An operator $L : V \to W$ is **linear** from a vector space V of functions to another vector space W of functions if

$$L[af + bg] = aL[f] + bL[g] \qquad \text{for every } a, b \in \mathbb{R} \text{ and } f, g \in V.$$

1. (linear partial differential operators) Show the following partial differential operators are linear. State clearly a natural vector space of functions for the domain V and codomain W of each operator.

   (a) (The Lewy Operator)

   $$L\begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} u_x - v_y + 2yu_z + 2xv_z \\ v_x + u_y + 2yv_z - 2xu_z \end{pmatrix}.$$

   Here we are using subscript notation for (partial) derivatives.

   (b) (anisotropic Laplacian)

   $$A[u] = \sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j} u.$$

   Here, we are using multi-index notation for derivatives and $\mathbf{e}_j$ is the j-th standard unit basis vector. Note: We should also require $a_j : U \to \mathbb{R}$ for $j = 1, \ldots n$ are given **positive** functions on some domain $U \subset \mathbb{R}^n$. You may further restrict the coefficients $a_j = a_j(x)$ in order to determine/specify the codomain W of this operator. The isotropic spacial case $a_1 = a_2 = \cdots = a_n \equiv 1$ of this operator gives the Laplacian

   $$\Delta u = \nabla^2 u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

   (c) (heat operator)

   $$H[u] = u_t - k\Delta u.$$

   Here the positive constant $k = \alpha^2$ is called the **diffusivity**, and the operator is also called the diffusion operator.

   (d) (wave operator)

   $$\Box u = u_{tt} - k\Delta u.$$

   Here the positive constant $k = c^2$ is called the **square of the propogation speed**. The wave operator is also sometimes called the D'Alembertian after Jean D'Alembert.

Solution:

(a) (Lewy operator) A natural domain for this operator, especially given the historical significance, is $V = C^1(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ consisting of pairs (u, v) of C^1 functions on all of $\mathbb{R}^3$. Of course, a natural alternative would be $V = C^1(U) \times C^1(U)$ where

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U is an open subset of $\mathbb{R}^3$. In any case, it should be clear from the definition, that with one of these choices for V, say the first one, a reasonable choice for the codomain is $W = C^0(\mathbb{R}^3) \times C^0(\mathbb{R}^3)$ the vector space of pairs of continuous functions. As for the linearity:

$$
\begin{aligned}
L\begin{bmatrix} au+b\tilde{u} \\ av+b\tilde{v} \end{bmatrix} &= \begin{pmatrix} (au+b\tilde{u})_x - (av+b\tilde{v})_y + 2y(au+b\tilde{u})_z + 2x(av+b\tilde{v})_z \\ (av+b\tilde{v})_x + (au+b\tilde{u})_y + 2y(av+b\tilde{v})_z - 2x(au+b\tilde{u})_z \end{pmatrix} \\
&= a\begin{pmatrix} u_x - v_y + 2yu_z + 2xv_z \\ v_x + u_y + 2yv_z - 2xu_z \end{pmatrix} + b\begin{pmatrix} \tilde{u}_x - \tilde{v}_y + 2y\tilde{u}_z + 2x\tilde{v}_z \\ \tilde{v}_x + \tilde{u}_y + 2y\tilde{v}_z - 2x\tilde{u}_z \end{pmatrix} \\
&= aL\begin{bmatrix} u \\ v \end{bmatrix} + bL\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}.
\end{aligned}
$$

(b) For the anisotropic Laplacian we can take $V = C^2(U)$ with U and open subset of $\mathbb{R}^n$ and $W \in C^0(U)$. You know it's $\mathbb{R}^n$ because there are n terms in the summation. Linearity is given by

$$
\begin{aligned}
A[au+bv] &= \sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j}(au+bv) \\
&= \sum_{j=1}^{n} a_j(x)(aD^{2\mathbf{e}_j}u + bD^{2\mathbf{e}_j}v) \\
&= a\sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j}u + b\sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j}v \\
& aA[u] + bA[v].
\end{aligned}
$$

(c) For the heat operator, we may wish to get fancy and require the domain to contain functions which are C^1 in time and C^2 in space. People who are fancy operators on the heat equation do this sort of thing, and the notation is something like $V = C^2_1(U \times [0,T))$. We could also just be simple and use $V = C^2(U \times [0,T))$. It's nice for you to know the domain here $U \times [0,T)$ which is called a **heat domain** where $U \subset \mathbb{R}^n$. Sometimes $t = 0$ will be excluded, and sometimes T will be included. Also sometimes $T = \infty$. In any case, the codomain is simpler: $W = C^0(U \times [0,T))$. For linearity, we can use the previous part to apply to the Laplacian:

$$H[au+bv] = (au+bv)_t - k\Delta(au+bv) = a_t + bv_t - ka\Delta u - kb\Delta v = aH[u] + bH[v].$$

(d) Similarly, the wave operator is composed of two terms each of which is essentially a Laplacian of some sort. In fact, we didn't use the positivity of the coefficients in part (b), so we could just say this is a special case. But we'll give the details. Why not? $V = C^2(\mathcal{U})$ where $\mathcal{U} \subset \mathbb{R}^{n+1}$ or maybe $V = C^2(U \times [0,T))$. Setting $\mathcal{U} = U \times [0,T)$, we have $W = C^0(\mathcal{U})$, and

$$\Box(au+bv) = (au+bv)_{tt} - k\Delta(au+bv) = a\Box u + b\Box v.$$

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2. Consider the heat equation $u_t = k\Delta u + f$ on (with forcing) $B_1(0) \times [0, \infty) = \{(x, y, t) : x^2 + y^2 < 1 \text{ and } t \geq 0\}$.

   (a) Let $w(\xi, \eta, \tau) = u(\alpha\xi, \alpha\eta, \beta\tau)$ where α and β are positive constants. Determine the domain of w, and compute

   $$w_\tau(\xi, \eta, \tau) \qquad \text{and} \qquad \Delta w(\xi, \eta, \tau) = w_{\xi\xi} + w_{\eta\eta}.$$

   (b) (scaling in time) Say you know how to solve $w_\tau - \Delta w = f_0(\xi, \eta, \tau)$ for any $f_0 \in C^0(B_1(0) \times [0, \infty)$ with a particular initial condition

   $$w(x, y, 0) = g_0(x, y)$$

   and a homogeneous boundary condition

   $$w\Big|_{x^2+y^2=1} = 0 \qquad \text{for all time } \tau \geq 0.$$

   Explain how to solve

   $$\begin{cases} u_t - k\Delta u = f & \text{on } B_1(0) \times [0, \infty) \\ u(x, y, 0) = g_0(x, y), & (x, y) \in B_1(0) \\ u\Big|_{x^2+y^2=1} = 0, & \text{for all time } t \geq 0 \end{cases}$$

   for $k \neq 1$ by scaling in time. Hint: Use the idea of part (a).

   (c) (scaling in space) If $w_t = \Delta w$ on $W = B_5(0) \times [0, \infty)$ find the PDE satisfied by $u(x, y, t) = w(5x, 5y, t)$ on $B_1(0) \times [0, \infty)$.

   (d) (anisotropic heat diffusion) Find an appropriate domain on which to solve the heat equation $w_t = \Delta w$ which allows you to solve the anisotropic heat equation

   $$u_t = 3u_{xx} + 2u_{yy} \qquad \text{on } B_1(0) \times [0, \infty).$$

   Which initial and boundary conditions can you handle? Hint: Scale in space by different factors in different directions, i.e., anisotropically.

Solution:

(a) If $w(\xi, \eta, \tau) = u(\alpha\xi, \alpha\eta, \beta\tau)$, then this suggests (actually it does more than suggest) the change of variables

$$x = \alpha\xi, \; y = \alpha\eta, \text{ and } t = \beta\tau.$$

Since the domain of u (in x and y) has $x^2 + y^2 < 1$, we see that ξ and η should satisfy $\xi^2 + \eta^2 < 1/\alpha^2$. This means the new domain should be

$$B_{1/\alpha}(0) \times [0, \infty).$$

Scaling the time doesn't make any visible change in this expression. In retrospect, I should have given you $B_1(0) \times [0, T)$ for u. Then you would need

$$B_{1/\alpha}(0) \times [0, T/\beta) \qquad \text{for the domain of } w.$$

Also, $w_\xi = \alpha u_x$. That is, $w_\xi(\xi, \eta, \tau) = \alpha u_x(\alpha\xi, \alpha\eta, \beta\tau)$. Similarly, with these same arguments $w_{\xi\xi} = \alpha^2 u_x x$ and $w_\tau = \beta u_t$. Thus,

$$w_\tau = \beta u_t \qquad \text{and} \qquad \Delta w = \alpha^2 \Delta u.$$

Those are the derivatives requested.

(b) (scaling in time) Now, here we kind of want to go in the opposite direction. We want to start with a solution u of the given problem and then express u in terms of a solution of the appropriate problem for w. Note that our flexibility is with the forcing $f_0 = f_0(\xi, \eta, \tau) = f_0(x, y, \tau)$, so we want to idenfity that function in order to be successful. Notice that we've sort of put $\alpha = 1$ since it's suggested we only scale in time, and we have $\xi = x$ and $\eta = y$. This should work.

The next important thing to notice is that the u equation has a scalar k in front of the Laplacian, but w equation does not. So we need to pick a time scaling to take care of that. Recalling part (a) above, it makes sense to take $\beta = 1/k$. Then, for example,

$$w_\tau = \frac{1}{k} u_t = \Delta u + \frac{1}{k} f = \Delta w + \frac{1}{k} f.$$

This tells us, recalling the arguments, that we want

$$f_0(x, y, \tau) = \frac{1}{k} f(x, y, \tau/k). \tag{1}$$

Now, we can only solve for w if we have a particular intitial condition $w(x, y, 0) = g_0(x, y)$. So we had better get the same intitial condition for $u(x, y, t) = w(x, y, kt)$. Fortunately, we're given that $u(x, y, 0) = g_0(x, y)$, so that's all good. Also, the homogeneous boundary condition translates flawlessly between the two problems. Thus, we can solve

$$\begin{cases} w_t = \Delta w + \frac{1}{k} f(x, y, \tau/k) & \text{on } B_1(0) \times [0, \infty) \\ w(x, y, 0) = g_0(x, y), & (x, y) \in B_1(0) \\ w\big|_{x^2+y^2=1} = 0, & \text{for all time } t \geq 0 \end{cases}$$

With this solution in hand, we set $u(x, y, t) = w(x, y, t/\beta) = w(x, y, kt)$ as mentioned above. Then

$$u_t(x, y, t) = k w_\tau = k\Delta w + f = k\Delta u + f \tag{2}$$

which is what we wanted. The initial and boundary conditions are satisfied as well, as mentioned above, so (2) is the key calculation. Also (1) is the key specification. Those two lines are really the heart of the problem.

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(c) This one is pretty straightforward:

$$u_t = w_t = \Delta w = \frac{1}{25}\,\Delta u.$$

So the equation is $u_t = \Delta u/25$.

(d) (last part) Consider $u(x, y, t) = w(x/\sqrt{3}, y/\sqrt{2}, t)$ where w solves

$$\begin{cases} w_t = \Delta w & \text{on } \mathcal{E} \times [0, \infty) \\ w(\xi, \eta, 0) = g_0(\xi, \eta), & (\xi, \eta) \in \mathcal{E} \\ w\Big|_{(\xi,\eta)\in\partial\mathcal{E}} = h_0(\xi, \eta, t), & \text{for all time } t \geq 0 \end{cases}$$

where $\mathcal{E} = \{(\xi, \eta) : 3\xi^2 + 2\eta^2 < 1\}$ is a domain with boundary the ellipse

$$\frac{\xi^2}{1/3} + \frac{\eta^2}{1/2} = 1.$$

Then $u_{xx} = w_{\xi\xi}/3$ and $u_{yy} = w_{\eta\eta}/2$, so

$$3u_{xx} + 2u_{yy} = w_{\xi\xi} + w_{\eta\eta} = w_t = u_t,$$

and we have a solution of the heat equation on $B_1(0) \times [0, \infty)$. The initial condition satisfied by this solution is

$$u(x, y, 0) = g_0(x/\sqrt{3}, y/\sqrt{2}),$$

and the boundary condition is

$$u\Big|_{x^2+y^2=1} = h_0(x/\sqrt{3}, y/\sqrt{2}), \text{ for all time } t \geq 0.$$

These are the initial and boundary conditions you can handle.

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A function $u : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^n$ and $\mathbf{p} \in U$ is **differentiable** at $\mathbf{p}$ if there is a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{w}\to\mathbf{0}} \frac{u(\mathbf{p}+\mathbf{w}) - u(\mathbf{p}) - L(\mathbf{w})}{|\mathbf{w}|} = 0. \tag{3}$$

The linear map $L : \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of u at $\mathbf{p}$ and is denoted by $du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$.

3. Let $u : U \to \mathbb{R}$ be differentiable at $\mathbf{p} \in U$.
   (a) show the first partial derivatives $D_j u(\mathbf{p})$ exist for $j = 1, 2, \dots, n$.
   (b) Express the linear function $L : \mathbb{R}^n \to \mathbb{R}$ for which (3) holds in terms of the **gradient vector**
   $$Du(\mathbf{p}) = (D_1 u(\mathbf{p}), D_2 u(\mathbf{p}), \dots, D_n u(\mathbf{p})).$$
   (c) Let $U = (0,1) \times (0,1) \subset \mathbb{R}^2$ and consider the specific function $u : U \to \mathbb{R}$ by
   $$u(x,\xi) = \begin{cases} x(1-\xi), & 0 \le x \le \xi \\ (1-x)\xi, & \xi \le x \le 1. \end{cases}$$
   Determine the points in U at which u is differentiable.
   (d) Let u be the specific function given in the last part of this problem. Reexpress u in the form
   $$u(x,\xi) = \begin{cases} u_1(x,\xi), & 0 \le \xi \le x \\ u_2(x,\xi), & x \le \xi \le 1. \end{cases}$$
   (e) What can you say about the regularity of the specific function u from the previous two parts? Hint: You can start by showing $u \in C^0(\overline{U})$. You can also find some subdomains U_1 and U_2 of U on which the functions u_1 and u_2 are C^∞.

Solution:

(a) Recall that a partial derivative is defined as

$$\frac{\partial u}{\partial x_j}(\mathbf{p}) = \lim_{h\to 0} \frac{u(\mathbf{p}+h\mathbf{e}_j) - u(\mathbf{p})}{h}.$$

If, for example, we take $w = h\mathbf{e}_j$ in the definition of differentiability we have

$$\lim_{h\to 0} \frac{u(\mathbf{p}+h\mathbf{e}_j) - u(\mathbf{p}) - L(h\mathbf{e}_j)}{|h|} = 0.$$

Taking $h > 0$ so that $h \searrow 0$, we get

$$\lim_{h\searrow 0} \left\{ \frac{u(\mathbf{p}+h\mathbf{e}_j) - u(\mathbf{p})}{h} - L(\mathbf{e}_j) \right\} = 0.$$

Notice that we used linearity in the second term and the h is no longer present there. Thus, we have

$$\lim_{h \searrow 0} \frac{u(\mathbf{p} + h\mathbf{e}_j) - u(\mathbf{p})}{h} = L(\mathbf{e}_j).$$

Similarly, if we take $h < 0$ so that $h \nearrow 0$, then we get

$$\lim_{h \nearrow 0} \left\{ -\frac{u(\mathbf{p} + h\mathbf{e}_j) - u(\mathbf{p})}{h} + L(\mathbf{e}_j) \right\} = 0.$$

Notice the value of the limit from this direction is the same:

$$\lim_{h \nearrow 0} \frac{u(\mathbf{p} + h\mathbf{e}_j) - u(\mathbf{p})}{h} = L(\mathbf{e}_j).$$

Putting these two together, we have

$$\lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{e}_j) - u(\mathbf{p})}{h} = L(\mathbf{e}_j).$$

This means

$$D_j u(\mathbf{p}) = \frac{\partial u}{\partial x_j}(\mathbf{p}) = L(\mathbf{e}_j) \qquad \text{exists.}$$

(b) The matrix of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^1$ is a $1 \times n$ row vector obtained by putting the values $L(\mathbf{e}_j)$ in the "columns" (of length one). That is,

$$L(\mathbf{w}) = (D_1 u(\mathbf{p}), D_2 u(\mathbf{p}), \ldots, D_n u(\mathbf{p})) \cdot \mathbf{w} = Du(\mathbf{p}) \cdot \mathbf{w}.$$

(c) Clearly, if $(x_0, \xi_0) \in U_1$ where U_1 is the trianglar domain

$$U_1 = \{(x, \xi) : 0 < x < \xi < 1\},$$

then $Du = (1 - \xi, -x)$. Since $Du \in C^\infty(U_1)$ we know u is differentiable at each point in this region. Similarly, if $(x_0, \xi_0) \in U_2$ where U_2 is the trianglar domain

$$U_2 = \{(x, \xi) : 0 < \xi < x < 1\},$$

then $Du = (-\xi, -x)$. Since $Du \in C^\infty(U_2)$ we know u is differentiable at each point in this region. We have used here the following theorem from calculus:

> A function with continuous first partials in an open set ($u \in C^1(U)$) is differentiable.

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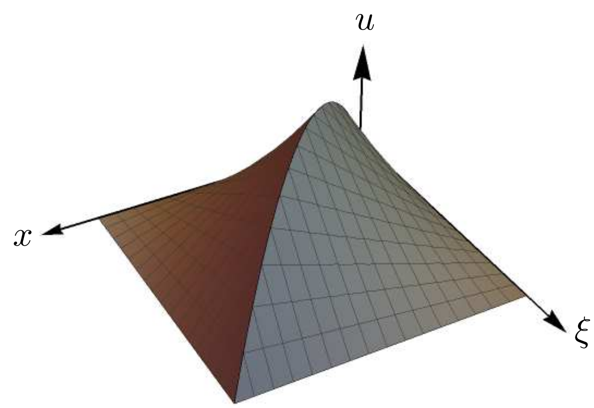
This result is stated at the end of section 3 of chapter 4 of Boas. It is stated much more clearly and with a tolerably good proof (if memory serves) in Thomas' elementary calculus text.

It is more or less obvious, that the definition of u gives a differentiable function at each boundary point of the square $[0,1]\times[0,1]$ except the corners $(0,0)$ and $(1,1)$. More generally, we suspect the function u is not differentiable along the diagonal $x=\xi$. Using numerical software to plot the graph of u confirms this suspicion:



Probably the easiest way to confirm the non-differentiability of u along the diagonal $x=\xi$ is to use part (a) where we have shown a differentiable function has first partial derivatives. Thus, we attempt to show u does not have a well-defined, say x partial at a point $\mathbf{p}=(x_0,x_0)$. The **differentiability quotient** of (3) becomes:

$$\frac{u(\mathbf{p}+\mathbf{w})-x_0(1-x_0)-L(\mathbf{w})}{|\mathbf{w}|}.$$

Taking $\mathbf{w}=h\mathbf{e}_1$ with $h>0$, we have $\mathbf{p}+\mathbf{w}=(x_0+h,x_0)$ and $x=x_0+h>\xi=x_0$. Therefore, we get

$$\lim_{h\searrow 0}\frac{(1-x_0-h)x_0-(1-x_0)x_0}{h}=-x_0.$$

On the other hand, taking $\mathbf{w}=h\mathbf{e}_1$ with $h<0$, we have $x=x_0+h<\xi=x_0$ and

$$\lim_{h\nearrow 0}\frac{(x_0+h)(1-x_0)-x_0(1-x_0)}{h}=1-x_0.$$

Since these one sided limits are clearly different, the limit of the difference quotient does not exist, there is no first partial derivative at these points, and u is not differentiable by part (a). Note that nothing particularly different happens at the corners $(0,0)$ and $(1,1)$, except that one of the one-sided limits (which are also one-sided partials) vanishes, which can be observed in the illustration. The behavior near the point $(1,1)$ is clearly visible, and one sees $u_x(1^-,1)=0$

while $u_x(1^+, 1) < 0$. This, however, it not quite adequate to show there is **no extension** of u to a function which is differentiable at, say $(1, 1)$. The computation does not show this at the corners because, for example in the case $\mathbf{p} = (1, 1)$, when calculating the first one sided limit we used points $\mathbf{p} + h\mathbf{e}_1 = (1, 1) + (h, 0) = (1 + h, 0)$ outside the closed square $[0, 1] \times [0, 1]$ and we can infer no formula for such points. What we have, is that under the assumption u_x and u_ξ exist, we must have $u_x(1, 1) = 0$ according to the limit from the left. We can also calculate the limit of the difference quotient associated with u_ξ from the left:

$$\lim_{h \nearrow 0} \frac{(1 + h)(0) - 0}{h} = 0.$$

Thus, under the assumption of differentiability we obtain $u_\xi(1, 1) = 0$ as well. But then we know the gradient must vanish $Du(1, 1) = 0$ and the linear approximation must also vanish $L(\mathbf{w}) = Du(1, 1) \cdot \mathbf{w} \equiv 0$. Here, note that we have used the expression from part (b). Finally, then, we may calculate the limit as $h \nearrow 0$ of the differentiability quotient along the diagonal:

$$\begin{aligned} \lim_{h \nearrow 0} \frac{u(1 + h, 1 + h) - u(1, 1) - L(h, h)}{|h|} &= \lim_{h \nearrow 0} -\frac{u(1 + h, 1 + h) - u(1, 1) - 0}{h} \\ &= \lim_{h \nearrow 0} -\frac{(1 + h)[1 - (1 + h)] - 0}{h} \\ &= -1 \neq 0. \end{aligned}$$

This shows there is no extension of u to a function which is differentiable at the point $(1, 1)$. Naturally, the same conclusion holds at the origin $\mathbf{p} = (0, 0)$.

(d) This is simply a matter of slicing the square vertically rather than horizontally: The specification $0 \le \xi \le x$ corresponds to the triangular region U_2 above with boundary on the x-axis. And similarly, $x \le \xi \le 1$ corresponds to the region triangular region U_1 with boundary on the ξ-axis. Thus,

$$u(x, \xi) = \begin{cases} u_1(x, \xi), & 0 \le \xi \le x \\ u_2(x, \xi), & x \le \xi \le 1. \end{cases}$$

where $u_1(x, \xi) = (1 - x)\xi$ and $u_2 = x(1 - \xi)$. Thus,

$$u(x, \xi) = \begin{cases} (1 - x)\xi, & 0 \le \xi \le x \\ x(1 - \xi), & x \le \xi \le 1. \end{cases}$$

(e) We know u is continuous on the open triangles U_1 and U_2 from the calculus theorem:

Differentiability implies continuity.

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See Thomas' Calculus. Each of the piecewise functions defining u is also individually in $C^\infty(\mathbb{R}^2)$, and it is easily checked that the formulas coincide along the diagonal. Thus, $u \in C^0(\overline{U})$. Considering the **Lipschitz quotient**

$$\frac{|u(\mathbf{p}) - u(\mathbf{q})|}{|\mathbf{p} - \mathbf{q}|}$$

with $\mathbf{p} \neq \mathbf{q}$, we can take $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ and one of the following possibilities holds: If $\mathbf{p}, \mathbf{q} \in U_1$, then

$$\begin{aligned}\frac{|u(\mathbf{p}) - u(\mathbf{q})|}{|\mathbf{p} - \mathbf{q}|} &= \frac{|p_1(1-p_2) - q_1(1-q_2)|}{\sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}} \\ &= \frac{|(1-p_2)(p_1-q_1) - (1-q_2)(q_1-p_1) + q_1(1-p_2) - p_1(1-q_2)|}{\sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}} \\ &\leq \frac{2|\mathbf{p}-\mathbf{q}| + |q_1 - p_1 - q_1 p_2 + p_1 q_2|}{|\mathbf{p}-\mathbf{q}|}.\end{aligned}$$

We have used the triangle inequality and the fact that $|1-p_2| \leq 1$ and $|1-q_2| < 1$. Also, note that

$$|q_1 - p_1 - q_1 p_2 + p_1 q_2| \leq |q_1 - p_1| + |p_2(p_1 - p_2) + p_1(q_2 - p_2)| \leq 3|\mathbf{p} - \mathbf{q}|.$$

Therefore, in this case

$$\frac{|u(\mathbf{p}) - u(\mathbf{q})|}{|\mathbf{p} - \mathbf{q}|} \leq 5. \tag{4}$$

A second case is when $\mathbf{p} \in U_1$ and $\mathbf{q} \in U_2$. In this case, consider the reflection $\hat{\mathbf{q}} = (q_2, q_1)$ of $\mathbf{q}$ about the diagonal $x = \xi$. Notice that we have $p_1 \leq p_2$ because $\mathbf{p} \in U_1$. Also, $q_2 \leq q_1$, so $\hat{\mathbf{q}} \in U_1$. Moreover, $u(\hat{\mathbf{q}}) = q_2(1-q_1) = (1-q_1)q_2 = u(\mathbf{q})$. Thus, by the previous case

$$\frac{|u(\mathbf{p}) - u(\hat{\mathbf{q}})|}{|\mathbf{p} - \hat{\mathbf{q}}|} \leq 5,$$

unless $\hat{\mathbf{q}} = \mathbf{p}$. If $\hat{\mathbf{q}} = \mathbf{p}$, however, we have $q_2 = p_1$ and $q_1 = p_2$, thus $u(\mathbf{q}) = (1-q_1)q_2 = p_1(1-p_2) = u(\mathbf{p})$ so the Lipschitz quotient vanishes. Also, if both $\mathbf{p}$ and $\mathbf{q}$ are on the diagonal, then $\mathbf{p} = (p_1, p_1)$ and $\mathbf{q} = (q_1, q_1)$, so the Lipschitz quotient satisfies

$$\frac{|u(\mathbf{p}) - u(\hat{\mathbf{q}})|}{|\mathbf{p} - \hat{\mathbf{q}}|} = \frac{|p_1(1-p_2) - q_1(1-q_2)|}{|\mathbf{p} - \hat{\mathbf{q}}|},$$

and the argument of the previous case applies. Thus, we are reduced to the case $\mathbf{p} \neq \hat{\mathbf{p}}$ and at least one of the points $\mathbf{p}$ or $\mathbf{q}$ does not lie on the diagonal. In this

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final case, there is a unique point $\mathbf{p}*$ on the intersection of the segment between $\mathbf{p}$ and $\mathbf{q}$ and the diagonal. In fact, noting that

$$(1-t)\mathbf{p}+t\mathbf{q} = ((1-t)p_1+tq_1, (1-t)p_2+q_2) = (p_1+(q_1-p_1)t, p_2-(p_2-q_2)t),$$

we see

$$\mathbf{p}^* = \left(1 - \frac{p_2-p_1}{q_1-q_2+p_2-p_1}\right)\mathbf{p} + \frac{p_2-p_1}{q_1-q_2+p_2-p_1}\,\mathbf{q}$$

and the value

$$t^* = \frac{p_2-p_1}{q_1-q_2+p_2-p_1}$$

of the parameter t on the segment is well-defined because $\max\{q_1-q_2, p_2-p_1\} > 0$. Thus, we note

$$\begin{aligned} |\mathbf{p}-\hat{\mathbf{q}}| &\le |\mathbf{p}-\mathbf{p}^*| + |\mathbf{p}^*-\hat{\mathbf{q}}| \\ &= |\mathbf{p}-\mathbf{p}^*| + |\mathbf{p}^*-\mathbf{q}| \\ &= |\mathbf{p}-\mathbf{q}|. \end{aligned}$$

Therefore, we can apply the previous case to the Lipschitz quotient for $\mathbf{p}$ and $\hat{\mathbf{q}}$ to see Then

$$\begin{aligned} \frac{|u(\mathbf{p})-u(\mathbf{q})|}{|\mathbf{p}-\mathbf{q}|} &\le \frac{|u(\mathbf{p})-u(\hat{\mathbf{q}})|}{|\mathbf{p}-\hat{\mathbf{q}}|} \\ &\le 5. \end{aligned}$$

We have established (4) in this second case as well. The remaining cases are symmetric with those considered above, so the estimate (4) holds in general, and we can say $u \in \text{Lip}(\overline{U})$.

It can be further verfied that u has a weak derivative given by

$$Du(x,\xi) = \begin{cases} ((1-\xi), -x), & 0 \le x \le \xi \\ (-\xi, (1-x)), & \xi < x \le 1. \end{cases}$$

We will not give the details here.

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An ODE Problem

4. This problem is about the (homogeneous and autonomous) second order linear ODE $y'' + y = 0$. You may recall that the **initial value problem**

$$\begin{cases} y'' + y = 0, \\ y(x_0) = y_0, \\ y'(x_0) = y_0' \end{cases} \tag{5}$$

has a **unique** solution. That solution, furthermore, is in $C^\infty(\mathbb{R})$.

(a) Write down/find the unique solution for (5).

(b) Show the solution you've written down is unique by completing the following steps:

1. Let u be another solution with $u \in C^2(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. That is, u satisfies (5). Let $w = y - u$, and find an initial value problem satisfied by w.
2. Let $z = w'$ and $m = w^2 + z^2$. Notice that $m \in C^1(x_0 - \delta, x_0 + \delta)$, and show that m satisfies
$$\begin{cases} m'(x) = 0 \text{ for } |x - x_0| < \delta, \\ m(x_0) = 0. \end{cases}$$
3. If there is some x with $|x - x_0| < \delta$ such that $m(x) > 0$, then prove there is some ξ between x and x_0 such that $m'(\xi) \neq 0$. Hint: Mean Value Theorem.
4. Conclude that $m(x) \equiv 0$ for x with $|x - x_0| < \delta$. In particular, $w(x) \equiv 0$ and $u(x) \equiv y(x)$ for all x with $|x - x_0| < \delta$.

(c) Show the **two point boundary value problem**

$$\begin{cases} y'' + y = 0, \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{6}$$

does **not** have a unique solution. Find infinitely many different solutions.

The point of this last part is that the uniqueness for a boundary value problem (even for ODEs) cannot be taken for granted. Even if you are able to write down a solution, you can not always be sure your solution is unique.

(d) What is the general situation for existence and uniqueness for the two point boundary value problem
$$\begin{cases} y'' + y = 0, \\ y(a) = 0, \ y(b) = c \end{cases}$$
where a, b, and c are real numbers with $a < b$?

Solution:

(a) $y = y_0 \cos(x - x_0) + y_0' \sin(x - x_0)$.

(b) Assume $u'' + u = 0$ with $u(x_0) = y_0$, $u'(x_0) = y_0'$ for some $u \in C^2(x_0 - \delta, x_0 + \delta)$.

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1. Set $w = y - u$. Then

$$\begin{cases} w'' + w = 0 \\ w(x_0) = 0 = w'(x_0). \end{cases}$$

2. Let $z = w'$ and $m = w^2 + z^2$. Then $m(x_0) = [w(x_0)]^2 + [w'(x_0)]^2 = 0$, and $m \in C^1(x_0 - \delta, x_0 + \delta)$ with

$$m' = 2ww' + 2w'w'' = 2(ww' - w'w) \equiv 0.$$

3. For any $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$, we have by the mean value theorem that there is some x^* between x and x_0 with

$$0 = |m'(x^*)| = \frac{|m(x) - m(x_0)|}{|x - x_0|}.$$

4. Since $m' \equiv 0$, we must have $m(x) \equiv m(x_0) = 0$. That is, $u \equiv y$.

Note: This approach differs slightly from the suggested steps in the problem, but it is essentially the same.

(c) $y(x) = \sin x$ is a solution. $u(x) = 3 \sin x$ is also a solution. In facts, $y(x) = c \sin x$ is a solution for any $c \in \mathbb{R}$.

(d) Starting with $y(x) = c_1 \cos(x - a) + c_2 \sin(x - a)$ as a general solution, the first boundary condition gives $c_1 = 0$. Thus, we have a one parameter family of possible solutions:

$$y(x) = c_2 \sin(x - a).$$

The question is thus reduced to the algebraic equation $y(b) = c_2 \sin(b - a) = c$ for c_2. There are three possibilities:

1. If $\sin(b - a) \neq 0$, then there is **a unique solution**

$$y(x) = \frac{c}{\sin(b - a)} \sin(x - a).$$

2. If $\sin(b - a) = 0$ and $c \neq 0$, there is **no solution**.
3. If $\sin(b - a) = 0$ and $c = 0$, then there are **infinitely many solutions**

$$y(x) = c_2 \sin(x - a).$$

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Laplace's Equation in One and Two Dimensions

Laplace's equation in two dimensions for a function $u = u(x, y)$ is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{7}$$

Laplace's equation in one dimension is

$$u'' = 0 \tag{8}$$

for $u = u(x)$. You might not think there is anything interesting to say about this ODE, and maybe you are right, but let's see.

5. It is natural to think about the solution of (8) as the solution of an **initial value problem** so that

$$u(x) = u_0'(x - x_0) + u_0 \qquad \text{where } u_0 = u(x_0).$$

(a) Show the following: The **two point boundary value problem** for (8),

$$\begin{cases} u'' = 0, \\ u(a) = u_a, \; u(b) = u_b, \end{cases} \tag{9}$$

has a **unique** solution for any real numbers a, b, u_a, and u_b with $a < b$.

(b) Did you write down a formula for the solution in part (a)? If not, write one/it down.

(c) What unique solution of (9) do you find when $u_a = 0$ and $u_b = 0$?

(d) Can you find a solution $u = u(x, y)$ of the boundary value problem

$$\begin{cases} \Delta u = 0, \\ u\big|_{\partial U} \equiv 0. \end{cases} \tag{10}$$

for (7) defined on an open set $U \subset \mathbb{R}^2$?

(e) Show the solution you found in part (d) is unique if the domain U is bounded by completing the following steps:

1. Assume there is a solution $v \in C^2(\overline{U})$ of (10) with $v(\mathbf{p}) > 0$ for some $\mathbf{p} \in U$. Let $w(\mathbf{x}) = v(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2$ where ϵ is a positive constant. Show that for $\epsilon > 0$ small enough, $w(\mathbf{x}) > 0$ for $\mathbf{x} \in \overline{U}$. Hint: There is some $M > 0$ for which $w \geq v(\mathbf{p}) - \epsilon M^2$.
2. Let $\mathbf{q} \in U$ be a point such that

$$c = v(\mathbf{q}) - w(\mathbf{q}) = \max_{\mathbf{x} \in U}[v(\mathbf{x}) - w(\mathbf{x})].$$

Why does such a point $\mathbf{q}$ exist and what is the sign of the constant c?

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3. Show that $\Delta v(\mathbf{q}) < 0$ (which is a contradiction). Hint: The function $\phi(\mathbf{x}) = w(\mathbf{x}) + c - v(\mathbf{x})$ vanishes at $\mathbf{x} = \mathbf{q}$. Combining this with the previous part tells you something about $\Delta\phi(\mathbf{q})$.

Solution:

(a) Integrating the equation from a to x, we **must** have

$$u'(x) \equiv u'(a) \qquad \text{(constant)}$$

and integrating again from a to x, we must have

$$u(x) = u(a) + u'(a)(x-a) = u_a + u'(a)(x-a).$$

The second endpoint condition gives

$$u(b) = u_a + u'(a)(b-a).$$

Since $a < b$, this means

$$u'(a) = \frac{u_b - u_a}{b-a} \qquad \text{and} \qquad u(x) = u_a + \frac{u_b - u_a}{b-a}(x-a). \tag{11}$$

This is the only possible solution of the two point boundary value problem. And it is easily checked that it is a solution.

(b) I already wrote it down in (11).

(c) If $u_a = u_b = 0$, then $u \equiv 0$.

(d) The function $u = u(x,y) \equiv 0$ is a solution for this PDE too (no matter what domain U is used).

(e) Let $M > 0$ be a bound for U. That is, $|\mathbf{x}| \le M$ for all $x \in U$. Then

$$\begin{aligned} w(\mathbf{x}) &= v(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2 \\ &\ge v(\mathbf{p}) - \epsilon(|\mathbf{x}| + |\mathbf{p}|)^2 \\ &\ge v(\mathbf{p}) - \epsilon(4M^2). \end{aligned}$$

Thus, if $\epsilon < v(\mathbf{p})/(8M^2)$, then $w(\mathbf{x}) > v(\mathbf{p})/2 > 0$ for all $\mathbf{x} \in U$, and $w(\mathbf{x}) \ge v(\mathbf{p})/2 > 0$ for all $x \in \overline{U}$.

(f) We are assuming $v \in C^2(\overline{U})$ with

$$v(\mathbf{p}) > 0 \qquad \text{and} \qquad v\big|_{\mathbf{x}\in\partial U} \equiv 0.$$

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This means in particular, $v - w \in C^0(\overline{U})$. A continuous function on a closed and bounded set has a maximum value. That is, there is some $\mathbf{q} \in \overline{U}$ for which

$$c = v(\mathbf{q}) - w(\mathbf{q}) = \max_{\overline{U}}[v(\mathbf{x}) - w(\mathbf{x})].$$

Since $\mathbf{p} \in U$ and we are assuming $v(\mathbf{p}) - w(\mathbf{p}) = v(\mathbf{p}) > 0$, we know $c > 0$. Also, since

$$v\Big|_{\mathbf{x}\in\partial U} \equiv 0 \qquad \text{and} \qquad w\Big|_{\mathbf{x}\in\partial U} > 0$$

we know

$$(v - w)\Big|_{\mathbf{x}\in\partial U} < 0.$$

Therefore, the maximum is not taken on ∂U. It must be a positive maximum $c > 0$ with $q \in U$.

(g) Consider $f(x, y) = w(x, y) + c - v(x, y)$. This is a non-negative function on U with an interior minimum $f(\mathbf{q}) = 0$. This means, in particular, that

$$Df(\mathbf{q}) = (0, 0).$$

Thus, the first order Taylor approximation of f at $\mathbf{q} = (q_1, q_2)$ gives

$$f(x, y) = f_{xx}(\mathbf{q}^*)(x - q_1)^2 + 2f_{xy}(\mathbf{q}^*)(x - q_1)(x - q_2) + f_{yy}(\mathbf{q}^*)(x - q_2)^2$$

where $\mathbf{q}^*$ is a point on the segment from $\mathbf{q}$ to $\mathbf{x}$. By taking $\mathbf{x} = (q_1 + h, q_2)$ we find

$$0 \le f_{xx}(\mathbf{q}^*)h^2.$$

That is, $f_{xx}(\mathbf{q}^*) \ge 0$ for some point $\mathbf{q}^* = (h^*, 0)$ with $0 < h^* < h$. Taking the limit as $h \searrow 0$, we conclude

$$f_{xx}(\mathbf{q}) \ge 0.$$

Similarly, taking $\mathbf{x} = (q_1, q_2 + k)$ for $k > 0$, we have

$$f_{yy}(q_1, q_2 + k^*) \ge 0$$

where $0 < k^* < k$. Taking $k \searrow 0$, we know

$$f_{yy}(\mathbf{q}) \ge 0$$

Therefore, $0 \le \Delta f = \Delta w - \Delta v$, and we have

$$\Delta v(\mathbf{q}) \le \Delta w(\mathbf{q}) = -4\epsilon < 0.$$

Here we have computed Δw using the formula $w(x, y) = v(\mathbf{p}) - \epsilon[(x_1 - p_1)^2 + (x_2 - p_2)^2]$. This is a contradiction of the assumption that v is a solution of $\Delta v = 0$, and establishes the uniqueness. (Technically, we have shown $v \le 0$. But $-v$ is also a solution, and the argument shows $-v \le 0$. That is, $v \ge 0$ as well.

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Laplace's Equation on a Rectangle

The next two problems give two different approaches to solving the boundary value problem for Laplace's equation (7) on a rectangle $U = (a, b) \times (c, d)$ in the plane. Both approaches can lead to viable solution methods, but we will only concentrate on some of the preliminaries.

6. In view of problem 2, you should not be surprised that it is enough to restrict to the special case $U = [0, L] \times [0, M]$ where L and M are positive numbers. This just makes some computations easier.

   (a) Verify that $u(x, y) = \sin(2\pi x/L) \sinh(2\pi y/L)$ solves

   $$\begin{cases} \Delta u = 0, \\ u(x,0) = 0,\ u(L,y) = 0,\ u(x,L) = g(x),\ u(0,y) = 0 \end{cases} \tag{12}$$

   if $g = g(x)$ happens to be a suitable function.

   (b) Solve

   $$\begin{cases} \Delta v = 0, \\ v(x,c) = 0,\ v(b,y) = 0,\ v(x,d) = 0,\ v(a,y) = \sinh[2\pi(b-a)/(d-c)] \sin[2\pi(y-c)/(d-c)] \end{cases}$$

   for $a < b$ and $c < d$. (Note: This problem had some correction/typo in the original formulation.)

   (c) Returning to (12) substitute a "separated variables" solution $u(x, y) = A(x)B(y)$ into Laplace's equation $\Delta u = 0$. Rearrange what you get into a form

   $$\Phi(x) = \Psi(y) \tag{13}$$

   for some functions Φ (involving only A and depending only on x) and Ψ (involving only B and depending only on y).

   (d) Take the partial derivative with respect to x of both sides of (13). What does this tell you about the function $\Phi = \Phi(x)$?

   (e) Find "ordinary differential equations" which must be satisfied by A and B if $u(x, y) = A(x)B(y)$ is a solution of $\Delta u = 0$. (I put ODEs in quotes here because you cannot determine the equation completely, but you should be able to determine the form of the equation, e.g., second order linear, homogeneous, etc., but there may be parameters which you cannot determine (yet).)

   (f) If you substitute the separated variables solution $u(x, y) = A(x)B(y)$ into the boundary values specified in (12), what can you conclude about the boundary values of A on $[a, b]$ and B on $[c, d]$ under the assumption that $u = AB$ is not identically zero?

   (g) Find all possible separated variables solutions $u(x, y) = A(x)B(y)$.

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Solution:

(a) Since a negative sign comes out when we differentiate sin twice and no negative sign come out when we differentiate sinh (and the scalars $2\pi/L$ are the same), it is clear u is a solution of the PDE. Also, the boundary conditions are clearly satisfied if

$$g(x) = \sinh(2\pi)\sin(2\pi x/L).$$

(b) Notice that in part (a) we had a top horizontal segment with the non-homogeneous boundary condition. Here we have a left vertical side. This suggests simply translating and switching the sin and sinh. Let's see if that works:

$$u(x, y) = \sinh[2\pi(x - a)/(d - c)]\sin[2\pi(y - c)/(d - c)].$$

Of course, we've got to have the same scaling factors to get a solution of the PDE, and the boundary condition suggests using $2\pi/(d - c)$. Yes. This will certainly solve the PDE. Also, when we put in $y = c$ or $y = d$ we get zero. When we put in $x = b$, we have a problem but we're getting $u(a, y) = 0$ as it is. I see the problem: In part (a) we had the horizontal side **off the axis** with the non-homogeneous condition: $y = L$. Here, we have the vertical side **on the axis**, so we've done the wrong translation in the sinh factor. Let's try instead

$$u(x, y) = \sinh[2\pi(x - b)/(d - c)]\sin[2\pi(y - c)/(d - c)].$$

Now everything works, but we're off by a sign. This is no problem to fix:

$$u(x, y) = \sinh[2\pi(b - x)/(d - c)]\sin[2\pi(y - c)/(d - c)].$$

Notice, when we differentiate with respect to x, the negative sign from the chain rule from $[2\pi(b - x)/(d - c)]$ comes out twice (because we differentiate twice). So the PDE is still satisfied. And we've got all the boundary conditions too.

(c) With substitution we get $A''B + AB'' = 0$ which we write as

$$\Phi(x) = -\frac{A''}{A} = \frac{B''}{B} = \Psi(y).$$

This is okay, of course, as long as $AB \neq 0$, and it's okay at any points (x, y) where $A(x)B(y) \neq 0$.

(d) We get

$$\frac{d}{dx}\left(-\frac{A''}{A}\right) = 0$$

which tells me

$$-\frac{A''}{A} = \lambda \qquad \text{(constant)}$$

Similarly, $B'' = \lambda B$ with the same constant λ.

(e) Hence $A'' = -\lambda A$ and $B'' = \lambda B$.

(f) $B(0) = 0$, $A(L) = 0$, $B(L) = g(x)/A(x)$, $A(0) = 0$. The condition $g(x)/A(x) = B(L)$ is the odd man out here. The function $g = g(x)$ is given, and presumably, it either helps us find a solution or not, but it's not clear that it should. In particular, we know we need $A(0) = A(L) = 0$, and if we take A as a multiple of g as this suggests, then either $g(0) = g(L) = 0$ or it does not. And we don't really know. So, let's set that aside for a moment. The other conditions look promising:

$$\begin{cases} A'' = -\lambda A \\ A(0) = 0 = A(L). \end{cases}$$

This looks like a two point boundary value problem (except for the unknown constant λ. We need cases:

1. If $\lambda < 0$, then $A = \alpha \cosh \mu x + \beta \sinh \mu x$ where $\mu = \sqrt{-\lambda}$. The condition $A(0) = 0$ implies $\alpha = 0$, and the condition $A(L) = 0$ then implies $\beta = 0$ unless $\mu = 0$ (which amounts to the same thing). There are no interesting solutions in this case.
2. If $\lambda = 0$, then $A = \alpha x + \beta$. Again the boundary conditions imply $\alpha = \beta = 0$ in short order. Nothing to see here.
3. If $\lambda > 0$, then we get $A = \alpha \cos \mu x + \beta \sin \mu x$ where $\mu = \sqrt{\lambda}$. The condition $A(0) = 0$ gives $\alpha = 0$ again, but then we get

$$\beta \sin(\mu L) = 0.$$

This gives nontrivial solutions when $\mu_k = k\pi/L$ for $k = 1, 2, 3, \ldots$. That is, $\lambda = k^2\pi^2/L^2$ and (more importantly)

$$A_k(x) = \beta_k \sin(k\pi x/L).$$

The B ode then gives $B_k(y) = c_1 \cosh(k\pi x/L) + c_2 \sinh(k\pi x/L)$. The one good boundary condition we got, namely $B(0) = 0$, implies $c_1 = 0$, so we get "solutions" that look like

$$u_k(x, y) = \sin(k\pi x/L) \sinh(k\pi y/L).$$

These are the "separated variables solutions." Of course, they don't solve the whole problem immediately, but they satisfy everything except the non-homogeneous boundary condition $u(x, L) = g(x)$. Two final notes:

(a) Either the separated variables solutions satisfy $u(x, L) = g(x)$ or not. That depends on g. See parts (a) and (b) above.

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(b) Any constant multiple of these separated variables solutions satisfies all the conditions except (maybe) $u(x, L) = g(x)$, and any linear combination of them

$$\sum a_k \sin(k\pi x/L) \sinh(k\pi y/L)$$

does as well. This means we can solve this problem whenver $g(x)$ is a linear combination of the functions $\sin(k\pi x/L)$. That turns out to be pretty good.

Poisson's Equation

Here we consider a more general version of (12)

$$\begin{cases} \Delta u = 0, \\ u(x,0) = g_1(x),\ u(L,y) = g_2(y),\ u(x,M) = g_3(x),\ u(0,y) = g_4(y). \end{cases} \tag{14}$$

7. Let us assume that $g_1, g_2, g_3, g_4 \in C^\infty(\mathbb{R})$ with

$$\begin{aligned} g_1(0) &= g_4(0), \\ g_1(L) &= g_2(0), \\ g_2(M) &= g_3(L), \text{ and} \\ g_3(0) &= g_4(M). \end{aligned}$$

It turns out that under these (boundary) conditions there are functions $g \in C^\infty(\overline{U})$ for which

$$g(x,0) = g_1(x),\ g(L,y) = g_2(y),\ g(x,M) = g_3(x),\ g(0,y) = g_4(y). \tag{15}$$

In fact, when we can solve (14) then the solution u will be such a function. For the moment, it's not even so obvious that such a smooth function g (of any sort) exists. But we will assume such a function exists for now.

(a) Rewrite the boundary conditions of (14) in terms of g satisfying (15) using only **one equation**. Hint: Look at the boundary condition in (10) from Problem 5.

(b) Let $w = u - g$ and, assuming u satisfies (14), find the boundary value problem satisfied by w. (The PDE you should find is a form of what is called **Poisson's equation**.

The point of this last part is that the boundary value problem (14) for Laplace's equation can be replaced by a boundary value problem for Poisson's equation with **homogeneous boundary values**.

(c) Let $U = (0, \ln 2) \times (0, \pi)$. Assume you can solve

$$\begin{cases} \Delta w = (1 + x/\ln 2) \sin y, \\ w\big|_{\partial U} \equiv 0. \end{cases} \tag{16}$$

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Use the solution of this problem to solve

$$\begin{cases} \Delta u = 0, \\ u\big|_{\partial U} \equiv (1 + x/\ln 2)\sin y. \end{cases} \tag{17}$$

(d) Use Problem 5 part (e) to show the boundary value problem (16) can have at most one solution $w \in C^2(U) \cap C^0(\overline{U})$.

Solution:

(a)

$$u\big|_{\partial U} = g \qquad \text{or} \qquad u\big|_{\partial U} = g\big|_{\partial U}.$$

(b) $\Delta w = \Delta u - \Delta g = -\Delta g$.

$$\begin{cases} \Delta w = -\Delta g \text{ on } U \\ w\big|_{\partial U} \equiv 0. \end{cases}$$

(c) Let $u(x,y) = w(x,y) + (1 + x/\ln 2)\sin y$. Then

$$\Delta u = \Delta w - (1 + x/\ln 2)\sin y = 0.$$

And

$$u\big|_{\partial U} = (1 + x/\ln 2)\sin y.$$

Thus, u solves (17).

(d) Assume w and v are both solutions of problem (16). Set $\phi = w - v$. Then $\Delta\phi = 0$ and $\phi\big|_{\partial U} \equiv 0$. Therefore,

$$\begin{cases} \Delta \phi = 0 \text{ on } U \\ \phi\big|_{\partial U} \equiv 0. \end{cases}$$

Since $U = (0, \ln 2) \times (0, \pi)$ is a bounded open subset of $\mathbb{R}^2$, our result from Problem 5 applies, and we know $\phi \equiv 0$ is the unique solution. Thus, $w \equiv v$ and (16) has at most one (classical) solution.

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Fourier Series

Let's think a little bit about Fourier series. Let $f : [0, L] \to \mathbb{R}$ be a continuous function. A **Fourier sine series** for f has the form

$$f(x) = \sum_{j=1}^{\infty} f_j \sin \frac{j\pi x}{L}. \tag{18}$$

where the **numbers** $f_1, f_2, f_3, \ldots$ are called the **Fourier coefficients** of f and the functions $\sin(\pi x/L)$, $\sin(2\pi x/L)$, $\sin(3\pi x/L)$,... make up what is called the **Fourier sine basis**. We do not need to worry too much about convergence of the series. For a continuous function, if the coefficients are chosen correctly, then the series will converge to $f(x)$ at least on $(0, L)$, and we can manipulate the series, at least as far as integrating term by term, pretty freely.

8. This problem is about computing Fourier coefficients.

   (a) Compute

   $$\int_0^L \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx.$$

   Hint: Something special happens when $j = k$ because (you can show)

   $$\int_0^L \sin^2 \frac{j\pi x}{L}\, dx = \int_0^L \cos^2 \frac{j\pi x}{L}\, dx \qquad \text{and} \qquad \sin^2 \frac{j\pi x}{L} + \cos^2 \frac{j\pi x}{L} = 1.$$

   Something even more special happens when $j \neq k$.

   (b) Multiply both sides of (18) by $\sin(k\pi x/L)$ and integrate both sides from $x = 0$ to $x = L$. Use the result to find a formula for the Fourier coefficients.

   (c) Consider the specific example $f : [0, L] \to \mathbb{R}$ by

   $$f(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(x - L)/(a - L), & a \le x \le L. \end{cases}$$

   Draw the graph $\{(x, f(x)) : 0 \le x \le L\}$ of f. What can you say about the regularity of f?

   (d) Let f be the specific function from the last part of this problem. Find the Fourier sine series expansion of f.

   (e) The trigonometric polynomial

   $$P_n(x) = \sum_{j=1}^{n} f_j \sin \frac{j\pi x}{L}$$

   is (called) the n**-th Fourier sine approximation of** f. Use mathematical software (Matlab, Mathematica, Maple, etc.) to plot $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_{10}(x)$, and $P_{100}(x)$ for the specific example from the last two parts.

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Solution:

(a) If $j \neq k$, then express the trigonometric functions in terms of exponentials:

$$\begin{aligned}
\int_0^L \sin\frac{j\pi x}{L}\sin\frac{k\pi x}{L}\,dx &= \frac{1}{(2i)^2}\int_0^L (e^{ij\pi x/L} - e^{-ij\pi x/L})(e^{ik\pi x/L} - e^{-ik\pi x/L})\,dx \\
&= -\frac{1}{4}\int_0^L (e^{i(j+k)\pi x/L} - e^{i(j-k)\pi x/L} - e^{i(k-j)\pi x/L} + e^{-i(j+k)\pi x/L})\,dx \\
&= -\frac{1}{4}\Bigg[\frac{L}{i(j+k)\pi}e^{i(j+k)\pi x/L} - \frac{L}{i(j-k)\pi}e^{i(j-k)\pi x/L} \\
&\qquad - \frac{L}{i(k-j)\pi}e^{i(k-j)\pi x/L} - \frac{L}{i(j+k)\pi}e^{-i(j+k)\pi x/L})\Bigg]\Bigg|_0^L \\
&= -\frac{L}{2\pi}\left[\frac{1}{j+k}\sin[(j+k)\pi x/L] + \frac{1}{k-j}\sin[(j-k)\pi x/L]\right]\Bigg|_0^L \\
&= 0.
\end{aligned}$$

If $j = k$, then

$$\begin{aligned}
\int_0^L \sin\frac{j\pi x}{L}\sin\frac{k\pi x}{L}\,dx &= \int_0^L \sin^2\frac{j\pi x}{L}\,dx \\
&= \frac{1}{2}\left(\int_0^L \sin^2\frac{j\pi x}{L}\,dx \int_0^L \cos^2\frac{j\pi x}{L}\,dx\right) \\
&= \frac{1}{2}\int_0^L 1\,dx \\
&= L/2.
\end{aligned}$$

To see that

$$\int_0^L \sin^2\frac{j\pi x}{L}\,dx = \int_0^L \cos^2\frac{j\pi x}{L}\,dx,$$

Note that $\sin^2$ is π periodic, so that

$$\begin{aligned}
\int_0^L \sin^2\frac{j\pi x}{L}\,dx &= \int_{L/(2j)}^{L+L/(2j)} \sin^2\frac{j\pi x}{L}\,dx \\
&= \int_0^L \sin^2\left(\frac{j\pi(\xi + L/(2j))}{L}\right)\,d\xi \\
&= \int_0^L \sin^2\left(\frac{j\pi\xi}{L} + \frac{\pi}{2}\right)\,d\xi \\
&= \int_0^L \cos^2\frac{j\pi\xi}{L}\,d\xi.
\end{aligned}$$

We have used the change of variables $\xi = x - L/(2j)$.

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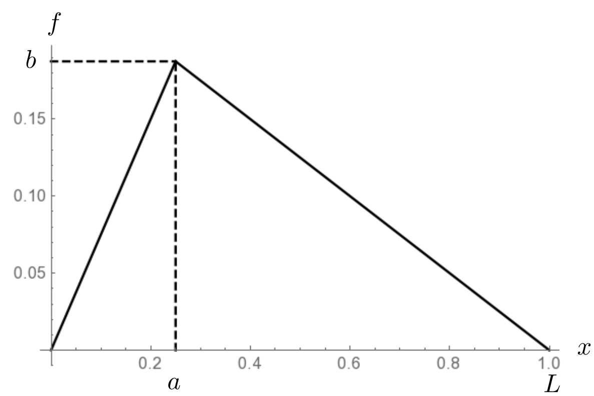
(b)

$$\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx = \int_0^L \sum_{j=1}^{\infty} f_j \sin\frac{k\pi x}{L}\sin\frac{j\pi x}{L}\,dx$$
$$= \sum_{j=1}^{\infty} f_j \int_0^L \sin\frac{k\pi x}{L}\sin\frac{j\pi x}{L}\,dx$$
$$= \frac{L}{2} f_k.$$

Note that we have integrated the series termwise. Thus,

$$f_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx.$$

(c) Here is a plot of a tent function.



(d) According to the formula above

$$f_j = \frac{2}{L}\int_0^L f(x)\sin\frac{j\pi x}{L}\,dx$$
$$= \frac{2}{L}\left[\frac{b}{a}\int_0^a x\sin\frac{j\pi x}{L}\,dx + \frac{b}{a-L}\int_a^L (x-L)\sin\frac{j\pi x}{L}\,dx\right].$$

We need to integrate by parts. I'll calculate for each separate integral:

$$\int_0^a x\sin\frac{j\pi x}{L}\,dx = -\frac{L}{j\pi}x\cos\frac{j\pi x}{L}\Big|_{x=0}^{a} + \frac{L}{j\pi}\int_0^a \cos\frac{j\pi x}{L}\,dx$$
$$= -\frac{L}{j\pi}a\cos\frac{j\pi a}{L} + \frac{L^2}{(j\pi)^2}\sin\frac{j\pi x}{L}\Big|_{x=0}^{a}$$
$$= -\frac{L}{j\pi}a\cos\frac{j\pi a}{L} + \frac{L^2}{(j\pi)^2}\sin\frac{j\pi a}{L}.$$

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$$\int_a^L (x-L)\sin\frac{j\pi x}{L}\,dx = -\frac{L}{j\pi}(x-L)\cos\frac{j\pi x}{L}\Big|_{x=a}^{L} + \frac{L}{j\pi}\int_a^L \cos\frac{j\pi x}{L}\,dx$$
$$= \frac{L}{j\pi}(a-L)\cos\frac{j\pi a}{L} + \frac{L^2}{(j\pi)^2}\sin\frac{j\pi x}{L}\Big|_{x=a}^{L}$$
$$= \frac{L}{j\pi}(a-L)\cos\frac{j\pi a}{L} - \frac{L^2}{(j\pi)^2}\sin\frac{j\pi a}{L}.$$

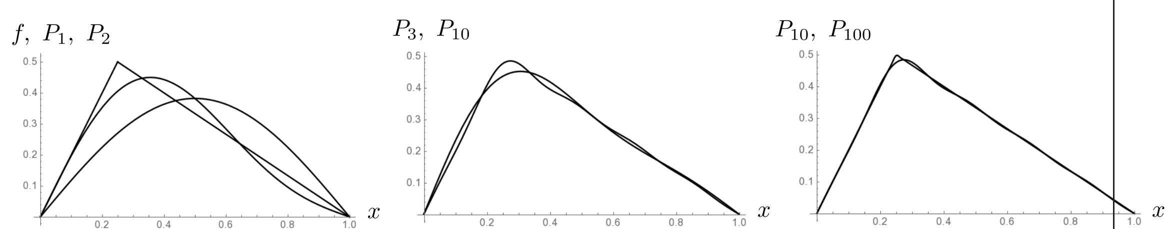
Substituting these values for the integrals in the calcuation we started before, we find

$$f_j = \frac{2b}{L}\left(\frac{1}{a} - \frac{1}{a-L}\right)\frac{L^2}{(j\pi)^2}\sin\frac{j\pi a}{L}$$
$$= -\frac{2b}{a(a-L)}\frac{L^2}{(j\pi)^2}\sin\frac{j\pi a}{L}.$$

Thus, the Fourier sine series expansion for this function is

$$f(x) = \frac{2bL^2}{a(L-a)}\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}\sin\frac{j\pi a}{L}\sin\frac{j\pi x}{L}. \tag{19}$$

(e) Here are the plots of some Fourier approximations for f with the values $a = 1/4$, $b = 1/2$ and $L = 1$:



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One Dimensional Poisson's Equation

The one dimensional Poisson's equation is $u'' = f(x)$, and again, we are back to a relatively easy ODE.

9. It is possible to use the method of Homework Assignment 1 (problems 1 and 2) to solve

$$\begin{cases} u'' = f(x), \\ u(0) = 0, \ u(L) = 0, \end{cases} \tag{20}$$

and you should write that solution down, but we are going to solve this boundary value problem a different way.

(a) Expand f as a Fourier sine series, so that the equation becomes

$$u'' = \sum_{j=1}^{\infty} f_j \sin \frac{j\pi x}{L}.$$

Solve infinitely many 1-D Poisson equations $u_j'' = f_j \sin(j\pi x/L)$ with homogeneous boundary conditions, so that the solution u can be written as a series

$$u(x) = \sum_{j=1}^{\infty} u_j.$$

(b) Substitute the Fourier coefficients f_j so that u takes the form

$$u(x) = \int_0^L G(x, \xi)\, f(\xi)\, d\xi \tag{21}$$

where $G = G(x, \xi)$ is a function of two variables defined as a series on $[0, L] \times [0, L]$. The function G is called the **Green's function** for the problem (20).

(c) Determine G explicitly and classify the regularity of G as a function of two variables. Hint(s): You can either restrict to horizontal lines $\xi =$ constant or vertical lines $x =$ constant, and use the previous problem (Problem 8) on Fourier series or, alternatively, you can use the solution obtained by the method of Homework Assignment 1 (problems 1 and 2) and put that solution (involving integrals) into the form (21) to find G directly by explicit integration. It's not a bad idea to do both.

Solution:

(a) As we calculated in the previous problem, we know

$$f_j = \frac{2}{L} \int_0^L f(x) \sin \frac{j\pi x}{L}\, dx. \tag{22}$$

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If we write

$$u'' = \sum u_j'' = \sum f_j \sin \frac{j\pi x}{L},$$

then

$$u_j(x) = -f_j \, \frac{L^2}{(j\pi)^2} \, \sin \frac{j\pi x}{L}.$$

Thus,

$$u(x) = -\sum_{j=1}^{\infty} f_j \, \frac{L^2}{(j\pi)^2} \, \sin \frac{j\pi x}{L}$$

should be a solution.

(b) Now, if we substutite the value for f_j from (22), this becomes

$$\begin{aligned} u(x) &= -\sum_{j=1}^{\infty} \frac{2}{L} \int_0^L f(\xi) \sin \frac{j\pi \xi}{L} \, d\xi \, \frac{L^2}{(j\pi)^2} \, \sin \frac{j\pi x}{L} \\ &= -\frac{2}{L} \int_0^L f(\xi) \left(\sum_{j=1}^{\infty} \frac{L^2}{(j\pi)^2} \, \sin \frac{j\pi \xi}{L} \, \sin \frac{j\pi x}{L} \right) d\xi. \end{aligned}$$

According to the form suggested in the problem, we can identify

$$G(x, \xi) = -\frac{2}{L} \sum_{j=1}^{\infty} \frac{L^2}{(j\pi)^2} \, \sin \frac{j\pi \xi}{L} \, \sin \frac{j\pi x}{L}.$$

(c) For a first approach, let us compare to (19):

$$\frac{2bL^2}{a(L-a)} \sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} \sin \frac{j\pi a}{L} \sin \frac{j\pi x}{L}.$$

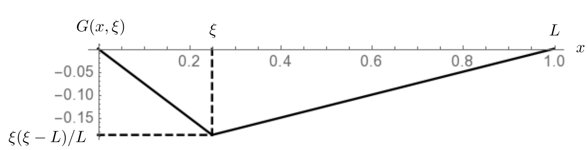
This suggests that we take $a = \xi$. Then these will match identically if

$$-\frac{1}{L} = \frac{b}{a(L-a)} = \frac{b}{\xi(L-\xi)}.$$

That is,

$$b = -\frac{\xi(L-\xi)}{L}.$$

If this is the case, then we see that each restriction of $G(x, \xi)$ to a line $\xi =$ constant is a piecewise affine function as indicated in the illustration below.



Thus, we can use the formula for the tent function in problem 8 with these values of a and b to see

$$G(x,\xi) = \begin{cases} -(L-\xi)x/L, & 0 \le x \le \xi \\ (x-L)\xi/L, & \xi \le x \le L. \end{cases} \tag{23}$$

This finishes this problem and the exam. (Whew!)

But there is a second approach to this problem. Let's see if we can pull that off:

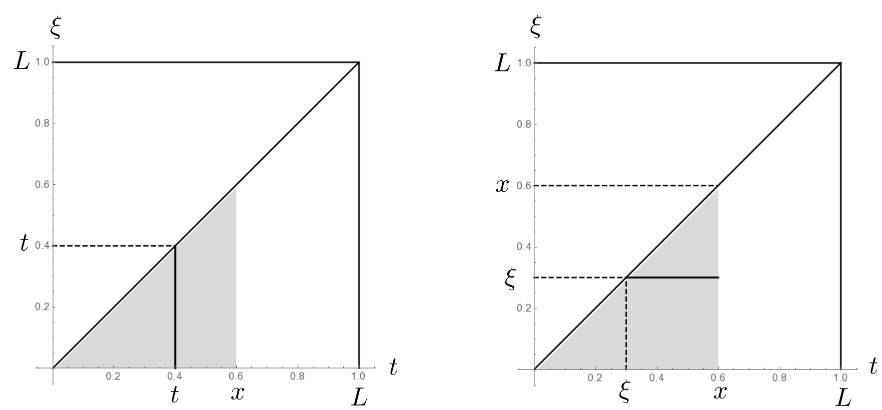
We integrate the ODE $u'' = f$ directly starting from $x = 0$ to get

$$u'(x) = u'(0) + \int_0^x f(\xi)\, d\xi.$$

Integrating a second time we have

$$u(x) = u'(0)x + \int_0^x \int_0^t f(\xi)\, d\xi\, dt.$$

Now we want to switch the order of integration. This is not a change of variable nor integration by parts or any other manipulation, but simply a **change of the order of integration** according to **Fubini's theorem**, which says that if we have **iterated integrals** or **an iterated intgral** as we have here, then (under certain circumstances) the order of integration may be switched with the appropriate changes of limits of integration. To see how this works, note that the limits of integration determine a certain triangular region in the t, ξ plane:



On the left note the region $U_x = \{(t,\xi) : 0 < t < x,\ 0 < \xi < t\}$ described according to the limits in the integration: First one fixes t with $0 < t < x$. Then one considers the vertical segment $0 < \xi < t$. The union of these segments gives the region U_x. We say these segments **foliate or slice the region vertically**. Thus, these iterated integrals are built around slicing the domain vertically. Fubini's theorem tells us two things in this situation. First, the theorem is built around the idea that these iterated integrals represent a **single**

area integration over the region U_x. Thus, according to Fubini's theorem we can (and should) write

$$\int_0^x \int_0^t f(\xi)\, d\xi\, dt = \int_{(t,\xi)\in U_x} f(\xi).$$

Thus, the region U_x is a domain of integration in its own right where we consider the integrand $f(\xi)$ as a **function of two variables** $\phi(t,\xi) = f(\xi)$ which just happens to only depend on one of them.

Second, Fubini's theorem says we can slice horizontally instead of vertically. This approach to folating the domain is illustrated on the right. We fix ξ between 0 and x, and then the horizontal slice has limits $\xi < t < x$. Thus, the region U_x may be also written as

$$U_x = \{(t,\xi) : 0 < \xi < x,\ \xi < t < x\}.$$

Note that these changes of limits were exactly what was discussed in Problem 3 Part (d). In the end, Fubini's theorem tell's us we can write the (area) integral as iterated integrals with the order of integration for the iterated integrals in either order. The integrand remains unchanged:

$$u(x) = u'(0)x + \int_0^x \int_\xi^x f(\xi)\, dt\, d\xi.$$

Now, we may proceed to simplify this new iterated integral:

$$u(x) = u'(0)x + \int_0^x f(\xi) \int_\xi^x 1\, dt\, d\xi = u'(0)x + \int_0^x f(\xi)(x-\xi)\, d\xi.$$

There are essentially two more steps. One is to use the second boundary condition $u(L) = 0$ to conclude

$$u'(0) = -\frac{1}{L}\int_0^L f(\xi)(L-\xi)\, d\xi.$$

Thus, we have

$$u(x) = \int_0^x f(\xi)(x-\xi)\, d\xi - \frac{x}{L}\int_0^L f(\xi)(L-\xi)\, d\xi.$$

And finally, we need to combine these integrals so we have the form required for the Green's function involving a single integral from $\xi = 0$ to $\xi = L$. We accomplish this by using a characteristic function

$$\chi_{[0,x)}(\xi) = \begin{cases} 1, & \xi \in [0,x) \\ 0, & \xi \notin [0,x). \end{cases}$$

in the first integral. Thus,

$$\begin{aligned} u(x) &= \int_0^L f(\xi)(x-\xi)\chi_{(0,x)}\,d\xi - \int_0^L f(\xi)\frac{L-\xi}{L}x\,d\xi. \\ &= \int_0^L f(\xi)\left((x-\xi)\chi_{(0,x)} - \frac{L-\xi}{L}x\right)\,d\xi. \end{aligned}$$

Notice the values of the characteristic function at the endpoints do not effect the value fo the integral. Now, hopefully the function contained in the large parentheses is our Green's function:

$$\begin{aligned} (x-\xi)\chi_{(0,x)} - \frac{L-\xi}{L}x &= \begin{cases} x-\xi-(L-\xi)x/L, & \xi\in[0,x) \\ (\xi-L)x/L, & \xi\notin[0,x). \end{cases} \\ &= \begin{cases} (x-L)\xi/L, & 0\le\xi\le x \\ (\xi-L)x/L, & x\le\xi\le L. \end{cases} \end{aligned}$$

Switching the way we slice the domain once again, we see this agrees precisely with (23).

This direct integration approach can also be tackled using integration by parts as follows: Say we start with

$$u(x) = u'(0)x + \int_0^x \int_\xi^x f(\xi)\,dt\,d\xi. \tag{24}$$

Setting

$$\psi(t) = \int_0^t f(\xi)\,d\xi \qquad \text{and} \qquad d\phi = dt$$

we have

$$\psi'(t) = f(t) \qquad \text{and} \qquad \phi(t) = t.$$

Thus, integrating by parts, we get

$$\begin{aligned} \int_0^x \int_\xi^x f(\xi)\,dt\,d\xi &= \phi\psi\Big|_{t=0}^x - \int_0^x \phi\psi'\,dt \\ &= \left[t\int_0^t f(\xi)\,d\xi\right]\Big|_{t=0}^x - \int_0^x tf(t)\,dt \\ &= x\int_0^x f(\xi)\,d\xi - \int_0^x \xi f(\xi)\,d\xi \\ &= \int_0^x (x-\xi)f(\xi)\,d\xi. \end{aligned}$$

The rest goes pretty much as before, but let me see if I can clean that up a bit on the second pass:

$$u'(0) = \frac{1}{L}\int_0^L (\xi-L)f(\xi)\,d\xi.$$

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Therefore, substiting what we have found into (24), we get

$$\begin{aligned} u(x) &= \frac{x}{L}\int_0^L (\xi - L) f(\xi)\, d\xi + \int_0^x (x-\xi) f(\xi)\, d\xi \\ &= \int_0^L \left(\frac{\xi - L}{L} x + \chi_{[0,x]}(\xi)(x-\xi) \right) f(\xi)\, d\xi. \end{aligned}$$

Then, as above

$$\frac{\xi - L}{L} x + \chi_{[0,x]}(\xi)(x-\xi) = \begin{cases} \xi(x-L)/L, & 0 \le \xi \le x \\ (\xi - L)x/L, & x < \xi \le L, \end{cases}$$

which is easily seen to be the same as

$$\frac{\xi - L}{L} x + \chi_{[0,x]}(\xi)(x-\xi) = \begin{cases} (\xi - L)x/L, & 0 \le x \le \xi \\ \xi(x-L)/L, & \xi \le x \le L. \end{cases}$$