Math 6702, Final Exam Name and section: ____________

This exam is meant loosely to involve "designer-engineer interactions" and to involve the design of an awning. To prepare your mind for this exam, think along the following lines:

1. You are (supposed to be) an engineer.
2. An engineer rarely has complete control over the objective and details of the problem he is given. You may be able to choose the general field (electrical engineering, mechanical engineering, or civil engineering), and maybe the specific subfield (electrical motors, aerospace, or roads), but you will still almost certainly work with others and under constraints imposed by others.
3. An engineer is often faced with vaguely stated problems, conflicting design requirements, and incomplete descriptions.
4. You are expected to take these impossible or open-ended problems and then draw definitive conclusions, make insightful suggestions, and give helpful feedback:

   "You can't have both **this** and **that**, one possible compromise is..."
   "An advantage of **this modification** would be..."
   "You need to decide what you want for **this aspect**."
   "You can expect to run into trouble **here**."

5. Your objective should be to execute your task in such a way as to produce a viable design with precise specifications for manufacture. Let's see how inspired you can be when it comes to using PDEs in the design of an awning.

1. (Laplace's awning) A designer/architect wants to have a stamped steel awning manufactured that resembles the illustration in Figure 1.

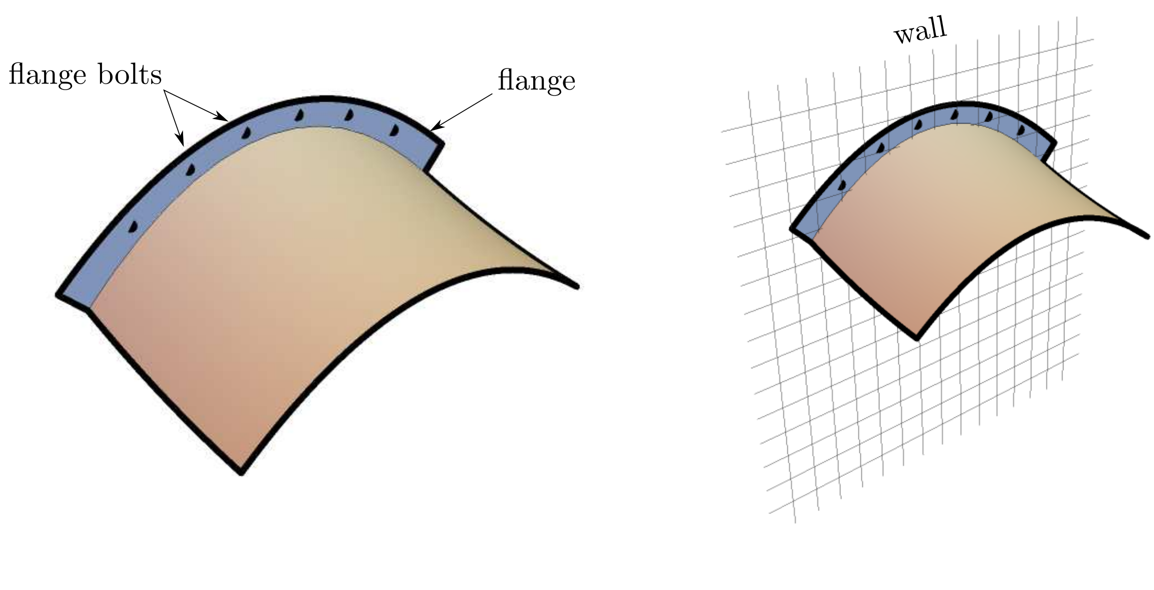

Figure 1: Rendering of an awning.

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The designer is keen on having two features:

1. The edge of the awning next to the building should contain a **flat flange** through which visible bolts pass fastening the awning to the building.
2. The stamped steel sheet of the awning should **not have a crease** around the flange but, as the architect understands it, the entire structure should be a C^1 surface.

In order to further illustrate and specify the concept the architect chooses coordinates so that the flange is horizontal (i.e., turns the awning on outside-end-up, flange down) and asks for a C^1 weak solution of Laplace's equation on the domain shown on the right in Figure 2.

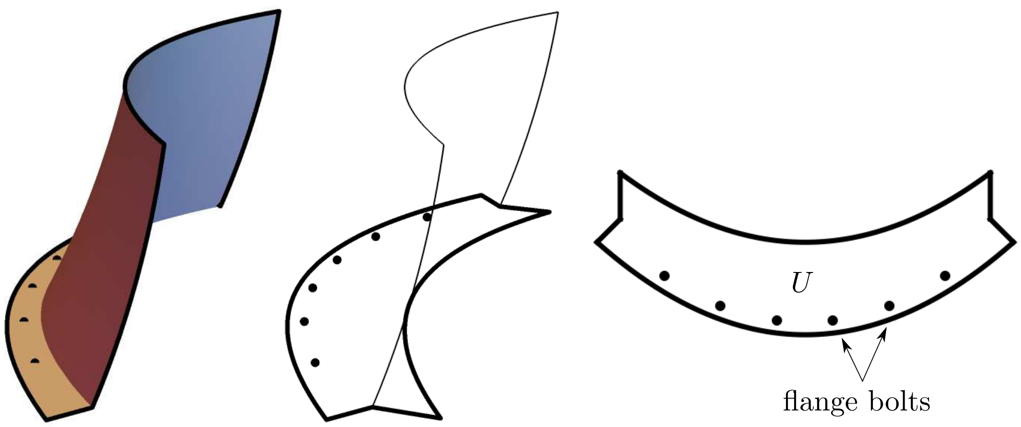

Figure 2: Awning as the graph of a C^1 weakly harmonic function u over its projection onto the building. The boundary values (center) and domain U of definition (right).

(a) What advice can you give the designer as an engineering consultant (who is expected to know something about weak solutions of Laplace's equation)?

(b) The designer has consulted with a manufacturer who can precast a rough blank and stamp it to specification with the flange horizontal at a cost proportional to the Dirichlet energy of the awning:

$$\mathcal{E}[u] = \int_U |Du|^2$$

where U is the projection domain of Figure 2. Note that if a particular design shape $u \in C^1(U) \cap W^{1,1}(U)$ minimizes the cost, then for any $\phi \in C_c^\infty(U)$ and $t \in \mathbb{R}$ one has

$$\mathcal{E}[u] \leq \mathcal{E}[u + t\phi]. \tag{1}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \mathcal{E}[u + t\phi]$$

where u is considered fixed and satisfies (1). Compute $f'(0)$ and attempt to give the designer any useful information you can offer concerning cost minimizing designs of the type he has in mind.

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2. (A hot new design) A possibly different designer wishes to create a stamped steel awning, without a flange, which is the graph of a function $u \in C^2(R) \cap C^2(\overline{R})$ over a rectangle R of fixed width a and indeterminate length b satisfying

$$\begin{cases} u_y = u_{xx} \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x, 0) = a^2/4 - x^2, \ u(\pm a/2, y) = 0, \ u(x, b) = a^2/16 - x^2/4. \end{cases} \tag{2}$$

   (a) What advice can you give this designer? The following suggestions may be helpful in backing up what you might say.

   (b) Consider any solution w of $w_y = w_{xx}$ and the function

   $$g(y) = \int_{-a/2}^{a/2} [w(x, y)]^2 \, dx,$$

   and calculate $g'(y)$. Note that $[w(x, y)]^2 \geq 0$. In particular, if $w(x, 0) = 0$ and $w(\pm a/2, y) \equiv 0$, then show $w(x, y) \equiv 0$.

   (c) Use the previous observation(s) to prove a uniqueness theorem for the equation $u_y = u_{xx}$ under appropriate assumptions. Hint: You should be able to figure out what is appropriate using your physical intuition/interpretation of the PDE.

   (d) The coordinate orientation of this awning (as you presumably noticed) is horizontal. One important design requirement is that the awning not have any convex portions that will collect water. Is this going to be a problem?

3. (farewell awning) A third designer wishes to create an awning with the same boundary values as the second designer but using a different PDE:

$$\begin{cases} u_{xx} = u_{yy} \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x, 0) = a^2/4 - x^2, \ u(\pm a/2, y) = 0, \ u(x, b) = a^2/16 - x^2/4. \end{cases} \tag{3}$$

   What can you say about this? In particular, does the design requirement that the awning contain no convex water collecting "bowls" put any restriction on the length?

4. (Something different) A fourth designer wishes to create an awning with roughly the same characteristics as the second and third designers but using the PDE

$$u_{xx} - 4u_{xy} + u_{yy} = 0.$$

   This designer leaves the boundary values to you (the engineer) subject to being determined over a rectangular domain $R = (-a/2, a/2) \times (0, b)$ by a function $u \in C^2(R) \cap C^0(\overline{R})$ and having no convex bowls. The width a is fixed, but the length is flexible as long as there are possibilities where b is "reasonably" long.

   (a) Can you help?

   (b) Suggestion: Consider a change of variables according to which $x = a_{11}\xi + a_{12}\eta$ and $y = a_{21}\xi + a_{22}\eta$ with the coefficients a_{ij} chosen so that the PDE for $w(\xi, \eta) = u(x, y)$ takes the form of an equation you know (something about).