Math 6702, Exam 2 Name and section: ____________________

1. (linear partial differential operators and Laplace's equation on a rectangle) Solve the following boundary value problem for $u \in C^2([0,1] \times [0,2])$:

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0,\ u(1,y) = \sinh\pi\,\sin\pi y,\ u(x,2) = \sinh 6\pi\,\sin 3\pi x,\ u(0,y) = 0. \end{cases} \tag{1}$$

Hint: Consider two separate boundary value problems with homogeneous boundary conditions on **three of the four** boundary segments of the rectangle. Solve each of these problems using separation of variables. Then use the linearity of the Laplace operator.

Solution: Consider

$$\begin{cases} \Delta u_1 = 0, \\ u_1(x,0) = 0,\ u_1(1,y) = \sinh\pi\,\sin\pi y,\ u_1(x,2) = 0,\ u_1(0,y) = 0. \end{cases} \tag{2}$$

This has solution

$$u_1(x,y) = \sinh\pi x\,\sin\pi y.$$

Consider also

$$\begin{cases} \Delta u_2 = 0, \\ u_2(x,0) = 0,\ u_2(1,y) = 0,\ u_2(x,2) = \sinh 6\pi\,\sin 3\pi x,\ u_2(0,y) = 0. \end{cases} \tag{3}$$

This has solution

$$u_2(x,y) = \sin 3\pi x\,\sinh 3\pi y.$$

The solution of the original problem is

$$u(x,y) = u_1(x,y) + u_2(x,y) = \sinh\pi x\,\sin\pi y + \sin 3\pi x\,\sinh 3\pi y.$$

Name and section: ___________________

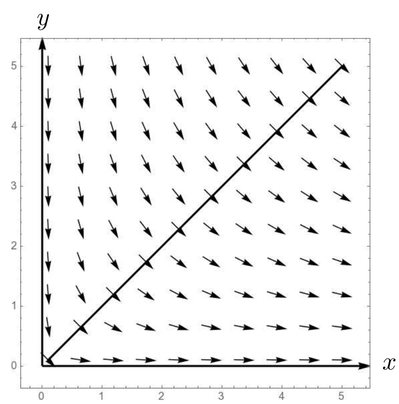
2. (first order linear PDE; method of characteristics) Solve the PDE

$$xu_x - yu_y + (x^2 + y^2)u = x^2 - y^2 \qquad \text{on } U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}.$$

"Solve" here means "Find all possible C^1 solutions." Your solution should depend on an arbitrary function which **you** will need to introduce. Knowing how to do that is part of the problem. (This is like if someone says: Solve $x'' = 0$. Then you know $x = at + b$ with two arbitrary constants a and b.)

Hint(s): Consider the **characteristic field** $\mathbf{v} = (x, -y)$ on the first quadrant U. Plot it with numerical software if necessary. Choose an appropriate non-characteristic curve.

Solution:



The characteristic field points down and to the right in the first quadrant U as indicated in the figure. Thus, $y = x$ is a natural choice for a non-characteristic curve. Thus, fixing a point (ξ, ξ) as an initial starting point, we seek a characteristic curve $\mathbf{r} = (x, y)$ satisfying

$$x' = x$$
$$y' = -y.$$

Thus, $x = \xi e^t$ and $y = \xi e^{-t}$. Substituting into the PDE we obtain a first order linear nonhomogeneous ODE for $u = u(x(t), y(t))$:

$$\frac{d}{dt}u + \xi^2(e^{2t} + e^{-2t})\,u = \xi^2(e^{2t} - e^{-2t}).$$

This can be written as

$$\frac{d}{dt}u + 2\xi^2 \cosh(2t)\,u = 2\xi^2 \sinh(2t).$$

There is an integrating factor

$$\mu = e^{\xi^2 \sinh(2t)},$$

Name and section: ____________________

and we get

$$(\mu u)' = 2\xi^2 \mu \sinh(2t) = 2\xi^2 \sinh(2t) e^{\xi^2 \sinh(2t)}.$$

Integrating gives:

$$e^{\xi^2 \sinh(2t)}\, u - u_0 = 2\xi^2 \int_0^t \sinh(2\tau) e^{\xi^2 \sinh(2\tau)}\, d\tau.$$

Thus,

$$u(x(t), y(t)) = e^{-\xi^2 \sinh(2t)} \left\{ u_0(\xi) + 2\xi^2 \int_0^t \sinh(2\tau) e^{\xi^2 \sinh(2\tau)}\, d\tau \right\}$$

where we have specified the value $u(\xi, \xi) = u_0(\xi)$ by an arbitrary function $u_0 \in C^1(0, \infty)$. It remains to determine ξ (and t) in terms of a point $(x, y) \in U$ according to

$$\begin{aligned} x &= \xi e^t \\ y &= \xi e^{-t}. \end{aligned}$$

Multiplying the equations t is eliminated: $\xi^2 = xy$. Thus, $\xi = \sqrt{xy}$. Dividing the equations ξ is eliminated: $e^{2t} = x/y$. Thus, $2t = \ln(x/y)$ and $t = \ln \sqrt{x/y}$. It is also possible to subtract the equations to obtain

$$2\sqrt{xy} \sinh t = x - y \qquad \text{so that} \qquad t = \sinh^{-1} \frac{1}{2} \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} \right).$$

In any case,

$$u(x, y) = e^{-[xy \sinh(\ln(x/y))]} \left\{ u_0(\sqrt{xy}) + 2xy \int_0^{\ln \sqrt{x/y}} \sinh(2\tau) e^{xy \sinh(2\tau)}\, d\tau \right\}.$$

This may also be written as

$$\begin{aligned} u(x, y) &= e^{-xy(x/y - y/x)/2} \left\{ u_0(\sqrt{xy}) + 2xy \int_0^{\ln \sqrt{x/y}} \sinh(2\tau) e^{xy \sinh(2\tau)}\, d\tau \right\} \\ &= e^{(y^2 - x^2)/2} \left\{ u_0(\sqrt{xy}) + 2xy \int_0^{\ln \sqrt{x/y}} \sinh(2\tau) e^{xy \sinh(2\tau)}\, d\tau \right\}. \end{aligned}$$

Name and section: ____________________

3. (one dimensional wave equation) Solve the initial value problem for the wave equation:

$$\begin{cases} u_{tt} = u_{xx} \text{ on } \mathbb{R} \times [0, \infty) \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases} \tag{4}$$

where $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$ to obtain d'Alembert's solution:

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

Hint(s): Factor the operator $\Box u = u_{tt} - u_{xx}$ as either

$$(u_t - u_x)_t + (u_t - u_x)_x \qquad \text{or} \qquad (u_t + u_x)_t - (u_t + u_x)_x.$$

Then solve two first order PDEs with appropriate Cauchy conditions. Incidentally, the initial conditions in (4) are Cauchy conditions for the wave equation.

Solution: As per the hint, let $w(x,t) = u_t(x,t) - u_x(x,t)$. Then $w(x,0) = u_t(x,0) - u_x(x,0) = v_0(x) - u_0'(x)$ and

$$w_t + w_x = 0.$$

Along the characteristic curve $\gamma(t) = (\xi + t, t)$ starting at $(\xi, 0)$, we find

$$\frac{d}{dt} w(\xi + t, t) = w_x(\xi + t, t) + w_t(\xi + t, t) = 0.$$

Therefore,

$$w(\xi + t, t) \equiv w(\xi, 0) = v_0(\xi) - u_0'(\xi).$$

Given an arbitrary point $(x, y) = (\xi + t, t)$, we have

$$w(x, y) = v_0(x - y) - u_0'(x - y).$$

We have solved for w. Applying this approach to the PDE $u_t - u_x = w$, we get

$$\frac{d}{dt} u(\xi - t, t) = -u_x(\xi - t, t) + u_t(\xi - t, t) = w(\xi - t, t) = v_0(\xi - 2t) - u_0'(\xi - 2t).$$

Therefore,

$$\begin{aligned} u(\xi - t, t) &= u(\xi, 0) + \int_0^t [v_0(\xi - 2\tau) - u_0'(\xi - 2\tau)]\, d\tau \\ &= u_0(\xi) + \frac{1}{2}[u_0(\xi - 2t) - u_0(\xi)] + \int_0^t v_0(\xi - 2\tau) d\tau \\ &= \frac{1}{2}[u_0(\xi - 2t) + u_0(\xi)] - \frac{1}{2}\int_{\xi}^{\xi - 2t} v_0(\eta) d\eta \\ &= \frac{1}{2}[u_0(\xi - 2t) + u_0(\xi)] + \frac{1}{2}\int_{\xi - 2t}^{\xi} v_0(\eta) d\eta. \end{aligned}$$

Name and section: ____________________

Given an arbitrary point $(x, y) = (\xi - t, t)$, we have

$$u(x, y) = \frac{1}{2}[u_0(x - y) + u_0(x + y)] + \frac{1}{2}\int_{x-y}^{x+y} v_0(\eta)d\eta.$$

This is d'Alembert's formula (1747).

Name and section: ______________________

The Divergence of a Vector Field in $\mathbb{R}^2$

This will give you a chance to integrate on curves. You'll need to integrate on curves. So the first part is a warm up involving integration on a curve. Remember, before you start it, that

$$\int_\Gamma f = \lim \sum f(p_j)\,\text{length}(\Gamma_j)$$

where $\{\Gamma_j\}$ is a partition of the curve Γ and $f : \Gamma \to \mathbb{R}$ is a real valued function; each point p_j is in the partition piece Γ_j and the limit is as the "diameter measure" (in this case length will work) of the largest partition piece tends to zero.

Also, the divergence for a vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ at a point $\mathbf{p} \in \mathbb{R}^2$ is defined as

$$\operatorname{div} \mathbf{v} = \lim_{U \to \{\mathbf{p}\}} \frac{1}{\text{area}(U)} \oint_{\partial U} \mathbf{v} \cdot \mathbf{n}$$

(when this limit exists). Here $\mathbf{n}$ is the **outward unit normal** to U and the little circle is put on the integral sign just to remind us that we're integrating over a **cycle** or, in this case, a simple closed curve.

Okay, let's do this.

4. A nice curve to consider (when thinking about integrating on a curve) is a single turn of a helix

$$\Gamma = \{(\cos t, \sin t, t) : t \in [0, 2\pi]\}.$$

Let's try to compute

$$\int_\Gamma f$$

where $f = f(x, y, z)$ is just some function I write down. This should illustrate how integration over a curve works in general. The first step is to write down a parameterization for the curve.

(a) Write down a parameterization $\gamma : [0, 2\pi] \to \mathbb{R}^3$ for the specific curve Γ given above and sketch the image. (Hint: Yes, this is as easy as it looks.)

(b) Now, this is perhaps a little harder: For the computation, we want to "change variables" from Γ to $[0, 2\pi]$. This requires a scaling factor:

$$\int_\Gamma f = \int_0^{2\pi} f \circ \gamma(t)\, \sigma\, dt.$$

What is the scaling factor σ for the specific helix parameterized in the previous part? And what is the scaling factor, in general, if a curve Γ is parameterized by $\gamma \in C^1([a, b] \to \mathbb{R}^n)$ on some interval $[a, b]$?

(c) Compute

$$\int_\Gamma f$$

for Γ the single turn of the helix above and $f(x, y, z) = x^2 + y^2 + z^2$.

(d) Consider a point $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ and a vector field $\mathbf{v} \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$. For positive numbers ϵ and δ, let

$$R = R_{\epsilon,\delta} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - p_1| < \epsilon \text{ and } |x_2 - p_2| < \delta\}$$

be a rectangular domain with outward unit normal $\mathbf{n}$. Draw R along with $\mathbf{n}$ and show

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = 2\delta \int_{p_1-\epsilon}^{p_1+\epsilon} \frac{\partial v_2}{\partial y}(x, p_2^*)\, dx + 2\epsilon \int_{p_2-\delta}^{p_2+\delta} \frac{\partial v_1}{\partial x}(p_1^*, y)\, dy$$

for some point $\mathbf{p}^* = (p_1^*, p_2^*) \in R$. Hint: Use the **mean value theorem** which tells you, for example, that if $v \in C^1(\mathbb{R}^2)$, then for $a < b$ and $y \in \mathbb{R}$, there is some $x^* \in (a, b)$ such that

$$v(b, y) - v(a, y) = \frac{\partial v}{\partial x}(x^*, y)\,(b - a).$$

(e) Compute

$$\lim_{\epsilon,\delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

Solution:

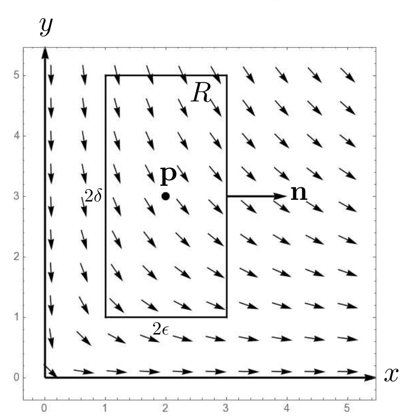
(a) $\gamma(t) = (\cos t, \sin t, t)$.

(b) $\sigma = \sqrt{1+1} = |\gamma'(t)|$.

(c) Therefore,

$$\int_\Gamma f = \int_0^{2\pi} (1+t^2)\sqrt{2}\, dt = \sqrt{2} \int_0^{2\pi} (1+t^2)\, dt = 2\pi\sqrt{2}(1 + 4\pi^2/3).$$

(d) Here is a drawing of a vector field around a point in the plane:



$$\int_{\partial R} \mathbf{v}\cdot\mathbf{n} = \int_{p_1-\epsilon}^{p_1+\epsilon} \mathbf{v}(x, p_2-\delta)\cdot(0,-1)\,dx + \int_{p_2-\delta}^{p_2+\delta} \mathbf{v}(p_1+\epsilon, y)\cdot(1,0)\,dy$$
$$+ \int_{p_1-\epsilon}^{p_1+\epsilon} \mathbf{v}(x, p_2+\delta)\cdot(0,1)\,dx + \int_{p_2-\delta}^{p_2+\delta} \mathbf{v}(p_1-\epsilon, y)\cdot(-1,0)\,dy$$
$$= \int_{p_1-\epsilon}^{p_1+\epsilon} [v_2(x, p_2+\delta) - v_2(x, p_2-\delta)]\,dx + \int_{p_2-\delta}^{p_2+\delta} [v_1(p_1+\epsilon, y) - v_1(p_1-\epsilon, y)]\,dy$$
$$= 2\delta \int_{p_1-\epsilon}^{p_1+\epsilon} \frac{\partial v_2}{\partial y}(x, p_2^*)\,dx + 2\epsilon \int_{p_2-\delta}^{p_2+\delta} \frac{\partial v_1}{\partial x}(p_1^*, y)\,dy$$

where we have obtained the values p_1^* and p_2^* from the mean value theorem.

(e) Since the area of R is $4\delta\epsilon$, we have

$$\lim_{\epsilon,\delta\to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v}\cdot\mathbf{n} = \lim_{\epsilon,\delta\to 0} \left(\frac{1}{2\epsilon} \int_{p_1-\epsilon}^{p_1+\epsilon} \frac{\partial v_2}{\partial y}(x, p_2^*)\,dx + \frac{1}{2\delta} \int_{p_2-\delta}^{p_2+\delta} \frac{\partial v_1}{\partial x}(p_1^*, y)\,dy \right)$$
$$= \frac{\partial v_2}{\partial y}(\mathbf{p}) + \frac{\partial v_1}{\partial x}(\mathbf{p}).$$

Name and section: ____________________

The $2D$ Heat Equation on $\mathcal{U} \subset \mathbb{R}^2$

5. Derive the heat equation (carefully and from scratch) as it applies to a laminar domain $\mathcal{U} \subset \mathbb{R}^2$. Start by listing/identifying all the quantities you will use with their units. Let's try this: I'll start you out and give you a sort of outline to follow. When I put an ellipsis $(\cdots)$, this will mean there are details for you to fill in—probably lots of them.

quantity	identification	units
$\theta_2 = \theta_2(x, y, t)$,	areal or laminar heat energy density	$[\theta_2] = \frac{[\text{energy}]}{L^2}$
$\vdots$		
Incidentally,	energy has units of work	$[\text{energy}] = [\text{force}]L = \frac{ML^2}{T^2}$
$\vdots$		
$\vec{\phi}_2 = \vec{\phi}_2$,	laminar heat flux field	$[\vec{\phi}_2] = \ldots$
$\vdots$		
$\vdots$		
$u = u(x, y, t)$,	temperature	$[u] = [\text{temperature}]$
$Du = Du(x, y, t)$,	temprature gradient	$[Du] = \ldots$
$\sigma = \sigma(x, y, u)$,	specific heat capacity	$[\sigma] = \ldots$
$K_2 = K_2(x, y, u)$,	laminar thermal conductivity	$[K_2] = \ldots$
$\vdots$		
$\vdots$		

Accounting of rate of change of total energy

$$\frac{d}{dt} \int_U \theta_2 = - \int_{\partial U} \vec{\phi}_2 \cdot \mathbf{n} + \int_U Q_2$$

$\ldots$

Law of specific heat $\ldots$

Fourier's law $\ldots$

$$\frac{\partial}{\partial t}[\sigma \rho_2 u] = \text{div}[K_2 Du] + Q_2.$$

$\ldots$

Finally, taking $\sigma \rho_2 = K_2$ (constant) and setting $f = Q_2/K_2$,

$$u_t = \Delta u + f.$$

Name and section: ____________________

Solution: Let $U \subset \mathcal{U}$.

quantity	identification	units
$\theta_2 = \theta_2(x,y,t)$,	areal or laminar heat energy density	$[\theta_2] = \frac{[\text{energy}]}{L^2}$
$\int_U \theta_2$	total energy in U at time t	$\left[\int_V \theta_2\right] = [\text{energy}]$
Incidentally,	energy has units of work	$[\text{energy}] = [\text{force}]L = \frac{ML^2}{T^2}$
$\vec{\phi}_2 = \vec{\phi}_2$,	laminar heat flux field	$[\vec{\phi}_2] = \frac{[\text{energy}]}{LT}$
$\int_{\partial U} \vec{\phi}_2 \cdot \mathbf{n}$	rate at which heat energy exits U	$\frac{[\text{energy}]}{T}$
Q_2	areal energy forcing rate density	$\frac{[\text{energy}]}{L^2T}$
$\int_U Q_2$	energy forcing rate on U	$\frac{[\text{energy}]}{T}$
$u = u(x,y,t)$,	temperature	$[u] = [\text{temperature}]$
$Du = Du(x,y,t)$,	temprature gradient	$[Du] = \frac{[\text{temperature}]}{L}$
$\rho_2 = \rho_2(x,y)$,	areal mass density	$[\rho] = \frac{M}{L^2}$
$\sigma = \sigma(x,y,u)$,	specific heat capacity	$[\sigma] = \frac{[\text{energy}]}{M[\text{temperature}]}$
$K_2 = K_2(x,y,u)$,	laminar thermal conductivity	$[K_2] = \frac{[\text{energy}]}{T[\text{temperature}]}$

Accounting of rate of change of total energy

The time derivative of the total energy in U should be given by the sum of the rates at which energy is entering U through ∂U and the heat energy internally generated/depleted by forcing:

$$\frac{d}{dt}\int_U \theta_2 = -\int_{\partial U} \vec{\phi}_2 \cdot \mathbf{n} + \int_U Q_2.$$

If θ_2 and $\vec{\phi}_2$ have adequate regularity, we can differentiate under the integral sign and apply the divergence theorem to obtain

$$\int_U \frac{\partial \theta_2}{\partial t} = -\int_U \operatorname{div} \vec{\phi}_2 + \int_U Q_2.$$

Name and section: ____________________

We can write this as

$$\int_{\mathcal{U}} \left[\frac{\partial \theta_2}{\partial t} + \operatorname{div} \vec{\phi}_2 - Q_2\right] \chi_U. \tag{5}$$

If the quantity

$$\frac{\partial \theta_2}{\partial t} + \operatorname{div} \vec{\phi}_2 - Q_2$$

is continuous and strictly positive at a point, then by taking U to be a small neighborhood of that point, we arrive at a contradiction of (5). The same reasoning applies if

$$\frac{\partial \theta_2}{\partial t} + \operatorname{div} \vec{\phi}_2 - Q_2$$

is strictly negative at a point. Therefore, under adequate regularity assumptions, for example $\theta_2 \in C^1(\mathcal{U})$, $\vec{\phi}_2 \in C^1(\mathcal{U} \to \mathbb{R}^2)$ and $Q_2 \in C^0(\mathcal{U})$, we have

$$\frac{\partial \theta_2}{\partial t} = -\operatorname{div} \vec{\phi}_2 + Q_2. \tag{6}$$

Forms of the Heat Equation

The relation (6) represents a first form of the heat equation for the unknown heat energy density $\theta_2 = \theta_2(x, y, t)$ and the unknown heat flux $\vec{\phi}_2 = \vec{\phi}_2(x, y, t)$. This constitutes a single PDE for three real valued functions of three variables—two spatial variables and time.

We proceed to express the three unknown functions of (6) in terms of a single real valued function. Let u be the temperature. We assume the following:

Law of specific heat: $\theta_2 = \sigma \rho_2 u$.

Fourier's law: $\vec{\phi}_2 = K_2 Du$.

Making these substutions in (6) under the assumption that there is no mass flow, we have

$$\rho_2 \frac{\partial}{\partial t}[\sigma u] = \operatorname{div}[K_2 Du] + Q_2.$$

In general, this is a nonlinear second order equation (PDE) for u.

Assuming further that σ is independent of t and the thermal conductivity is K_2 constant in space, this becomes

$$\rho_2 \sigma u_t = K_2 \Delta u + Q_2.$$

Finally, taking $\sigma \rho_2 = K_2$ (constant) and setting $f = Q_2/K_2$ we obtain the "standard" heat equation:

$$u_t = \Delta u + f.$$