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An operator $L : V \to W$ is **linear** from a vector space V of functions to another vector space W of functions if

$$L[af + bg] = aL[f] + bL[g] \qquad \text{for every } a, b \in \mathbb{R} \text{ and } f, g \in V.$$

1. (linear partial differential operators) Show the following partial differential operators are linear. State clearly a natural vector space of functions for the domain V and codomain W of each operator.

   (a) (The Lewy Operator)

   $$L\begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} u_x - v_y + 2yu_z + 2xv_z \\ v_x + u_y + 2yv_z - 2xu_z \end{pmatrix}.$$

   Here we are using subscript notation for (partial) derivatives.

   (b) (anisotropic Laplacian)

   $$A[u] = \sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j} u.$$

   Here, we are using multi-index notation for derivatives and $\mathbf{e}_j$ is the j-th standard unit basis vector. Note: We should also require $a_j : U \to \mathbb{R}$ for $j = 1, \ldots n$ are given **positive** functions on some domain $U \subset \mathbb{R}^n$. You may further restrict the coefficients $a_j = a_j(x)$ in order to determine/specify the codomain W of this operator. The isotropic spacial case $a_1 = a_2 = \cdots = a_n \equiv 1$ of this operator gives the Laplacian

   $$\Delta u = \nabla^2 u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

   (c) (heat operator)

   $$H[u] = u_t - k\Delta u.$$

   Here the positive constant $k = \alpha^2$ is called the **diffusivity**, and the operator is also called the diffusion operator.

   (d) (wave operator)

   $$\Box u = u_{tt} - k\Delta u.$$

   Here the positive constant $k = c^2$ is called the **square of the propogation speed**. The wave operator is also sometimes called the D'Alembertian after Jean D'Alembert.

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2. Consider the heat equation $u_t = k\Delta u + f$ (with forcing) on

$$B_1(0) \times [0,\infty) = \{(x,y,t) : x^2 + y^2 < 1 \text{ and } t \geq 0\}.$$

(a) Let $w(\xi,\eta,\tau) = u(\alpha\xi, \alpha\eta, \beta\tau)$ where α and β are positive constants. Determine the domain of w, and compute

$$w_\tau(\xi,\eta,\tau) \qquad \text{and} \qquad \Delta w(\xi,\eta,\tau) = w_{\xi\xi} + w_{\eta\eta}.$$

(b) (scaling in time) Say you know how to solve $w_\tau - \Delta w = f_0(\xi,\eta,\tau)$ for any $f_0 \in C^0(B_1(0) \times [0,\infty))$ with a particular initial condition

$$w(\xi,\eta,0) = g_0(\xi,\eta)$$

and a homogeneous boundary condition

$$w\Big|_{\xi^2+\eta^2=1} = 0 \qquad \text{for all time } \tau \geq 0.$$

Explain how to solve

$$\begin{cases} u_t - k\Delta u = f & \text{on } B_1(0) \times [0,\infty) \\ u(x,y,0) = g_0(x,y), & (x,y) \in B_1(0) \\ u\Big|_{x^2+y^2=1} = 0, & \text{for all time } t \geq 0 \end{cases}$$

for $k \neq 1$ by scaling in time. Hint: Use the idea of part (a).

(c) (scaling in space) If $w_t = \Delta w$ on $W = B_5(0) \times [0,\infty)$ find the PDE satisfied by $u(x,y,t) = w(5x,5y,t)$ on $B_1(0) \times [0,\infty)$.

(d) (anisotropic heat diffusion) Find an appropriate domain on which to solve the heat equation $w_t = \Delta w$ which allows you to solve the anisotropic heat equation

$$u_t = 3u_{xx} + 2u_{yy} \qquad \text{on } B_1(0) \times [0,\infty).$$

Which initial and boundary conditions can you handle? Hint: Scale in space by different factors in different directions, i.e., anisotropically.

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A function $u : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^n$ and $\mathbf{p} \in U$ is **differentiable** at $\mathbf{p}$ if there is a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{w}\to\mathbf{0}} \frac{u(\mathbf{p}+\mathbf{w}) - u(\mathbf{p}) - L(\mathbf{w})}{|\mathbf{w}|} = 0. \tag{1}$$

The linear map $L : \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of u at $\mathbf{p}$ and is denoted by $du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$.

3. Let $u : U \to \mathbb{R}$ be differentiable at $\mathbf{p} \in U$.
   (a) show the first partial derivatives $D_j u(\mathbf{p})$ exist for $j = 1, 2, \ldots, n$.
   (b) Express the linear function $L : \mathbb{R}^n \to \mathbb{R}$ for which (1) holds in terms of the **gradient vector**
   $$Du(\mathbf{p}) = (D_1 u(\mathbf{p}), D_2 u(\mathbf{p}), \ldots, D_n u(\mathbf{p})).$$
   (c) Let $U = (0,1) \times (0,1) \subset \mathbb{R}^2$ and consider the specific function $u : U \to \mathbb{R}$ by
   $$u(x,\xi) = \begin{cases} x(1-\xi), & 0 \le x \le \xi \\ (1-x)\xi, & \xi \le x \le 1. \end{cases}$$
   Determine the points in U at which u is differentiable.
   (d) Let u be the specific function given in the last part of this problem. Reexpress u in the form
   $$u(x,\xi) = \begin{cases} u_1(x,\xi), & 0 \le \xi \le x \\ u_2(x,\xi), & x \le \xi \le 1. \end{cases}$$
   (e) What can you say about the regularity of the specific function u from the previous two parts? Hint: You can start by showing $u \in C^0(\overline{U})$. You can also find some subdomains U_1 and U_2 of U on which the functions u_1 and u_2 are C^∞.

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An ODE Problem

4. This problem is about the (homogeneous and autonomous) second order linear ODE $y'' + y = 0$. You may recall that the **initial value problem**

$$\begin{cases} y'' + y = 0, \\ y(x_0) = y_0, \\ y'(x_0) = y_0' \end{cases} \tag{2}$$

has a **unique** solution. That solution, furthermore, is in $C^\infty(\mathbb{R})$.

(a) Write down/find the unique solution for (2).

(b) Show the solution you've written down is unique by completing the following steps:

1. Let u be another solution with $u \in C^2(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. That is, u satisfies (2). Let $w = y - u$, and find an initial value problem satisfied by w.
2. Let $z = w'$ and $m = w^2 + z^2$. Notice that $m \in C^1(x_0 - \delta, x_0 + \delta)$, and show that m satisfies
$$\begin{cases} m'(x) = 0 \text{ for } |x - x_0| < \delta, \\ m(x_0) = 0. \end{cases}$$
3. If there is some x with $|x - x_0| < \delta$ such that $m(x) > 0$, then prove there is some ξ between x and x_0 such that $m'(\xi) \neq 0$. Hint: Mean Value Theorem.
4. Conclude that $m(x) \equiv 0$ for x with $|x - x_0| < \delta$. In particular, $w(x) \equiv 0$ and $u(x) \equiv y(x)$ for all x with $|x - x_0| < \delta$.

(c) Show the **two point boundary value problem**

$$\begin{cases} y'' + y = 0, \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{3}$$

does **not** have a unique solution. Find infinitely many different solutions.

The point of this last part is that the uniqueness for a boundary value problem (even for ODEs) cannot be taken for granted. Even if you are able to write down a solution, you can not always be sure your solution is unique.

(d) What is the general situation for existence and uniqueness for the two point boundary value problem

$$\begin{cases} y'' + y = 0, \\ y(a) = 0, \ y(b) = c \end{cases}$$

where a, b, and c are real numbers with $a < b$?

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Laplace's Equation in One and Two Dimensions

Laplace's equation in two dimensions for a function $u = u(x, y)$ is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4}$$

Laplace's equation in one dimension is

$$u'' = 0 \tag{5}$$

for $u = u(x)$. You might not think there is anything interesting to say about this ODE, and maybe you are right, but let's see.

5. It is natural to think about the solution of (5) as the solution of an **initial value problem** so that

$$u(x) = u_0'(x - x_0) + u_0 \qquad \text{where } u_0 = u(x_0).$$

   (a) Show the following: The **two point boundary value problem** for (5),

$$\begin{cases} u'' = 0, \\ u(a) = u_a, \ u(b) = u_b, \end{cases} \tag{6}$$

   has a **unique** solution for any real numbers a, b, u_a, and u_b with $a < b$.

   (b) Did you write down a formula for the solution in part (a)? If not, write one/it down.

   (c) What unique solution of (6) do you find when $u_a = 0$ and $u_b = 0$?

   (d) Can you find a solution $u = u(x, y)$ of the boundary value problem

$$\begin{cases} \Delta u = 0, \\ u\big|_{\partial U} \equiv 0. \end{cases} \tag{7}$$

   for (4) defined on an open set $U \subset \mathbb{R}^2$?

   (e) Show the solution you found in part (d) is unique if the domain U is bounded by completing the following steps:

   1. Assume there is a solution $v \in C^2(\overline{U})$ of (7) with $v(\mathbf{p}) > 0$ for some $\mathbf{p} \in U$. Let $w(\mathbf{x}) = v(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2$ where ϵ is a positive constant. Show that for $\epsilon > 0$ small enough, $w(\mathbf{x}) > 0$ for $\mathbf{x} \in \overline{U}$. Hint: There is some $M > 0$ for which $w \geq v(\mathbf{p}) - \epsilon M^2$.
   2. Let $\mathbf{q} \in U$ be a point such that

$$c = v(\mathbf{q}) - w(\mathbf{q}) = \max_{\mathbf{x} \in U} [v(\mathbf{x}) - w(\mathbf{x})].$$

      Why does such a point $\mathbf{q}$ exist and what is the sign of the constant c?
   3. Show that $\Delta v(\mathbf{q}) < 0$ (which is a contradiction). Hint: The function $\phi(\mathbf{x}) = w(\mathbf{x}) + c - v(\mathbf{x})$ vanishes at $\mathbf{x} = \mathbf{q}$. Combining this with the previous part tells you something about $\Delta\phi(\mathbf{q})$.

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Laplace's Equation on a Rectangle

The next two problems give two different approaches to solving the boundary value problem for Laplace's equation (4) on a rectangle $U = (a, b) \times (c, d)$ in the plane. Both approaches can lead to viable solution methods, but we will only concentrate on some of the preliminaries.

6. In view of problem 2, you should not be surprised that it is enough to restrict to the special case $U = [0, L] \times [0, M]$ where L and M are positive numbers. This just makes some computations easier.

   (a) Verify that $u(x, y) = \sin(2\pi x/L) \sinh(2\pi y/L)$ solves

   $$\begin{cases} \Delta u = 0, \\ u(x, 0) = 0, \ u(L, y) = 0, \ u(x, L) = g(x), \ u(0, y) = 0 \end{cases} \tag{8}$$

   if $g = g(x)$ happens to be a suitable function.

   (b) Solve

   $$\begin{cases} \Delta v = 0, \\ v(x, c) = 0, \ v(b, y) = 0, \ v(x, d) = 0, \\ v(a, y) = \sinh(2\pi(b - a)/(d - c)) \sin[2\pi(y - c)/(d - c)] \end{cases}$$

   for $a < b$ and $c < d$.

   (c) Returning to (8) substitute a "separated variables" solution $u(x, y) = A(x)B(y)$ into Laplace's equation $\Delta u = 0$. Rearrange what you get into a form

   $$\Phi(x) = \Psi(y) \tag{9}$$

   for some functions Φ (involving only A and depending only on x) and Ψ (involving only B and depending only on y).

   (d) Take the partial derivative with respect to x of both sides of (9). What does this tell you about the function $\Phi = \Phi(x)$?

   (e) Find "ordinary differential equations" which must be satisfied by A and B if $u(x, y) = A(x)B(y)$ is a solution of $\Delta u = 0$. (I put ODEs in quotes here because you cannot determine the equation completely, but you should be able to determine the form of the equation, e.g., second order linear, homogeneous, etc., but there may be parameters which you cannot determine (yet).)

   (f) If you substitute the separated variables solution $u(x, y) = A(x)B(y)$ into the boundary values specified in (8), what can you conclude about the boundary values of A on $[0, L]$ and B on $[0, L]$ under the assumption that $u = AB$ is not identically zero?

   (g) Find all possible separated variables solutions $u(x, y) = A(x)B(y)$.

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Poisson's Equation

Here we consider a more general version of (8)

$$\begin{cases} \Delta u = 0, \\ u(x,0) = g_1(x),\ u(L,y) = g_2(y),\ u(x,M) = g_3(x),\ u(0,y) = g_4(y). \end{cases} \tag{10}$$

7. Let us assume that $g_1, g_2, g_3, g_4 \in C^\infty(\mathbb{R})$ with

$$\begin{aligned} g_1(0) &= g_4(0), \\ g_1(L) &= g_2(0), \\ g_2(M) &= g_3(L), \text{ and} \\ g_3(0) &= g_4(M). \end{aligned}$$

It turns out that under these (boundary) conditions there are functions $g \in C^\infty(\overline{U})$ for which

$$g(x,0) = g_1(x),\ g(L,y) = g_2(y),\ g(x,M) = g_3(x),\ g(0,y) = g_4(y). \tag{11}$$

In fact, when we can solve (10) then the solution u will be such a function. For the moment, it's not even so obvious that such a smooth function g (of any sort) exists. But we will assume such a function exists for now.

(a) Rewrite the boundary conditions of (10) in terms of g satisfying (11) using only **one equation**. Hint: Look at the boundary condition in (7) from Problem 5.

(b) Let $w = u - g$ and, assuming u satisfies (10), find the boundary value problem satisfied by w. (The PDE you should find is a form of what is called **Poisson's equation**.

The point of this last part is that the boundary value problem (10) for Laplace's equation can be replaced by a boundary value problem for Poisson's equation with **homogeneous boundary values**.

(c) Let $U = (0, \ln 2) \times (0, \pi)$. Assume you can solve

$$\begin{cases} \Delta w = (1 + x/\ln 2)\sin y, \\ w\big|_{\partial U} \equiv 0. \end{cases} \tag{12}$$

Use the solution of this problem to solve

$$\begin{cases} \Delta u = 0, \\ u\big|_{\partial U} \equiv (1 + x/\ln 2)\sin y. \end{cases} \tag{13}$$

(d) Use Problem 5 part (e) to show the boundary value problem (12) can have at most one solution $w \in C^2(U) \cap C^0(\overline{U})$.

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Fourier Series

Let's think a little bit about Fourier series. Let $f : [0, L] \to \mathbb{R}$ be a continuous function. A **Fourier sine series** for f has the form

$$f(x) = \sum_{j=1}^{\infty} f_j \sin \frac{j\pi x}{L}. \tag{14}$$

where the **numbers** $f_1, f_2, f_3, \ldots$ are called the **Fourier coefficients** of f and the functions $\sin(\pi x/L)$, $\sin(2\pi x/L)$, $\sin(3\pi x/L)$,... make up what is called the **Fourier sine basis**. We do not need to worry too much about convergence of the series. For a continuous function, if the coefficients are chosen correctly, then the series will converge to $f(x)$ at least on $(0, L)$, and we can manipulate the series, at least as far as integrating term by term, pretty freely.

8. This problem is about computing Fourier coefficients.
   (a) Compute
   $$\int_0^L \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx.$$
   Hint: Something special happens when $j = k$ because (you can show)
   $$\int_0^L \sin^2 \frac{j\pi x}{L}\, dx = \int_0^L \cos^2 \frac{j\pi x}{L}\, dx \qquad \text{and} \qquad \sin^2 \frac{j\pi x}{L} + \cos^2 \frac{j\pi x}{L} = 1.$$
   Something even more special happens when $j \neq k$.
   (b) Multiply both sides of (14) by $\sin(k\pi x/L)$ and integrate both sides from $x = 0$ to $x = L$. Use the result to find a formula for the Fourier coefficients.
   (c) Consider the specific example $f : [0, L] \to \mathbb{R}$ by
   $$f(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(x - L)/(a - L), & a \le x \le L. \end{cases}$$
   Draw the graph $\{(x, f(x)) : 0 \le x \le L\}$ of f. What can you say about the regularity of f?
   (d) Let f be the specific function from the last part of this problem. Find the Fourier sine series expansion of f.
   (e) The trigonometric polynomial
   $$P_n(x) = \sum_{j=1}^{n} f_j \sin \frac{j\pi x}{L}$$
   is (called) the n**-th Fourier sine approximation of** f. Use mathematical software (Matlab, Mathematica, Maple, etc.) to plot $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_{10}(x)$, and $P_{100}(x)$ for the specific example from the last two parts.

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One Dimensional Poisson's Equation

The one dimensional Poisson's equation is $u'' = f(x)$, and again, we are back to a relatively easy ODE.

9. It is possible to use the method of Homework Assignment 1 (problems 1 and 2) to solve

$$\begin{cases} u'' = f(x), \\ u(0) = 0, \ u(L) = 0, \end{cases} \tag{15}$$

and you should write that solution down, but we are going to solve this boundary value problem a different way.

(a) Expand f as a Fourier sine series, so that the equation becomes

$$u'' = \sum_{j=1}^{\infty} f_j \sin \frac{j\pi x}{L}.$$

Solve infinitely many 1-D Poisson equations $u_j'' = f_j \sin(j\pi x/L)$ with homogeneous boundary conditions, so that the solution u can be written as a series

$$u(x) = \sum_{j=1}^{\infty} u_j.$$

(b) Substitute the Fourier coefficients f_j so that u takes the form

$$u(x) = \int_0^L G(x,\xi)\, f(\xi)\, d\xi \tag{16}$$

where $G = G(x,\xi)$ is a function of two variables defined as a series on $[0,L] \times [0,L]$. The function G is called the **Green's function** for the problem (15).

(c) Determine G explicitly and classify the regularity of G as a function of two variables. Hint(s): You can either restrict to horizontal lines $\xi =$ constant or vertical lines $x =$ constant, and use the previous problem (Problem 8) on Fourier series or, alternatively, you can use the solution obtained by the method of Homework Assignment 1 (problems 1 and 2) and put that solution (involving integrals) into the form (16) to find G directly by explicit integration. It's not a bad idea to do both.