned about in calculus. If you're really honest, you might call this the integration you probably didn't really understand when you took calculus. But don't worry, you have an opportunity to understand it now.

Integration starts with a **set** and a **function**. Let's call the set I. The function should, at least under most circumsaances, be real valued:

$$f : I \to \mathbb{R}.$$

We often use the symbol I to denote an interval, but that is not the case here, though certainly an interval will work, and you should keep that example in mind. As in most cases, it is required that the set have some structure. In this case, let us describe the structure required for the set I by saying we want I to be a **domain of integration**. I could go through an axiomatic description of what it means to be a domain of integration, but I will postpone that discussion in detail and just give some informal description of certain properties we need the set I to possess and give some examples.

Informally, the basic property required of a domain of integration is that it can be "cut up" into finitely many "pieces," these pieces need to (be able to) get small in **two ways.** We call the pieces $\mathcal{P} = \{I_1, I_2, \dots, I_k\}$ a **partition** of the domain of integration I, and it is roughly required that

$$I = \cup_{j=1}^k I_j \qquad \text{and} \qquad I_j \cap I_\ell = \phi \qquad \text{for} \qquad j \ne \ell. \tag{29}$$

One way the partition pieces need to be able to get small is in **diameter**. In particular, we call the number

$$\max\{\text{diam}(I_1), \text{diam}(I_2), \dots, \text{diam}(I_k)\}$$

the "norm" of the partitition and write

$$\|\mathcal{P}\| = \max\{\text{diam}(I_1), \text{diam}(I_2), \dots, \text{diam}(I_k)\}.$$

The other way, is somewhat more central to the overall process of integration. This is called **measure**. It is very important to understand in some way that one integrates with respect to a measure. You know many examples of measures, but you may very well have never considered a measure as a mathematical object in its own right. One example is length or **length measure**. If our domain of integration happens to be an interval, then it is very likely we will want to integrate with respect to length measure. In the elementary version of integration in calculus, this use of length measure is tacitly indicated by tacking the differential symbol "dx" onto the back of the integral notation. In any case, a measure is a function defined on subsets of the domain of integration. For simplicity, let us assume it is possible to measure **all subsets** of I. The collection of all subsets of I is called the power set of I and is denoted by $\wp(I)$. This gives the domain of the measure

$$\mu : \wp(I) \to [0, \infty).$$

Thus, for each subset A of I, $\mu(A)$ is the measure of A. If we can understand the length of a subset A of an interval, say $\mathbb{R}$, then the length measure might be denoted by length(A). It is a curious fact that it is not possible to understand the length measure of every subset of an interval in the real line $\mathbb{R}$, but this is a disturbing detail we will set aside for the moment.

In summary, we have a domain of integration I and two functions

$$\text{diam} : \wp(I) \to [0, \infty) \qquad \text{and} \qquad \mu : \wp(I) \to [0, \infty)$$

which we can think of as defined on all subsets of I called diameter and measure. Describing all the precise properties and limitations of these functions and their interaction with partition pieces is what is necessary to give an axiomatic discussion of domains of integration. Even more informally, you can simply think of a domain of integration as **a set upon which you can integrate a real valued function** $f : I \to \mathbb{R}$.

Any time you have a domain of integration I and a partition $\mathcal{P} = \{I_j\}_{j=1}^k$, you can choose a point x_j^* in each partition piece I_j, assuming the piece is not empty and has at least one point in it—which I imagine it is reasonable for us to assume—and form a **Riemann sum**

$$\sum_{j=1}^k f(x_j^*)\ \mu(I_j).$$

The points $x_1^*, x_2^*, \dots, x_k^*$ are called **evaluation points**. Thus, we multiply the measure of a partition piece $\mu(I_j)$ times the "height" $f(x_j^*)$ determined by some point in the piece, and add up the results. Please compare this sum to the Reimann sum

$$\sum_{j=0}^{k-1} f(x_j)\ (x_{j+1} - x_j) = \sum_{j=0}^{k-1} f(x_j)\ \text{length}(x_j, x_{j+1})$$

from calculus I where $f : (a, b) \to \mathbb{R}$ and the partition is determined by

$$a = x_0 < x_1 < \cdots < x_k = b.$$

Here the measure of an interval is its length (and the diameter also happens to be the same as the measure, but we haven't explicitly brought diameter into the picture yet). Also, the numbering is a little bit different.

With Riemann sums in hand, we are ready to try to integrate...at least in theory. We say a number L is the integral of f over the domain of integration I and write

$$L = \int_I f$$

if

$$L = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{k} f(x_j^*)\ \mu(I_j).$$

This is a touch of a complicated construction, but let me try to put into words some of what is going on (and you may want to try to understand and internalize if you want to understand integration). For any $\epsilon > 0$, there is some $\delta > 0$ so that given

1. **any partition** $\mathcal{P} = \{I_j\}_{j=1}^k$ with $\|\mathcal{P}\| < \delta$ and
2. **any evaluation points** $x_j^* \in I_j$ for $j = 1, 2, \dots, k$

there holds

$$\left| \sum_{j=1}^{k} f(x_j^*)\ \mu(I_j)\ -\ L \right| < \epsilon.$$

Notice that condition **1** is where the diameter of the partition pieces comes in.

From some point of view, the kind of integration I have described above is pretty useful and necessary to understand in much of the analysis of physical problems. We can see that more when we get into, for example, the derivation of the heat equation.

Of course, there is the whole matter of **calculating values** of integrals, and the whole concept of what the integral actually **is** as described above is often not immediately useful for that. Going in that direction, one usually wants to reduce everything, in some way or another, back to some integral or integrals over an interval of the kind you learned about in Calculus I. There definitely are some ways in which the basic understanding above can be pretty useful. Let's start with some simple examples.

Say you want to integrate on a sphere $\partial B_r(\mathbf{p}) \subset \mathbb{R}^n$, and you want to change variables to integrate instead on a unit sphere. This activity assumes there is a real valued function $f : \partial B_r(\mathbf{p}) \to \mathbb{R}$ defined on the sphere. Thus, we start with the integral

$$\int_{\partial B_r(\mathbf{p})} f \qquad \text{or} \qquad \int_{\mathbf{x}\in\partial B_r(\mathbf{p})} f(\mathbf{x})$$

and we consider the map/change of variables $\psi : \mathbb{S}^{n-1} \to \partial B_r(\mathbf{p}) \subset \mathbb{R}^n$ by

$$\psi(\mathbf{x}) = \mathbf{p} + r\mathbf{x}. \tag{30}$$

The j-th entry of ψ is

$$\psi_j = p_j + rx_j. \tag{31}$$

At this point, we have to be a little careful. Notice carefully the domain (and codomain) of the mapping ψ. Looking at the formula (30) one sees another change of variables on a different domain, namely $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ by the same formula. The components are the same for this "larger" map, and one can compute a traditional total derivative. Specifically,

$$\frac{\partial \psi_j}{\partial x_k} = r\delta_{jk} = \begin{cases} r, & j = k \\ 0, & j \neq k \end{cases}$$

and the total derivative is given by

$$D\Psi = \left(\frac{\partial \psi_j}{\partial x_k}\right) = \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & r \end{pmatrix}.$$

The Jacobian scaling factor for $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is given by the absolute value of the determinant of this total derivative matrix and has value $\mathbf{J} = r^n$. But this is not the correct Jacobian scaling factor for the change of variables between hypersurfaces.

Intuitively, one can say simply that one of the dimensions in the dilation given by $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ "doesn't count" when one restricts to the hypersurface $\mathbb{S}^{n-1}$. Similarly, one may simply "see" intuitively that dilation of a hypersphere of dimension $n-1$ by a factor r imposes an $(n-1)$ dimensional measure scaling of r^{n-1}. These suggestions do lead to the correct value $J = r^{n-1}$, but it would be more useful to have a specific formula of more general applicability. Say we had a change of variables $\psi : \mathcal{T} \to \mathcal{S}$ where $\mathcal{S}$ and $\mathcal{T}$ are two general hypersurfaces, and we want to find

$$\int_{\mathcal{S}} f = \int_{\mathcal{T}} f \circ \psi \ J. \tag{32}$$

What is the Jacobian scaling factor in this case?

In order to describe how to find a formula that applies to such a mapping of hypersurfaces, it is perhaps helpful to consider two other special cases. One has already been mentioned: If $\psi : \Omega \to U$ is a mapping of full dimension open subsets Ω and U in $\mathbb{R}^n$, then the associated n-dimensional measure scaling is

$$J = |\det D\psi|. \tag{33}$$

If $\psi : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ and $n < m$, then $D\psi$ is an $m \times n$ matrix which is not square, so certainly the formula (33) doesn't make sense. The correct formula in this case is

$$J = \sqrt{\det(D\psi^T \ D\psi)}.$$

Notice that $D\psi^T \ D\psi$ is an $n \times n$ matrix. This matrix is also symmetric and positive definite. As a consequence $\det(D\psi^T \ D\psi) > 0$. This second formula is typically what one uses for integration on a (hyper)surface. Say, $\mathcal{S} \subset \mathbb{R}^m$ is parameterized by a mapping $X : U \to \mathbb{R}^m$ with $U \subset \mathbb{R}^n$, then

$$\int_{\mathcal{S}} f = \int_U f \circ X \ \sqrt{\det(DX^T \ DX)}.$$

Now, returning to the situation involving $\psi : \mathcal{T} \to \mathcal{S}$, say we have $\mathcal{S} \subset \mathbb{R}^m$ is parametrized by $X : U \to \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$ is parameterized by $Y : V \to \mathbb{R}^n$ with U and V both open subsets of $\mathbb{R}^\ell$ so that $\mathcal{S}$ and $\mathcal{T}$ are both ℓ dimensional submanifolds[3] of their respective ambient spaces. Then noting that $\mathcal{S} = X(U)$ and $\mathcal{T} = Y(V)$, the scaling factor from U to $\mathcal{S}$ is

$$\sqrt{\det(DX^T \ DX)} \tag{34}$$

[3]The special case of hypersurfaces here would be when we have $m = n$ and $\ell = n - 1$.

and the scaling factor from V to $\mathcal{T}$ is

$$\sqrt{\det(DY^T\ DY)}.$$

Letting J denote the scaling factor from $\mathcal{T}$ to $\mathcal{S}$ determined by ψ, we should not be surprised to see the scaling factor associated with the composition $X^{-1}\circ\psi\circ Y : V \to U$ is given by

$$\frac{\sqrt{\det(DY^T\ DY)}}{\sqrt{\det(DX^T\ DX)}}\ J.$$

This composition however is a mapping of full dimension open subsets of $\mathbb{R}^\ell$. Thus, we have

$$\frac{\sqrt{\det(DY^T\ DY)}}{\sqrt{\det(DX^T\ DX)}}\ J = \det D(X^{-1}\circ\psi\circ Y)$$

or

$$J = \frac{\sqrt{\det(DX^T\ DX)}}{\sqrt{\det(DY^T\ DY)}}\ \det D(X^{-1}\circ\psi\circ Y). \tag{35}$$

So this is a formula which can be used in (32) and adapted to obtain

$$\begin{aligned}\int_{\mathbf{q}\in\partial B_r(\mathbf{p})} f(\mathbf{q}) &= \int_{\mathbf{x}\in\mathbb{S}^{n-1}} f(\mathbf{p}+r\mathbf{x})\ J\\ &= \int_{\mathbf{x}\in\mathbb{S}^{n-1}} f(\mathbf{p}+r\mathbf{x})\ r^{n-1}\\ &= r^{n-1}\int_{\mathbf{x}\in\mathbb{S}^{n-1}} f(\mathbf{p}+r\mathbf{x}).\end{aligned}$$

To see this, let $X : U \to \mathbb{R}^n$ parameterize $\partial B_r(\mathbf{p})$. The reason (34) works to give the scaling factor associated with this map is that if one takes a small cube C_ϵ of side length ϵ centered at a point $\mathbf{u}\in U$, then the $(n-1)$ dimensional measure of C_ϵ is

$$\mu^{n-1}(C_\epsilon) = \epsilon^{n-1}$$

and the measure of the image $X(C_\epsilon)$ is

$$\mathcal{H}^{n-1}(X(C_\epsilon)) \approx \epsilon^n\sqrt{\det(DX(\mathbf{u})^T\ DX(\mathbf{u}))}.$$

Note that μ^{n-1} is a full-dimension measure in $U\subset\mathbb{R}^{n-1}$, and $\mathcal{H}^{n-1}$ is $(n-1)$ dimensional Hausdorff measure in $\mathbb{R}^n$. Also, the formula

$$\epsilon^n\sqrt{\det(DX(\mathbf{u})^T\ DX(\mathbf{u}))}$$

gives precisely the measure of $dX_{\mathbf{u}}(C_\epsilon)$ where $dX_{\mathbf{u}} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the differential map at $\mathbf{u}$ which is a linear map with matrix having column vectors $dX_{\mathbf{u}}(\mathbf{e}_j) = DX(\mathbf{u})\mathbf{e}_j$ for $j = 1, 2, \ldots, n-1$, and

$$\epsilon^n \sqrt{\det(DX(\mathbf{u})^T \ DX(\mathbf{u}))}$$

is the volume of the parallelopiped spanned by the vectors

$$\{\epsilon DX(\mathbf{u})\mathbf{e}_1 = dX_{\mathbf{u}}(\epsilon\mathbf{e}_1), \epsilon DX(\mathbf{u})\mathbf{e}_2 = dX_{\mathbf{u}}(\epsilon\mathbf{e}_2), \ldots, \epsilon DX(\mathbf{u})\mathbf{e}_{n-1} = dX_{\mathbf{u}}(\epsilon\mathbf{e}_{n-1}).$$

More precisely,

$$\lim_{\epsilon \searrow 0} \frac{\epsilon^n \sqrt{\det(DX(\mathbf{u})^T \ DX(\mathbf{u}))}}{\mathcal{H}^{n-1}(X(C_\epsilon))} = 1.$$

One last point before we make the crucial observation is perhaps worth making: When we say $X : U \to \mathbb{R}^n$ parameterizes $\partial B_r(\mathbf{p})$, we can't expect to parameterize the entire hypersurface. Nevertheless, we can parameterize a *neighborhood* around a particular point $X(\mathbf{u}) \in \mathbb{R}^n$, and this is good enough to determine the Jacobian scaling factor for integration on $\mathcal{S} = \partial B_r(\mathbf{p})$.

In this case, we do not need to know precisely the scaling for X, but simply that such a scaling is determined. Once we have X, we can find a parameterization $Y : U \to \mathbb{S}^{n-1}$ for $\mathbb{S}^{n-1}$ near the corresponding point $\psi^{-1} \circ X(\mathbf{u})$. Specifically,

$$Y = \frac{1}{r}(X - \mathbf{p}).$$

This is just a composition of the affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ by $A(\mathbf{x}) = (\mathbf{x} - \mathbf{p})/r$ and the parameterization X. If we want to know the scaling factor for Y, we can look at the vectors

$$dY_{\mathbf{u}}(\mathbf{e}_1), dY_{\mathbf{u}}(\mathbf{e}_2), \ldots, dY_{\mathbf{u}}(\mathbf{e}_{n-1}).$$

These vectors are

$$\frac{1}{r} dX_{\mathbf{u}}(\mathbf{e}_1), \frac{1}{r} dX_{\mathbf{u}}(\mathbf{e}_2), \ldots, \frac{1}{r} dX_{\mathbf{u}}(\mathbf{e}_{n-1}),$$

and the volume of the parallelopiped they span is clearly

$$\frac{1}{r^{n-1}} \sqrt{\det(DX(\mathbf{u})^T \ DX(\mathbf{u}))}.$$

We conclude

$$\sqrt{\det(DX(\mathbf{u})^T \ DX(\mathbf{u}))} = r^{n-1} \sqrt{\det(DY(\mathbf{u})^T \ DY(\mathbf{u}))}.$$

This puts us in a position to use formula (35) which gives

$$J = r^{n-1}\ \det D(X^{-1} \circ \psi \circ Y).$$

Remember that ψ was a map from $\mathbb{S}^1$ to $\partial B_r(\mathbf{p})$ given by $\psi(\mathbf{x}) = \mathbf{p} + r\mathbf{x}$. Also, ψ was the restriction of the map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ given by the same formula. We are interested here in the map

$$X^{-1} \circ \psi \circ Y : U \to U.$$

Since $Y = A \circ X$ and $\psi \circ A$ is the identity, we can write

$$X^{-1} \circ \psi \circ Y = X^{-1} \circ \psi \circ A \circ X = X^{-1} \circ X = \mathrm{id}_U.$$

The identity on U is of course just the restriction of the identity (linear) map on $\mathbb{R}^{n-1}$ and the associated scaling factor is

$$\det D(X^{-1} \circ \psi \circ Y) \equiv 1.$$

At length, we have established $J \equiv r^{n-1}$ with a formula though we have never written down anything explicity for the parameterizations X or Y of the spheres. This could be done, but we didn't do it, and we didn't really need to do it.

B.1 n volume of the n ball

As another kind of example, let us consider the integral with respect to the full-dimension measure over $B_1(\mathbf{0})$ of the constant function $f \equiv 1$, that is, the n dimensional volume of the ball $B_1(\mathbf{0}) \subset \mathbb{R}^n$. I mentioned earlier the use of Fubini's theorem. A generalized version of Fubini's theorem gives

$$\int_{B_1(\mathbf{0})} 1 = \int_0^1 \left(\int_{\partial B_r(\mathbf{0})} 1 \right) dr = \int_0^1 \left(\int_{\partial B_1(\mathbf{0})} r^{n-1} \right) dr.$$

Notice the Jacobian scaling factor r^{n-1} associated with the change of variables $\psi : \mathbb{S}^1 \to \partial B_r(\mathbf{0})$ by $\psi(\mathbf{x}) = r\mathbf{x}$. Continuing we have

$$\int_{B_1(\mathbf{0})} 1 = \int_0^1 r^{n-1} \left(\int_{\mathbb{S}^1} 1 \right) dr.$$

remembering our enigmatic name $n\omega_n$ for the $(n-1)$ dimensional measure of the unit sphere, we can write

$$\int_{B_1(\mathbf{0})} 1 = n\omega_n \int_0^1 r^{n-1}\, dr = \omega_n.$$

This makes the name much less enigmatic. We can start instead with the definition

$$\omega_n = \int_{B_1(\mathbf{0})} 1$$

as the n dimensional measure of the unit ball $B_1(\mathbf{0}) \subset \mathbb{R}^n$ and then calculate the $(n-1)$ dimensional measure

$$\mathcal{H}^{n-1}(\mathbb{S}^{n-1})$$

of the $(n-1)$ sphere $\partial B_1(\mathbf{0}) = \mathbb{S}^{n-1}$ to find the formula

$$\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} 1 = n\omega_n.$$

C Open sets, closed sets, closure and compactness

A **metric space** is a set X accompanied by a **distance function**, that is a function $d : X \times X \to [0, \infty)$ satisfying

M1 (symmetric) $d(x, y) = d(y, x)$ for all $x, y \in X$.

M2 (positive definite) $d(x, x) = 0$ for all $x \in X$ and if $d(x, y) = 0$, then $x = y$.

M3 (triangle inequality) For all $x, y, z \in X$, there holds

$$d(x, y) \le d(x, z) + d(z, y).$$

C.1 Examples and abstract structure

The real numbers $\mathbb{R}$ is a metric space with $d(x, y) = |x - y|$ (the absolute value of the difference). The real numbers $\mathbb{R}$ admits a great deal of additional "structure." Algebraically, $\mathbb{R}$ is a **group** under addition, a **ring** and a **field** under (addition and) multiplication, and a **vector space** under multiplication and addition. I will not define all these algebraic terms here. You are should be familiar with all of these structures, though you may not know the names. The properties involved in formally defining the abstract structures listed here are known to you as "arithmetic."

The real numbers also have analytic structure. Analytically $\mathbb{R}$ is **Archimedian** as a field, **ordered**, **Dedekind complete**, and **complete** or **Cauchy complete** as a metric space and has the **Heine-Borel** property. You probably know most of these properties, though you may not know the names or the technical details. You may not know very well about metric or Cauchy completeness, and there is another analytic property we may discuss a bit further later: $\mathbb{R}$ is a **measure space** with a specified collection $\mathfrak{M}$ of subsets called the measurable sets in which the measure of an interval with endpoints a and b is its length $|b-a|$.

Returning to metric space structure, the set

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R}\}$$

is a metric space with

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

Any subset of a metric space is a metric space with the restriction metric.

C.2 Open sets in a metric space

Given $r > 0$ and a point x in a metric space X one may define the special set called a **ball of radius** r **and center** x by

$$B_r(x) = \{\xi \in X : d(\xi, x) < r\}.$$

A subset $U \subset X$ in a metric space X is **open** if for each point $x \in U$, there is some $r > 0$ so that $B_r(x) \subset U$.

A subset $A \subset X$ in a metric space X is **closed** if the complement $A^c = X \backslash A$ is open.

Exercise 4 Prove the following:

(a) The ball $B_r(x)$ is open.

(b) The entire metric space X is open.

(c) The empty set is open.

(d) Any union of open sets is open: Let $\{U_\alpha\}_{\alpha \in \Gamma}$ where Γ is an indexing set by a collection of open sets in X, then

$$\bigcup_{\alpha \in \Gamma} U_\alpha = \{x \in X : x \in U_\alpha \text{ for some } \alpha \in \Gamma\}$$

is open.

(e) A finite intersection of open sets is open: If $k \in \mathbb{N}$ and $U_1, U_2, \dots, U_k$ are open sets, then

$$\bigcap_{j=1}^{k} U_j = \{x \in X : x \in U_j \text{ for every } j = 1, 2, \dots, k$$

is open

(f) The entire metrix space X is closed.

(g) The empty set is closed.

(h) Any intersection of closed sets is closed.

(i) A finite union of closed sets is closed.

(j) Given $x \in X$ and $r > 0$ the set

$$\{\xi \in X : d(\xi, x) \leq r\} \tag{36}$$

is closed.

Note: The set $B_r(x)$ is often called an **open ball**, but one does not need to know about open sets to define such a set. This is important because we used this special kind of set to define what it means to be open. The set defined in (36) is often called a **closed ball**. It is also the **closure** of the open ball as we will discuss presently.

Exercise 5 (counterexamples)

(a) Find a collection of open subsets of $\mathbb{R}$ with intersection that is not open.

(b) Find a collection of closed subsets of $\mathbb{R}$ with union that is not closed.

C.3 Closure

We have already defined what it means for a set to be closed in a metric space X. Given any set $A \subset X$, there is a unique smallest closed subset of X containing A. This set is

$$\bigcap_{\substack{C \text{ closed} \\ C \supset A}} C,$$

that is, the intersection of all closed sets containing A. This set is called the **closure** of A and is denoted by $\overline{A}$.

Exercise 6 Show

$$\overline{B_r(x)} = \{\xi \in X : d(\xi, x) \leq r\}.$$

C.4 Compact sets

A collection $\{U_\alpha\}_{\alpha\in\Gamma}$ of open sets is said to be an **open cover** of a set A if

$$A \subset \bigcup_{\alpha\in\Gamma} U_\alpha.$$

A **compact set** is a set K with the following property: Given any open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of K, there exist finitely many sets

$$U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}$$

in the given open cover for which

$$K \subset \bigcup_{j=1}^{k} U_{\alpha_j}.$$

That is, there may be any (infinite) number of sets in the original open cover, but only finitely many of them are required to cover the set K.

In the set of real numbers $\mathbb{R}$ and in the Euclidean spaces $\mathbb{R}^n$ the compact sets are characterized by being closed and bounded.

A set A in a metric space X is **bounded** if there is some $R > 0$ and some $x \in X$ for which

$$A \subset B_R(x).$$

Exercise 7 Show that every compact set in a metric space is closed and bounded.

It is not true in general that a closed and bounded set in a metric space must be compact.

Exercise 8 Give an example of a metric space and a closed and bounded subset of that metric space which is not compact.

A space in which the compact sets are characterized by being closed and bounded is said to have the **Heine-Borel property**. The assertion that every closed and bounded subset of $\mathbb{R}^n$ is compact is called the Heine-Borel theorem.

C.5 Metric completeness

A sequence $\{x_j\}_{j=1}^{\infty}$ is said to be **Cauchy** or to be a **Cauchy sequence** if for any $\epsilon > 0$, there is some $N > 0$ such that

$$j, k > N \qquad \text{implies} \qquad d(x_j, x_k) < \epsilon.$$

Note that this definition says nothing about the existence of a limit for the sequence.

Exercise 9 Show that every sequence which converges to a limit $L \in X$ in the sense that for any $\epsilon > 0$ there is some $N > 0$ such that

$$j > N \qquad \text{implies} \qquad d(x_j, L) < \epsilon$$

is a Cauchy sequence.

Exercise 10 Give an example of a metric space X and a sequence $\{x_j\}_{j=1}^{\infty} \subset X$ for which the sequence is Cauchy but does not converge to an element $L \in X$.

A space in which every Cauchy sequence converges is said to be (metrically) **complete**. The space $\mathbb{R}^n$ is complete.

C.6 The Heine-Borel property for $\mathbb{R}^n$

Here is a sketch of the proof that a closed and bounded subset of $\mathbb{R}^n$ is compact. One can first assert that some specific **closed cubes**[4] is compact. Consider for $R > 0$ the closed cube

$$\begin{aligned} C &= [-R, R]^n \\ &= \prod_{j=1}^{n} [-R, R] \\ &= [-R, R] \times [-R, R] \times \cdots \times [-R, R] \\ &= \{\mathbf{x} = (x_1, x_2, \ldots, x_j) : |x_j| < R,\ j = 1, 2, \ldots, n\}. \end{aligned}$$

To see such a set is compact we argue by contradiction:

[4]The same argument given here can be adapated to any closed rectangular solid given as the cross product of any closed intervals. I've chosen this particular cube primarily so the notation is a bit simpler, and the argument is a bit easier to type.

Assume there is an open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of the cube C with the property that

$$C \not\subset \bigcup_{\alpha\in F} U_\alpha \qquad \text{whenever} \qquad F \text{ is a finite subset of } \Gamma. \tag{37}$$

Another way to say the condition of "not a subset" in (37) is

$$C \Bigg\backslash \bigcup_{\alpha\in F} U_\alpha \neq \phi.$$

Note that the closed cube C can be written as the union of finitely many closed cubes with all sidelengths half those of the original cube. More precisely denoting by $I = [-R, 0]$ the first half of $[-R, R]$ and by $J = [0, R]$ the second half, every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C$ satisfies $x_j \in I$ or $x_j \in J$ for each $j = 1, 2, \ldots, n$; letting

$$K_j = \begin{cases} I, & \text{if } x_j \in I \\ J, & \text{if } x_j \in J \end{cases}$$

we have

$$\mathbf{x} \in \prod_{j=1}^{n} K_j = K_1 \times K_2 \times \cdots \times K_n$$

where each K_j is a closed interval half as long as $[-R, R]$. There are finitely many such smaller cubes. In fact, since there are two choices for each factor interval K_j, there are precisely 2^n such smaller cubes. Denoting the smaller cubes by $C_1, C_2, \ldots, C_{2^n}$ we have for each $j = 1, 2, \ldots, 2^n$

$$C_i \subset C \subset \bigcup_{\alpha\in\Gamma} U_\alpha.$$

That is, $\{U_\alpha\}_{\alpha\in\Gamma}$ is an open cover of each of the smaller cubes.

If each C_j for $j = 1, 2, \ldots, 2^n$ admits a finite subcover $F_j \subset \{U_\alpha\}_{\alpha\in\Gamma}$, then we can union these finite subcovers

$$F = \cup_{j=1}^{2^n} F_j$$

and get a finite cover of the original cube C contradicting (37). This means at least one of the smaller cubes has the same property (37) we started with for the bigger cube C. Note that the cube gets smaller here, but the open cover is the same open cover. Repeating this process we find that for this particular open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ there exists a sequence of cubes

$$C_1, C_2, C_3, \ldots$$

satisfying the following four properties:

(i) The cubes are **nested**:

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

(ii) The cubes are **small** in the sense that if $\mathbf{x}, \mathbf{y} \in C_m$, then

$$|\mathbf{y} - \mathbf{x}| < \frac{R\sqrt{n}}{2^{m-1}}.$$

This condition has a name. We say the **diameter** of the cube C_m is smaller than or equal to $R\sqrt{n}/2^{m-1}$.

(iii) Each cube C_m has the fixed collection $\{U_\alpha\}_{\alpha\in\Gamma}$ as an open cover:

$$C_m \subset \bigcup_{\alpha\in\Gamma} U_\alpha,$$

but

(iv) Each cube C_m does not admit a finite subcover from $\{U_\alpha\}_{\alpha\in\Gamma}$:

$$C_m \backslash \bigcup_{\alpha\in F} U_\alpha \neq \phi \qquad \text{whenever} \qquad F \text{ is a finite subset of } \Gamma. \tag{38}$$

At this point, we need a lemma:

Lemma 1 If $A_1, A_2, A_3, \ldots$ is a sequence of nonempty nested closed sets in $\mathbb{R}^n$ satisfying

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

and

$$\lim_{m\to\infty} \operatorname{diam}(A_m) = 0$$

then the following hold:

(a) Any sequence of points $\{\mathbf{x}_m\}_{m=1}^{\infty}$ with $\mathbf{x}_m \in A_m$ is a Cauchy sequence.

(b) By completeness, any sequence $\{\mathbf{x}_m\}_{m=1}^{\infty}$ as in **(a)** has a limit

$$\mathbf{p} = \lim_{m\to\infty} \mathbf{x}_m.$$

(c) The limit $\mathbf{p}$ is the same in each case independent of the particular sequence, and

(d) The intersection of the sets $A_1, A_2, A_3, \ldots$ is nonempty and is given by

$$\bigcap_{m=1}^{\infty} A_j = \{\mathbf{p}\}.$$

Applying the lemma to the sequence of shrinking cubes $\{C_m\}_{m=1}^{\infty}$, we conclude

$$\bigcap_{m=1}^{\infty} C_m = \{\mathbf{p}\}$$

for some unique point $\mathbf{p} \in C$. There exists some set $U_* \in \{U_\alpha\}_{\alpha\in\Gamma}$ with $\mathbf{p} \in U_*$, and there is some $r > 0$ for which

$$B_r(\mathbf{p}) \subset U_*.$$

It follows that for m large enough the entire cube C_m with diameter smaller than or equal to $R\sqrt{n}/2^{m-1}$ and containing $\mathbf{p}$ lies entirely in $B_r(\mathbf{p}) \subset U_*$. This is a contradiction because no finite subcover of $\{U_\alpha\}_{\alpha\in\Gamma}$ covers C_m, and yet we have shown C_m lies in a single open set $U_* \in \{U_\alpha\}_{\alpha\in\Gamma}$.

We conclude the closed cube $C = [-R, R]^n$ is compact. Returning to the original question about a closed and bounded set $A \subset \mathbb{R}^n$, we can take $R > 0$ large enough so that $A \subset [-R, R]$ and apply the following result:

Lemma 2 If K is a compact set and A is a closed set with $A \subset K$, then A is compact.

Since $A \subset [-R, R]^n$ and A is closed, it follows that A must be compact.

D Divergence

The **divergence** is an operator on vector fields $\mathbf{v} \in C^1(I \to \mathbb{R}^n)$ where I is a domain of integration with some particular properties. One simple example of a domain of integration on which the divergence is well-defined is given by taking I to be a bounded open subset Ω of $\mathbb{R}^n$. Then

$$\operatorname{div} : C^1(\Omega \to \mathbb{R}^n) \to C^0(\Omega)$$

assigns a continuous real valued function $\operatorname{div} \mathbf{v} \in C^0(\Omega)$ to the vector field $\mathbf{v}$. I could start with a familiar formula for the divergence in rectangular coordinates in this special case of $I = \Omega \subset \mathbb{R}^n$, but that would have rather limited application. Instead, I will start with a general principle upon which the concept of divergence is based, and then we can evaluate the definition in various special cases to get coordinate expressions.

In all cases. the domain of integration I should allow the consideration of vector fields $\mathbf{v} : I \to V$ where V is some suitable vector space containing all possible values of the vector field. For example, if $I = \mathcal{S}$ is a surface in $\mathbb{R}^n$, then $\mathbb{R}^n$ provides a suitable codomain for a vector field on $\mathcal{S}$. Notice that I have been a little cavalier about the dimension of $\mathcal{S}$ compared to the dimension n here. The usual assumption would be that $\mathcal{S}$ is a two dimensional surface and $n \geq 3$ or at least $n \geq 2$. However one could also assume, and sometimes this is what some authors might mean, $\mathcal{S}$ is a *hyper* surface in $\mathbb{R}^n$ so the dimension of $\mathcal{S}$ is $n - 1$. In fact, we can take as domain of integration many subminifolds M of arbitrary dimension $m \leq n$ in $\mathbb{R}^n$ and talk about vector fields $\mathbf{v} : M \to \mathbb{R}^n$. The discussion below will apply to all these cases.

The general principle has two parts:

1. The divergence is based on **flux integrals**, and
2. The value of the divergence $\operatorname{div} \mathbf{v}(\mathbf{p})$ captures (or attempts to capture) a measure of the "outward flow" infinitesimally at a point $\mathbf{p} \in I$.

Thus, we start with a subdomain of integration $J \subset I$ with $\mathbf{p} \in J$. We must be able to make sense of the boundary ∂J within I and along ∂J we need a well-defined **outward unit normal field $\mathbf{n}$** $: I \to V$. With this in mind, we want to consider a "nice enough" domain of integration J so that ∂J is also a domain of integration and given any $\mathbf{v} \in C^0(I \to V)$ the integral

$$\int_{\partial J} \langle \mathbf{v}, \mathbf{n} \rangle$$

makes sense.