Math 6702, Exam 2 Name and section: ____________________

1. (linear partial differential operators and Laplace's equation on a rectangle) Solve the following boundary value problem for $u \in C^2([0,1] \times [0,2])$:

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0,\ u(1,y) = \sinh \pi\ \sin \pi y,\ u(x,2) = \sinh 6\pi\ \sin 3\pi x,\ u(0,y) = 0. \end{cases} \tag{1}$$

Hint: Consider two separate boundary value problems with homogeneous boundary conditions on **three of the four** boundary segments of the rectangle. Solve each of these problems using separation of variables. Then use the linearity of the Laplace operator.

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2. (first order linear PDE; method of characteristics) Solve the PDE

$$xu_x - yu_y + (x^2 + y^2)u = x^2 - y^2 \qquad \text{on } U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}.$$

"Solve" here means "Find all possible C^1 solutions." Your solution should depend on an arbitrary function which **you** will need to introduce. Knowing how to do that is part of the problem. (This is like if someone says: Solve $x'' = 0$. Then you know $x = at + b$ with two arbitrary constants a and b.)

Hint(s): Consider the **characteristic field v** $= (x, -y)$ on the first quadrant U. Plot it with numerical software if necessary. Choose an appropriate non-characteristic curve.

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3. (one dimensional wave equation) Solve the initial value problem for the wave equation:

$$\begin{cases} u_{tt} = u_{xx} \text{ on } \mathbb{R} \times [0, \infty) \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases} \tag{2}$$

where $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$ to obtain d'Alembert's solution:

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

Hint(s): Factor the operator $\Box u = u_{tt} - u_{xx}$ as either

$$(u_t - u_x)_t + (u_t - u_x)_x \qquad \text{or} \qquad (u_t + u_x)_t - (u_t + u_x)_x.$$

Then solve two first order PDEs with appropriate Cauchy conditions. Incidentally, the initial conditions in (2) are Cauchy conditions for the wave equation.

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The Divergence of a Vector Field in $\mathbb{R}^2$

This will give you a chance to integrate on curves. You'll need to integrate on curves. So the first part is a warm up involving integration on a curve. Remember, before you start it, that

$$\int_\Gamma f = \lim \sum f(p_j)\,\text{length}(\Gamma_j)$$

where $\{\Gamma_j\}$ is a partition of the curve Γ and $f : \Gamma \to \mathbb{R}$ is a real valued function; each point p_j is in the partition piece Γ_j and the limit is as the "diameter measure" (in this case length will work) of the largest partition piece tends to zero.

Also, the divergence for a vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ at a point $\mathbf{p} \in \mathbb{R}^2$ is defined as

$$\text{div}\,\mathbf{v} = \lim_{U\to\{\mathbf{p}\}} \frac{1}{\text{area}(U)} \oint_{\partial U} \mathbf{v}\cdot\mathbf{n}$$

(when this limit exists). Here $\mathbf{n}$ is the **outward unit normal** to U and the little circle is put on the integral sign just to remind us that we're integrating over a **cycle** or, in this case, a simple closed curve.

Okay, let's do this.

4. A nice curve to consider (when thinking about integrating on a curve) is a single turn of a helix

$$\Gamma = \{(\cos t, \sin t, t) : t \in [0, 2\pi]\}.$$

Let's try to compute

$$\int_\Gamma f$$

where $f = f(x, y, z)$ is just some function I write down. This should illustrate how integration over a curve works in general. The first step is to write down a parameterization for the curve.

(a) Write down a parameterization $\gamma : [0, 2\pi] \to \mathbb{R}^3$ for the specific curve Γ given above and sketch the image. (Hint: Yes, this is as easy as it looks.)

(b) Now, this is perhaps a little harder: For the computation, we want to "change variables" from Γ to $[0, 2\pi]$. This requires a scaling factor:

$$\int_\Gamma f = \int_0^{2\pi} f\circ\gamma(t)\,\sigma\,dt.$$

What is the scaling factor σ for the specific helix parameterized in the previous part? And what is the scaling factor, in general, if a curve Γ is parameterized by $\gamma \in C^1([a, b] \to \mathbb{R}^n)$ on some interval $[a, b]$?

(c) Compute

$$\int_\Gamma f$$

for Γ the single turn of the helix above and $f(x, y, z) = x^2 + y^2 + z^2$. Hint: This may not be as easy as it looks at first.

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(d) Consider a point $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ and a vector field $\mathbf{v} \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$. For positive numbers ϵ and δ, let

$$R = R_{\epsilon,\delta} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - p_1| < \epsilon \text{ and } |x_2 - p_2| < \delta\}$$

be a rectangular domain with outward unit normal $\mathbf{n}$. Draw R along with $\mathbf{n}$ and show

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = 2\delta \int_{p_1-\epsilon}^{p_1+\epsilon} \frac{\partial v_2}{\partial y}(x, p_2^*)\, dx + 2\epsilon \int_{p_2-\delta}^{p_2+\delta} \frac{\partial v_1}{\partial x}(p_1^*, y)\, dy$$

for some point $\mathbf{p}^* = (p_1^*, p_2^*) \in R$. Hint: Use the **mean value theorem** which tells you, for example, that if $v \in C^1(\mathbb{R}^2)$, then for $a < b$ and $y \in \mathbb{R}$, there is some $x^* \in (a, b)$ such that

$$v(b, y) - v(a, y) = \frac{\partial v}{\partial x}(x^*, y)\,(b - a).$$

(e) Compute

$$\lim_{\epsilon,\delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

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The $2D$ Heat Equation on $\mathcal{U} \subset \mathbb{R}^2$

5. Derive the heat equation (carefully and from scratch) as it applies to a laminar domain $\mathcal{U} \subset \mathbb{R}^2$. Start by listing/identifying all the quantities you will use with their units. Let's try this: I'll start you out and give you a sort of outline to follow. When I put an ellipsis ($\cdots$), this will mean there are details for you to fill in—probably lots of them.

quantity	identification	units
$\theta_2 = \theta_2(x,y,t)$,	areal or laminar heat energy density	$[\theta_2] = \frac{[\text{energy}]}{L^2}$
$\vdots$		
Incidentally,	energy has units of work	$[\text{energy}] = [\text{force}]L = \frac{ML^2}{T^2}$
$\vdots$		
$\vec{\phi}_2 = \vec{\phi}_2$,	laminar heat flux field	$[\vec{\phi}_2] = \ldots$
$\vdots$		
$\vdots$		
$u = u(x,y,t)$,	temperature	$[u] = [\text{temperature}]$
$Du = Du(x,y,t)$,	temprature gradient	$[Du] = \ldots$
$\sigma = \sigma(x,y,u)$,	specific heat capacity	$[\sigma] = \ldots$
$K_2 = K(x,y,u)$,	laminar thermal conductivity	$[K_2] = \ldots$
$\vdots$		
$\vdots$		

Accounting of rate of change of total energy

$$\frac{d}{dt}\int_U \theta_2 = -\int_{\partial U} \vec{\phi}_2 \cdot \mathbf{n} + \int_U Q_2$$

$\ldots$

Law of specific heat $\ldots$

Fourier's law $\ldots$

$$\frac{\partial}{\partial t}[\sigma\rho_2 u] = \operatorname{div}[K_2 Du] + Q_2.$$

$\ldots$

Finally, taking $\sigma\rho_2 = K_2$ (constant) and setting $f = Q_2/K_2$,

$$u_t = \Delta u + f.$$