# A Set Theoretical Framework
# for
# Regularity

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By **regularity** we mean roughly **continuity and differentiability properties of functions**. There are various measures for continuity and differentiability and some of those are commonly represented by sets (specifically by sets of functions). We introduce some of these here, and generally collect notes related to Chapter 4 on differentiation in the text *Mathematical Methods in the Physical Sciences* by Mary Boas.

The book should be viewed as a kind of starting place for topics. It is usually helpful to view the topics in the book from a more general or abstract point of view, and I will attempt to provide that framework in the lectures.

## Contents

# 1 Continuity

The simplest and most familiar regularity classes of functions are the classes $C^k$ for $k = 0, 1, 2, \ldots$. Generally, *continuity* is represented by the *symbol* $C^0$. A real valued function $f$ defined on the real line $\mathbb{R}$ or an interval $I \subset \mathbb{R}$ is **continuous** at $x_0 \in I$ if the following condition holds:

Given $\epsilon > 0$, there is some $\delta > 0$ for which

$$\left.\begin{array}{l} x \in I, \text{ and} \\ |x - x_0| < \delta \end{array}\right\} \quad \Longrightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Such a function $f : I \to \mathbb{R}$ is simply **continuous** or $C^0$ if it is continuous at each point in $I$. Thus $C^0(I)$ denotes the collection of all continuous real valued functions on the interval $I$. The same set of functions may be denoted by

$$\begin{array}{c} C^0(a,b) \text{ if } I = (a,b), \\ C^0[a,b) \text{ if } I = [a,b), \\ C^0[a,b] \text{ if } I = [a,b]. \end{array}$$

A function $f : (a,b) \to \mathbb{R}$ is **differentiable** at $x_0 \in (a,b)$ if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

exists. A function is **differentiable** on $(a,b)$ if $f$ is differentiable at each point $x_0 \in (a,b)$. In this case (1) defines a function $f' : (a,b) \to \mathbb{R}$ called the **derivative** of $f$. Let us denote by $\text{Diff}(a,b)$ the collection of all differentiable real valued functions defined on $(a,b)$. Then

$$C^1(a,b) = \{f \in \text{Diff}(a,b) : f' \in C^0(a,b)\}.$$

If $I$ is any interval, then $C^1(I)$ denotes the collection of functions $f : I \to \mathbb{R}$ such that there is an open interval $J = (\alpha, \beta)$ with $I \subset J$ and a function $\phi \in C^1(J)$ such that the **restriction** of $\phi$ to $I$ satisfies

$$\phi\big|_I = f.$$

Naturally, the function $\phi$ is called an **extension** of $f$.

**Exercise 1** *Show that if $f \in \mathrm{Diff}(a,b)$, then $f \in C^0(a,b)$.*

Since differentiability implies continuity, we know $C^1(I) \subset C^0(I)$. Also, the functions with $k$ continuous derivatives are given inductively by

$$C^k(I) = \{f \in C^{k-1}(I) : f^{(k-1)} \in C^1(I)\}.$$

In the study of partial differential equations we consider functions $u = u(x_1, \ldots, x_n)$ of several variables. Usually these functions are real valued, but sometimes they are vector valued with

$$u : U \to \mathbb{R}^k \qquad \text{where} \qquad U \subset \mathbb{R}^n.$$

We wish to extend certain notions of continuity and differentiability to these more general functions. The sets corresponding to open intervals $(a,b)$ and more general sets like half open/half closed intervals $[a,b)$ are somewhat more complicated in higher dimensions.

At length, we will define the basic differentiability classes

$$C^0(U) \supset \mathrm{Diff}(U) \supset C^1(U) \supset C^2(U) \supset C^3(U) \supset \cdots \supset C^k(U) \supset \cdots$$

for $U \subset \mathbb{R}^n$ without too much difficulty. After that, roughly speaking, we would like to insert **fractional differentiability classes** in between each of the $C^k$ spaces. For example, we would like a continuum of classes $C^{0,\alpha}(U)$ for $0 < \alpha < 1$ such that

$$C^0(U) \supset C^{0,\alpha} \supset \mathrm{Diff}(U) \supset C^1(U)$$

so that $u \in C^{0,\alpha}$, say if $\alpha = 1/2$, means roughly that $u$ is "half differentiable." Similarly, we want classes $C^{k,\alpha}$ for every $k$ with

$$C^k(U) \supset C^{k,\alpha} \supset C^{k+1}(U)$$

where $u \in C^k$ means $u$ is $k$ times differentiable and the $k$-th partials of $u$ have a fractional measure $(\alpha)$ of differentiability. There are two main problems encountered in this construction. We may enumerate them as

1. The inclusion problem.

2. The metric problem.

The inclusion problem is the simpler one. It is simply that the inclusion $C^k \subset C^{k,\alpha}$ only holds under certain additional hypotheses. Thus, some understanding of what is needed to have the desired inclusion is required. The metric problem is somewhat more complicated to explain at this point because it requires an understanding of some more general structures. Roughly speaking, in the study of differential equations (and regularity in particular) it is extremely useful to be able to measure the **distance between functions**. The metric problem is that, again, the most natural definitions for the fractional differentiability of $C^{k,\alpha}$ do not lend themselves immediately to a reasonable notion of distance between functions. Fortunately, some aspects of the metric problem already arise in the $C^k$ spaces and even in $C^0$, so can be addressed in a relatively simple context. We take this up at the beginning of the next section on continuity. First we discuss some preliminaries concerning sets in $\mathbb{R}^n$.

# 2 Kinds of Sets

The set $\mathbb{R}^n$ consisting of ordered $n$-tuples of real numbers,

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

is called $n$ dimensional **Euclidean space**. A set $U \subset \mathbb{R}^n$ is said to be **open** if for each $\mathbf{p} = (p_1, \ldots, p_n) \in U$, there is some $r > 0$ so that the (open) ball

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\} \subset U.$$

In the definition of the open ball, the expression

$$|\mathbf{x} - \mathbf{p}| = \left( \sum_{j=1}^{n} (x_j - p_j)^2 \right)^{1/2}$$

is called the **Euclidean distance** from $\mathbf{x}$ to $\mathbf{p}$.

**Exercise 2** *Show that an open ball in $\mathbb{R}^n$ is open. Show that*

$$\overline{B_r(\mathbf{p})} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| \le r\}$$

*is not open.*

**Summary/Observation:** We have defined the $C^k$ differentiability classes $C^k(I)$ for $I$ an interval in $\mathbb{R}^1$. When we were done, we had

$$C^0(I) \supset C^1(I) \supset C^2(I) \supset C^3(I) \cdots .$$

Note that we might like to add to this inclusion $C^0(I) \supset \mathrm{Diff}(I) \supset C^1(I)$, but $\mathrm{Diff}(I)$ was only defined for $I = (a, b)$ an open interval. It was easy to define continuity $C^0$ for any interval. In fact, we could have defined $C^0(E)$ for any subset of $\mathbb{R}^1$, not just intervals, and we can define the class $C^0$ in a much much broader context which will be discussed below. But we needed an open interval to consider difference quotients and define differentiability. The notion of open sets given above may seem simple, but it is rather powerful. It is useful for us even back in $\mathbb{R}^1$, and we can use it to define $C^1(U)$ for any **open** set in $\mathbb{R}^1$. Then $C^1(E)$ can be defined for any set in $\mathbb{R}^1$ as the set of functions with a $C^1$ extension to some open set containing $E$. Then $C^k(E)$ may be defined inductively.

An interval $I \subset \mathbb{R}$ is said to be a **finite** interval if it is bounded below and bounded above, that is, if it has well-defined left and right endpoints $a \le b$ in $\mathbb{R}$. Note that in our discussion of intervals above infinite intervals like $[a, \infty)$ and $\mathbb{R} = (-\infty, \infty)$ were allowed. We now return to our immediate goal of describing the analogue of finite open intervals $(a, b)$ and finite closed intervals $[a, b]$ in higher dimensional Euclidean spaces.

An open set $U \subset \mathbb{R}^n$ is **connected** if whenever $V_1$ and $V_2$ are disjoint open sets in $\mathbb{R}^n$ with the property that $U \subset V_1 \cup V_2$, then either

$$V_1 \cap U = \phi \qquad \text{or} \qquad V_2 \cap U = \phi.$$

(Here $\phi$ is the empty set.)

**Exercise 3** *Show* $B_1(-1) \cup B_1(1)$ *is* **not** *connected in* $\mathbb{R}^1$. *In fact, the open connected sets in* $\mathbb{R}^1$ *are precisely the open intervals* $(a, b)$.

We will give another characterization of open connected sets in $\mathbb{R}^n$ below.

A set $E$ is **closed** in $\mathbb{R}^n$ if the complement $\mathbb{R}^n \backslash E$ is open.

**Exercise 4** *Show that the empty set is open and* $\mathbb{R}^n$ *is closed.*

Any union of open sets is open, and it follows that any intersection of closed sets is closed. Any set is a subset of the closed set $\mathbb{R}$, so the collection of closed sets containing any set $E$ is nonempty. This means the intersection of all closed sets containing a set $E$ is always a well-defined set. We call this intersection

$$\overline{E} = \bigcap_{\substack{C \text{ closed}, \\ E \subset C}} C$$

the **closure** of $E$.

**Exercise 5** *The closure of any set $E$ is closed. What is the closure of the empty set?*

A subset $U$ of $\mathbb{R}^n$ is **bounded** if there is a number $M$ for which

$$|\mathbf{x}| < M \qquad \text{whenever} \qquad \mathbf{x} \in U.$$

The expression

$$|\mathbf{x}| = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}$$

appearing in the definition of boundedness is called the **Euclidean norm** of $\mathbf{x}$.

> The sets in $\mathbb{R}^n$ corresponding to open intervals in $\mathbb{R}^1$ are the open connected sets. These can be much more complicated than open intervals.
>
> A **finite open interval** $(a, b) \subset \mathbb{R}^1$ is one for which $a, b \in \mathbb{R}$. Recall that we denote $\mathbb{R}$ by $(-\infty, \infty)$. The sets in $\mathbb{R}^n$ corresponding to finite open intervals in $\mathbb{R}^1$ are the open connected sets which are bounded.
>
> The sets in $\mathbb{R}^n$ corresponding to the finite closed intervals $[a, b] \subset \mathbb{R}^1$ are the closures of open connected sets which are bounded.

**Exercise 6** *If $E$ is bounded, then the closure $\overline{E}$ of $E$ is also bounded.*

We have discussed what it means for a set $E \subset \mathbb{R}^n$ to be open, closed, connected, and/or bounded. These are a good start concerning the elementary understanding of the properties of sets in $\mathbb{R}^n$. After a further discussion of continuity, we can add to this list the notions of being **pathwise connected** and **simply connected** which is are also sometimes useful.

# 3 Continuity II

If $U \subset \mathbb{R}^n$ and $u : U \to \mathbb{R}^k$, then our definition of continuity above adapts with a minimal change of notation: The function $u$ is **continuous** at $\mathbf{p} \in U$ if

Given $\epsilon > 0$, there is some $\delta > 0$ for which

$$\left.\begin{array}{l}\mathbf{x} \in U, \text{ and} \\ |\mathbf{x} - \mathbf{p}| < \delta\end{array}\right\} \qquad \Longrightarrow \qquad |u(\mathbf{x}) - u(\mathbf{p})| < \epsilon.$$

Note that we are using the Euclidean distance here instead of the absolute value. In fact this definition applies in much more generality which is very worthwhile to introduce.

The Euclidean norm $|\mathbf{x}| = |\mathbf{x} - \mathbf{0}|$ satisfies the following properties:

**(i non-negative homogeneity)** $|c\mathbf{x}| = |c||\mathbf{x}|$ for all $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

**(ii positive definite)** $|\mathbf{x}| = 0 \iff \mathbf{x} = \mathbf{0}$, and

**(iii the triangle inequality for norms)** $|\mathbf{x} + \mathbf{p}| \leq |\mathbf{x}| + |\mathbf{p}|$ for all $\mathbf{p}, \mathbf{x} \in \mathbb{R}^n$.

Given any **vector space** $V$ (over $\mathbb{R}$) we say $\|\cdot\| : V \to [0, \infty)$ is a **norm** if we have

**(i non-negative homogeneity)** $\|cv\| = |c|\|v\|$ for all $c \in \mathbb{R}$ and $v \in V$,

**(ii positive definite)** $\|v\| = 0 \iff v = \mathbf{0}$, and

**(iii the triangle inequality for norms )** $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

**Exercise 7** *Write down the definition of a* **vector space***.*

Solution: A vector space $V$ over a field $F$ is a set with an operation of **addition**, i.e., a function $+ : V \times V \to V$ whose values are written as $(v, w) \mapsto v + w$, and a **scaling**, i.e., a function $(a, v) \mapsto av \in V$ for $(a, v) \in F \times V$, such that $V$ is a **commutative group** under addition:

[i associative] $(v + w) + z = v + (w + z)$ for every $v, w, z \in V$.

[ii identity element] There is some $\mathbf{0} \in V$ such that $\mathbf{0} + v = v + \mathbf{0} = v$ for every $v \in V$. The vector $\mathbf{0}$ is called the **additive identity**.

[iii inverses] For each $v \in V$, there is some $w \in V$ with $v + w = w + v = \mathbf{0}$. The vector $w$ is called the **additive inverse** of $v$ and is denoted by $w = -v$.

[iv commutative] $v + w = w + v$ for all $v, w \in V$.

The scaling is associative in the sense that

$$(ab)v = a(bv) \qquad \text{for every } a, b \in F \text{ and } v \in V.$$

Finally, the addition and scaling should satisfy two distributive properties:

**(i distributive of scalars over vectors)** $a(v+w) = av+aw$ for $a \in F$ and $v, w \in V$.

**(i distributive of vectors over scalars)** $(a+b)v = av+bv$ for $a, b \in F$ and $v \in V$.

We could go into the algebraic properties possessed by a **field** in general, but let it suffice to say that the real numbers $\mathbb{R}$ and and the complex numbers $\mathbb{C}$ are fields. Roughly speaking there are two operations, addition and multiplication, in a field $F$; the field is a group under addition, and the nonzero elements $F^*$ are a group under multiplication. The Euclidean spaces $\mathbb{R}^n$ are vector spaces over $\mathbb{R}$.

**Exercise 8** *1. Show that $C^k(I)$ is a vector space for every $k = 0, 1, 2, \ldots$.*

*2. Show $C^0(U)$ is a vector space for $U \subset \mathbb{R}^n$.*

**Exercise 9** *Given any vector space $V$ (over $\mathbb{R}$) with a norm $\| \cdot \|$, show that*

$$d : V \times V \to [0, \infty) \quad \text{by} \quad d(v, w) = \|w - v\|$$

*satisfies*

**(i reflexive)** $d(v, w) = d(w, v)$ *for all* $v, w \in V$*,*

**(ii positive definite)** $d(v, w) = 0 \iff v = w$*, and*

**(iii satisfies the triangle inequality for distances)** $d(v, w) \leq d(v, z) + d(z, w)$ *for all* $v, w, z \in V$*.*

*The function $d$ is called the* **norm induced distance***.*

Notice that the properties satisfied by a norm induced distance do not depend on the vector space structure of $V$.

Given any set $X$ and a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

**(i reflexive)** $d(x, y) = d(y, x)$ for all $x, y \in X$,

**(ii positive definite)** $d(x, y) = 0 \iff x = y$, and

**(iii satisfies the triangle inequality for distances)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

we say $X$ is a **metric space** and $d$ is a **distance function** (or sometimes "metric").

In this way, we can adapt our definition of continuity to a much more general context:

If $X_1$ and $X_2$ are **metric spaces**, with distance functions $d_1$ and $d_2$ respectively, and $f : X_1 \to X_2$ is a function, then we say $f$ is continuous at $p \in X_1$ if the following condition holds:

Given $\epsilon > 0$, there is some $\delta > 0$ for which

$$\left.\begin{array}{l} x \in X_1, \text{ and} \\ d_1(x, p) < \delta \end{array}\right\} \qquad \Longrightarrow \qquad d_2(f(x), f(p)) < \epsilon.$$

The class $C^0(X_1 \to X_2)$ consists of all functions $f : X_1 \to X_2$ which are continuous at all points of $X_1$.

**Exercise 10** *Show that*

$$d(\mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n} |p_j - x_j|$$

*is a distance function on* $\mathbb{R}^n$*. How is this distance related to Euclidean distance?*

Returning to simpler cases, when the functions under consideration are real valued, we simply write $C^0(X_1)$ for $C^0(X_1 \to \mathbb{R})$. Even in the simplest case $C^0(I)$ where $I$ is in interval in $\mathbb{R}$ it is sometimes convenient to consider $C^0$ itself as a metric space. This is usually accomplished by setting

$$d(f, g) = \sup_{x \in I} |g(x) - f(x)|. \tag{2}$$

This brings up a minor technical difficulty.

**Exercise 11** *Find two continuous (real valued) functions $f$ and $g$ on the interval $I = (0, 1)$ such that the value $d(f, g)$ defined in (2) is not a finite number.*

There are various approaches to dealing with this difficulty. The ususal one is to simply realize that if you want $C^0$ to be a metric space, then you need to restrict to some smaller subset of functions. In essence, you can't (always) treat $C^0$ as a metric space. I will suggest a variant of this approach below with some additional details and a somewhat lengthy discussion. Before starting that, let me mention a couple other alternatives/special cases. One "quick fix" is to restrict attention to only compact (closed and bounded) sets. Then in $\mathbb{R}^n$ at least we have recourse to the following result:

**Theorem 1 (max/min theorem)** *If $K$ is a nonempty subset of $\mathbb{R}^n$ which is* **compact***, i.e., closed and bounded, and $u \in C^0(K)$, then there are points $\mathbf{p}$ and $\mathbf{q}$ in $K$ such that*

$$u(\mathbf{p}) \le u(\mathbf{x}) \le u(\mathbf{q}) \qquad \text{for every } \mathbf{x} \in K.$$

*The values $u(\mathbf{p})$ and $u(\mathbf{q})$ are called the minimum and maximum of $u$ on $K$ respectively. The points $\mathbf{p}$ and $\mathbf{q}$ are called minimum and maximum points respectively. The minimum and maximum values are unique, but the minimum and maximum points are not necessarily unique.*

**Exercise 12** *The max/min value theorem asserts that every continuous real valued function $f \in C^0[a, b]$ defined on a closed interval in $\mathbb{R}$ attains its maxiumum and minimum values at points in the interval $[a, b]$. Use this result to show that if $f, g \in C^0[a, b]$, then*

$$\sup_{x \in I} |g(x) - f(x)| < \infty.$$

More generally, $C^0(K)$ is a metric space with distance function

$$d(f, g) = \sup_{x \in K} |g(x) - f(x)| \qquad (3)$$

whenever $K$ is a compact subset of $\mathbb{R}^n$.

Another possibility is to allow distance functions with values in the **extended real numbers** $[0, \infty]$. On the one hand arithmetic in $[0, \infty]$, and even in $(-\infty, \infty]$, is easy to define setting $a + \infty = \infty$ when $a \in \mathbb{R}$. On the other hand, the "point at infinity" $\infty$ has no additive inverse in this arithmetic, and the informational value of the condition $d(f, g) = \infty$ is somewhat limited.

To introduce our approach, let us first note that the proposed distance function given (2) or (3) "looks like" it comes from a **norm**. Fortunately, we've already discussed the notion of a norm. Let us set

$$C_b^0(X) = \{u \in C^0(X) : \sup_{x \in X} |u(x)| < \infty\}.$$

**Exercise 13** *Show that of $X$ is any metric space, then $C_b^0(X)$ is a vector space and*

$$\|u\|_{C^0} = \sup_{x \in X} |u(x)| \text{ defines a norm on the subspace } C_b^0(X). \tag{4}$$

*In fact, given any set $A$ whatsoever the set of* **bounded real valued functions** *on $A$, sometimes denoted $B(A)$ is a vector space (over $\mathbb{R}$), and if we have $A = X$ is a metric space, then $C_b^0(X) \subset B(X)$ is a subspace. Actually,*

$$B(A) = \{f : A \to \mathbb{R} \text{ such that} \sup_{x \in A} |f(x)| < \infty\}$$

*is a normed space with the norm*

$$\|f\|_{C^0} = \sup_{x \in A} |f(x)| \tag{5}$$

*though this is not a space we will have much use for in this course.*

To summarize, we use the notation $C_b^0(X)$ to denote the normed space associated with $C^0(X)$ where $X$ is a metric space and (4) defines the norm so that

$$d_0(u, v) = \|u - v\|_{C^0} = \sup_{x \in X} |u(x) - v(x)|$$

defines the $C^0$ distance on $C_b^0(X)$. The norm (4) is variously called the *sup norm*, the $C^0$ norm, the *norm of uniform convergence*, and (borrowing from the intagrability classes we will talk about later) the $L^\infty$ norm. It is an important norm. Furthermore, the approach we have presented here is a precursor of the standard approach to the Hölder spaces $C^{k,\alpha}$ discussed below.

**Exercise 14** *$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ satisfies $f \in C^0(\mathbb{R}) \backslash C_b^0(\mathbb{R})$. (The point being: There are some very common and familiar and useful functions, like polynomials, which do not fall into natural normed spaces associated with differentiability. On the other hand, some other useful functions like* sin, cos, *and* $\tan^{-1}$ *are in $C_b^0(\mathbb{R})$.)*

As a final remark, the possibility of considering a space of functions like $\mathcal{F} = C^0_b(X)$ as a metric space in its own right should suggest that one might consider functions on $\mathcal{F}$ and the real valued continuous functions $C^0(\mathcal{F})$ in particular. In fact, one does consider such functions $f : \mathcal{F} \to \mathbb{R}$, and these are commonlly called **functionals**. A **continuous functional** is *a continuous function on a metric space of functions.* The subject called **Calculus of Variations** is about minimizing such functionals. We know enough now to formulate a simple problem in the calculus of variations:

> Given an open disk $U = B_1(\mathbf{0}) = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ in the plane, and a fixed function $u_0 \in C^1(\overline{U})$, find the function in
>
> $$\mathcal{A} = \left\{ u \in C^1(\overline{U}) : \text{ for which } \ u\big|_{\partial U} = u_0 \right\}$$
>
> with graph of least area. The area of the graph of $u$ is defined by
>
> $$\text{area}[u] = \int_U \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}.$$

Here we are considering the area functional area : $C^1(\overline{U}) \to \mathbb{R}$. Technically, we haven't made $C^1$ a metric space, but that is now easy because

$$\|u\|_{C^1} = \|u\|_{C^0} + \max\left\{ \left\|\frac{\partial u}{\partial x}\right\|_{C^0}, \left\|\frac{\partial u}{\partial x}\right\|_{C^0} \right\}$$

defines a norm. Technically, we haven't defined partial derivatives yet either, but I expect you have at least a passing familiarity with them and know these partial derivatives are given by limits of difference quotients

$$\frac{\partial u}{\partial x}(\mathbf{p}) = \lim_{h \to 0} \frac{u(p_1 + h, p_2) - u(\mathbf{p})}{h} \qquad \text{and} \qquad \frac{\partial u}{\partial y}(\mathbf{p}) = \lim_{h \to 0} \frac{u(p_1, p_2 + h) - u(\mathbf{p})}{h}$$

where $\mathbf{p} = (p_1, p_2)$ lies in an open set of $\mathbb{R}^2$ where $u$ is defined.

**Exercise 15** *Show that the space $C^1(U)$ consisting of functions with a $C^1$ extension to an open subset $V$ of $\mathbb{R}^2$ with $\overline{U} \subset V$ is a vector space and that*

$$[u]_{C^1} = \max\left\{ \left\|\frac{\partial u}{\partial x}\right\|_{C^0}, \left\|\frac{\partial u}{\partial x}\right\|_{C^0} \right\}$$

*satisfies the non-negative homogeneity and triangle inequality for norms on*

$$C_b^1(U) = \{u \in C^1(U) : \|u\|_{C^0} < \infty \text{ and } [u]_{C^1} < \infty\}.$$

*This function* $[\ \cdot\ ]_{C^1} : C_b^1(U) \to [0, \infty)$ *is called the* $C^1$ **seminorm**. *Why isn't* $u \mapsto [u]_{C^1}$ *a norm?*

# 4 Path Connectedness Simply Connected Sets

It should now be clear what it means for a function $\mathbf{r} : I \to \mathbb{R}^n$ where $I$ is an interval to be continuous. We have discussed the space $C^0(I \to \mathbb{R}^n)$. You should have also seen, at some point, differentiable functions $\mathbf{r} : I \to \mathbb{R}^n$. The image points of such a function are determined by $n$ separate real valued functions of one variable:

$$\mathbf{r}(t) = (r_1(t), r_2(t), \dots, r_n(t)). \tag{6}$$

When $n = 2$ the image is a parameterized curve in the plane and we might write $\mathbf{r}(t) = (x(t), y(t))$. When $n = 3$, the image is a parameterized curve in three dimensional Euclidean space, and we might write $\mathbf{r}(t) = (x(t), y(t), z(t))$. When $n > 3$, the image is still a curve. Unfortunately, continuous curves can be very complicated.

**Theorem 2 (space filling curve)** *There exists a continuous function* $\mathbb{R} : [0, 1] \to \mathbb{R}^3$ *such that for every* $\mathbf{x}$ *in the unit cube*

$$C = [0, 1] \times [0, 1] \times [0, 1] = \{(x, y, z) : 0 \le x, y, z \le 1\}$$

*there is some* $t \in [0, 1]$ *such that* $\mathbf{r}(t) = \mathbf{x}$. *The function* $\mathbf{r}$ *is surjective (onto).*

Fortunately, differentiable curves are not so badly behaved. We say $\mathbf{r} : (a, b) \to \mathbb{R}^n$ is **differentiable** if each of the functions $r_1, r_2, \dots, r_n$ appearing in (6) is differentiable. The **vector valued derivative** of such a vector valued function is given simply by

$$\mathbf{r}'(t) = (r_1'(t), r_2'(t), \dots, r_n'(t)).$$

To get an idea of the geometric meaning of this vector, consider the following vector valued difference quotient:

$$\frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}.$$

**Exercise 16** *Show that*

$$\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}=\mathbf{r}'(t).$$

The vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ can be pictured as the **secant vector** which has origin at the point $\mathbf{r}(t)$ and terminal point at $\mathbf{r}(t+h)$. See Figure 1. The difference quotient is

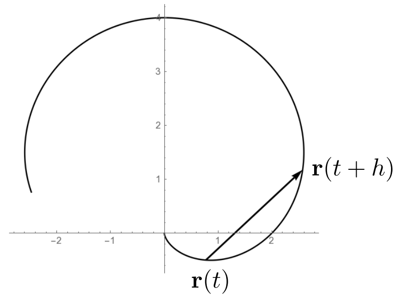


Figure 1: A curve in the plane with a secant vector from $\mathbf{r}(t)$ to $\mathbf{r}(t+h)$.

a real valued scaling of the secant vector, so it is still in the direction of a secant, and if there is a finite limit $\mathbf{r}'(t)$, then the resulting vector should be a tangent vector.

**Exercise 17** *Explain why the value of* $|\mathbf{r}'(t)|$ *is the rate of change of change of the position* $\mathbf{r}(t)$ *with respect to the parameter* $t$*, i.e., the speed.*

There are some special curves and particular parameterizations that are useful. One of those is a straight line from $\mathbf{p}$ to $\mathbf{q}$ parameterized as a **convex combination**. This is given by

$$\mathbf{r}(t)=(1-t)\mathbf{p}+t\mathbf{q}.$$

**Exercise 18** *Find* $\mathbf{r}(0)$*,* $\mathbf{r}(1)$*, and* $\mathbf{r}'(t)$*.*

A curve which is parameterized by a collection of **concatenated straight line segments** is called a **polygonal path**. For example,

$$\gamma(t)=\begin{cases}(t,0), & 0\le t\le 1\\ (1,t-1), & 1\le t\le 2\end{cases} \tag{7}$$

is a concatenated path.

**Exercise 19** *Sketch the image of the path* $\gamma$ *defined in (7).*

Generally, by a **path** we simply mean a continuous vector valued function on an interval, but as pointed out above, these can be very complicated. A path is said to be a $C^1$ path if each of the coordinate functions $r_1, r_2, \dots, r_n$ is $C^1$.

An set $U \subset \mathbb{R}^n$ is said to be **path connected** if given any two points $\mathbf{p}$ and $\mathbf{q}$ in $U$, there is a path $\gamma : [a, b] \to U$ with $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$. In this case, we say the path $\gamma$ connects $\mathbf{p}$ to $\mathbf{q}$.

The following results are the main objective of this section:

**Theorem 3** *If $U$ is an open subset of $\mathbb{R}^n$ and $U$ is path connected, then $U$ is connected.*

**Theorem 4** *If $U$ is an open subset of $\mathbb{R}^n$ and $U$ is connected, then $U$ is path connected.*

**Theorem 5** *If $U$ is an open subset of $\mathbb{R}^n$ and $U$ is connected, then any two points in $U$ can be connected by a polygonal path.*

**Theorem 6** *If $U$ is an open subset of $\mathbb{R}^n$ and $U$ is connected, then any two points in $U$ can be connected by a $C^1$ path.*

Finally, we explain what it means for a subset $U \subset \mathbb{R}^n$ to be **simply connected**. A path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$. A closed path is also called a **loop**.

A continuous function $H : [a, b] \times [0, 1] \to X$ is a **homotopy** or **continuous deformation** of one path $\mathbf{r} : [a, b] \to X$ into another path $\gamma : [a, b] \to X$ if

$$\begin{aligned} \mathbf{r}(t) &= H(t, 0) \quad \text{for } t \in [a, b], \text{ and} \\ \gamma(t) &= H(t, 1) \quad \text{for } t \in [a, b]. \end{aligned}$$

Here $X$ can be any metric space, but we have in mind particularly the case $X = U \subset \mathbb{R}^n$.

A set $U \subset \mathbb{R}^n$ is **simply connected** if given any loop $\mathbf{r} : [a, b] \to U$, there exists a homotopy $H : [a, b] \times [0, 1] \to U$ such that

$$\begin{aligned} \mathbf{r}(t) &= H(t, 0) \quad \text{for } t \in [a, b], \\ H(a, s) &= H(b, s) \quad \text{for } s \in [0, 1] \text{ so that each path } h : t \mapsto H(t, s) \text{ is a loop, and} \\ H(t, 1) &\equiv \mathbf{q} \quad \text{for some point } \mathbf{q} \in U. \end{aligned}$$

# 5 Continuity Measures
# Fractional Differentiability

A function $u : U \to \mathbb{R}$ defined on $U \subset \mathbb{R}^n$ is **Lipschitz continuous** at $\mathbf{p} \in U$ if the following condition holds:

There is some $\delta > 0$ and some $\Lambda$ for which

$$\left.\begin{array}{l} \mathbf{x} \in U \\ |\mathbf{x} - \mathbf{p}| < \delta \end{array}\right\} \qquad \Longrightarrow \qquad |u(\mathbf{x}) - u(\mathbf{p})| < \Lambda |\mathbf{x} - \mathbf{p}|. \tag{8}$$

**Exercise 20** *Show that if $u$ is Lipschitz continuous at $\mathbf{p}$, then $u$ is continuous at $\mathbf{p}$.*

Extending the pointwise Lipschitz condition given above to sets requires some care. A function $u : U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ is said to be **globally Lipschitz at p with respect to** $E \subset U$ if

$$\sup_{\mathbf{x} \in E \backslash \{\mathbf{p}\}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|} < \infty.$$

In this case, we can set

$$\Lambda = \sup_{\mathbf{x} \in E \backslash \{\mathbf{p}\}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|},$$

and we call $\Lambda$ the Lipschitz constant of $u$ at $\mathbf{p}$. The set of all Lipschitz functions on a set $U \subset \mathbb{R}^n$ usually refers to those functions which are globally Lipschitz at every point of $U$ and for which the **global Lipschitz constant**

$$[u]_{C^{0,1}} = \sup_{\mathbf{p} \in U} \sup_{\mathbf{x} \in E \backslash \{\mathbf{p}\}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|} = \sup_{\substack{\mathbf{p}, \mathbf{x} \in U \\ \mathbf{x} \neq \mathbf{p}}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|} < \infty.$$

Again, this is a **seminorm**, and it is usual to write

$$\mathrm{Lip}(U) = \{u : U \to \mathbb{R} \text{ such that } [u]_{C^{1,0}} < \infty\}.$$

For $u \in \mathrm{Lip}(U)$, the constant $[u]_{C^{0,1}}$ is called the **Lipschitz constant** for $u$.

**Exercise 21**

1. *Show $f(x) = |x|$ has $f \in \mathrm{Lip}(\mathbb{R})$ but $g(x) = x^2$ has $g \notin \mathrm{Lip}(\mathbb{R})$.*
2. *Show $u(\mathbf{x}) = |\mathbf{x}|$ has $u \in \mathrm{Lip}(\mathbb{R}^n)$.*

**Exercise 22** *In keeping with the discussion of continuity above, the space*

$$\mathrm{Lip}_b(U) = \{u : U \to \mathbb{R} \text{ such that } \|u\|_{C^0} < \infty \text{ and } [u]_{C^{1,0}} < \infty\}$$

*which is a normed (and metric) space with norm given by*

$$\|u\|_{C^{0,1}} = \|u\|_{C^0} + [u]_{C^{0,1}},$$

*is also referred to as the Lipschitz functions and sometimes denoted by* $\mathrm{Lip}(U)$ *but more commonly by* $C^{0,1}(U)$*. In particular, when* $U$ *is an open bounded set in* $\mathbb{R}^n$ *and one refers to* $\mathrm{Lip}(\overline{U})$ *there is no ambiguity because* $\mathrm{Lip}_b(\overline{U}) = C^{0,1}(U)$ *and* $\mathrm{Lip}(\overline{U})$ *are the same in that case; see below.*

*We have just described at least four different kinds of Lipschitz functions* $u : U \to \mathbb{R}$ *with respect to a set* $U \subset \mathbb{R}^n$*:*

1. *The functions which are (locally) Lipschitz at every point in* $U$*.*
2. *The functions which are uniformly (locally) Lipschitz at every point in* $U$*, that is, there is a single constant* $\Lambda$ *such that for each* $\mathbf{p} \in U$ *there is some* $\delta > 0$ *such that (8) holds.*
3. *The functions which are globally Lipschitz at each point of* $U$*.*
4. *The functions which are uniformly globally Lipschitz.* $\mathrm{Lip}(U)$*.*

*Let us denote these spaces of functions by* $\mathrm{Lip}_1(U)$*,* $\mathrm{Lip}_2(U)$*,* $\mathrm{Lip}_3(U)$*, and* $\mathrm{Lip}(U)$ *respectively.*

1. *Show there are sets* $U$ *for which each of the following hold:*
   - *(a)* $\mathrm{Lip}(U) \subsetneqq \mathrm{Lip}_3(U) \cap \mathrm{Lip}_2(U)$.
   - *(b)* $\mathrm{Lip}_3(U) \subsetneqq \mathrm{Lip}_1(U)$.
   - *(c)* $\mathrm{Lip}_2(U) \subsetneqq \mathrm{Lip}_1(U)$.
   - *(d)* $\mathrm{Lip}_3(U) \cap \mathrm{Lip}_2(U) \subsetneqq \mathrm{Lip}_3(U)$.
   - *(e)* $\mathrm{Lip}_3(U) \cap \mathrm{Lip}_2(U) \subsetneqq \mathrm{Lip}_2(U)$.
2. *Show that*

$$\mathrm{Lip}_2(\overline{U}) = \{u \in C^0(\overline{U}) : u \in \mathrm{Lip}(K) \text{ for each compact set } K \subset \overline{U}\}.$$

3. *Show that if $U$ is open, then*

$$\mathrm{Lip}_2(U) = \{u \in C^0(U) : u \in \mathrm{Lip}(K) \textit{ for each compact set } K \subset U\}.$$

4. *Show that if $\overline{U}$ is compact, then*

$$C^{0,1}(\overline{U}) = \mathrm{Lip}(\overline{U}) = \mathrm{Lip}_3(\overline{U}) = \mathrm{Lip}_2(\overline{U}) = \mathrm{Lip}_1(\overline{U}).$$

Sometimes the set

$$\mathrm{Lip}_{loc}(U) = \{u \in C^0(U) : u \in \mathrm{Lip}(K) \text{ for each compact set } K \subset U\}$$

is called the set of **locally Lipschitz functions on** $U$. The usual context for this is when $U$ is an open set. Otherwise, one may need to be particularly careful about the properties of such functions and the (continuity) properties of functionals on such a set in particular. There is also a corresponding space of locally Lipschitz functions $\mathrm{Lip}_{b,loc}(U) = C^{0,1}_{loc}(U)$ which are bounded on compact subsets of $U$. Again, this notation is used especially when $U$ is an open subset of $\mathbb{R}^n$.

The set of all (uniformly globally) Lipschitz functions on a subset $E$ of a normed vector space, and analogues of all the Lipschitz spaces discussed above, may be defined by simply replacing the Euclidean norm on $U$ in the discussion above with the norm on the vector space. The resulting space is denoted by $\mathrm{Lip}(E)$.

**Exercise 23** *Formulate a condition defining (uniformly globally) Lipschitz functions $f : X_1 \to X_2$ where $X_1$ and $X_2$ are metric spaces and define a seminorm $[\,\cdot\,]_{C^{0,1}}$. The space of all such functions is denoted* $\mathrm{Lip}(X_1 \to X_2)$*. In the special case $X_1 = X_2$ and $[f]_{C^{0,1}} < 1$, such a function is called a* **contraction mapping**.

Combining two results known as the Weierstrass' nondifferentiability theorem and Rademacher's differentiability theorem, we can state the following:

**Theorem 7** *There is a function $f \in C^0(\mathbb{R})$ which is not differentiable at any point, and any function $f \in C^{0,1}(\mathbb{R})$ is differentiable at* **most** *points, in the following sense: Given any $\epsilon > 0$, there exist open intervals $I_j = (a_j, b_j)$ for $j = 1, 2, 3, \ldots$ such that the set of points $N$ where $f$ is nondifferentiable satisfies*

$$N \subset \bigcup_{j=1}^{\infty} I_j$$

*and the sum of the lengths of the intervals satisfies*

$$\sum_{j=1}^{\infty}(b_j - a_j) < \epsilon.$$

*In particular, the length of* $\mathbb{R}$ *is infinite, but the points of nondifferentiability of* $f$ *must lie in a set with zero length.*

Rademacher's regularity theorem extends to $u \in C^{0,1}(U)$ for any open subset $U$ of $\mathbb{R}^n$.

## Hölder Continuity

A function $u : U \to \mathbb{R}$ defined on $U \subset \mathbb{R}^n$ is **Hölder continuous** at $\mathbf{p} \in U$ with **Hölder exponent** $\alpha \in (0,1)$ if the following condition holds:

There is some $\delta > 0$ and some $c$ for which

$$\left.\begin{array}{l} \mathbf{x} \in U \\ |\mathbf{x} - \mathbf{p}| < \delta \end{array}\right\} \quad \Longrightarrow \quad |u(\mathbf{x}) - u(\mathbf{p})| < c|\mathbf{x} - \mathbf{p}|^{\alpha}. \tag{9}$$

**Exercise 24** *Show that if* $u$ *is Hölder continuous at* $\mathbf{p}$*, then* $u$ *is continuous at* $\mathbf{p}$*.*

**Exercise 25** *Show that if* $u$ *is Lipschitz continuous at* $\mathbf{p}$*, then* $u$ *is Hölder continuous at* $\mathbf{p}$ *with any exponent* $0 < \alpha < 1$*.*

**Exercise 26** *Show* $f(x) = x^{1/3}$ *is Hölder continuous at* $x = 0$ *with exponent* $1/3$ *but not with exponent* $1/2$ *(nor satisfying (9) with* $\alpha = 1$*). How about* $u : \mathbb{R}^n \to \mathbb{R}$ *by* $u(\mathbf{x}) = |\mathbf{x}|^{1/3}$*?*

As with Lipschitz continuity there are various options for the most desirable notion of a Hölder contiuous function on a set. We can begin with the **uniformly globally Hölder functions** for which

$$[u]_{C^{0,\alpha}} = \sup_{\mathbf{p}\in U} \sup_{\mathbf{x}\in U\setminus\{\mathbf{p}\}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|^{\alpha}} = \sup_{\substack{\mathbf{p},\mathbf{x}\in U \\ \mathbf{x}\neq\mathbf{p}}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|^{\alpha}} \tag{10}$$

is finite and defines a seminorm on $C^{0,\alpha}(U)$ for which we should, perhaps, have an alternate name. There is not a standard one, but I might suggest $\text{Höl}^{\alpha}(U)$. In any case, this set has the usual problems concerning inclusion and the lack of a norm. As with the Lipschitz functions, one option for dealing with both of these problems is to consider the subspace

$$C_b^{0,\alpha}(U) = \text{Höl}_b^\alpha(U) = \{u \in C^0(U) : \|u\|_{C^0} < \infty \text{ and } [u]_{C^{0,\alpha}} < \infty\}.$$

This is a normed space with norm

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + [u]_{C^{0,\alpha}}.$$

Once we have this space of **bounded uniformly globally Hölder functions**, we may define the larger space

$$C_{loc}^{0,\alpha}(U) = \{u \in C^0(U) : u \in C^{0,\alpha}(K) \text{ for each compact set } K \subset U\}.$$

It is customary to let $C^{0,\alpha}(U)$ denote $C_{loc}^{0,\alpha}(U)$ whenever $U$ is open and to let $C^{0,\alpha}(\overline{U})$ denote $C^{0,\alpha}(\overline{U}) = C_b^{0,\alpha}(\overline{U})$ when $\overline{U}$ is compact. That is to say, many authors use these conventions.

**Exercise 27** *Let $f(x) = x^{1/3}$. Show $f \in [C_{loc}^{0,1/2}(0,1) \cap C_b^{0,1/3}(0,1)]$ but*

$$\sup_{x\in(0,1)} \sup_{t\in(0,1)\setminus\{x\}} \frac{|f(t) - f(x)|}{|t - x|^{1/2}} = \infty.$$

**Exercise 28** *Let $f(x) = x^2$ and $g(x) = \tan^{-1}(x)$.*

1. *Show that $f \in C_{loc}^{0,1/2}(\mathbb{R})$ but for each $x \in \mathbb{R}$*

$$\sup_{t\in\mathbb{R}\setminus\{x\}} \frac{|f(t) - f(x)|}{|t - x|^{1/2}} = \infty.$$

2. *Show that $g \in C_b^{0,1/2}(\mathbb{R})$ and for each $x \in \mathbb{R}$*

$$\sup_{t\in\mathbb{R}\setminus\{x\}} \frac{|g(t) - g(x)|}{|t - x|^{1/2}} < \infty.$$

**Exercise 29** *Say $u \in C^0(U)$ and $U$ is an open subset of $\mathbb{R}^n$. Show that $u \in C_{loc}^{0,\alpha}(U)$ if and only if*

*For each* $\mathbf{p} \in U$*, there is some* $\delta$ *and some* $c$ *such that*

$$\left.\begin{array}{l} \mathbf{x} \in U \\ |\mathbf{x} - \mathbf{p}| < \delta \end{array}\right\} \qquad \Longrightarrow \qquad |u(\mathbf{x}) - u(\mathbf{p})| < c|\mathbf{x} - \mathbf{p}|^\alpha.$$

*This condition is called local pointwise Hölder continuity on* $U$*.*

Having defined the Hölder classes $C^{0,\alpha}$ "between" $C^0$ and $C^1$, we are now in a position to define the Hölder fractional differentiability classes between $C^k$ and $C^{k+1}$ for the higher derivatives. Before I do that, let me (at least) state the definition of partial derivatives (which presumably you all know) and give some related notation which you probably don't know.

# 6 Partial Derivatives

The simplest case in which partial derivatives arise is that in which one has a function of two variables. Say $u = u(x, y)$ is a function of two variables defined in an open set $U$ with $\mathbf{p} = (p_1, p_2) \in U$. We defined the **first partial derivative** of $u$ in the $x$-direction at $\mathbf{p}$ to be

$$\frac{\partial u}{\partial x}(\mathbf{p}) = \lim_{h \to 0} \frac{u(p_1 + h, p_2) - u(p_1, p_2)}{h}$$

when this limit exists. We can also call this the first partial of $u$ in the $\mathbf{e}_1$ direction in honor of the standard basis vector $\mathbf{e}_1 = (1, 0)$. Notice that the limit above may also be written

$$\lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{e}_1) - u(\mathbf{p})}{h}. \qquad (11)$$

**Exercise 30** *Sketch the graph of* $u(x, y) = x^2 + y^2$*. Let* $\mathbf{p}$ *be a point in the first quadrant of the domain of* $u$*.*

1. *Along with your graph, sketch the curve*

$$\alpha(t) = \mathbf{p} + t\mathbf{e}_1 \qquad \text{for } t \in \mathbb{R}$$

*where where we interpret* $\mathbf{p}$ *as* $(p_1, p_2, 0) \in \mathbb{R}^3$ *and* $\mathbf{e}_1 = (1, 0, 0)$*.*

2. *Also sketch the curve*

$$\gamma(t) = (\mathbf{p} + t\mathbf{e}_1, u(\mathbf{p} + t\mathbf{e}_1))$$

*where here we interpret* $\mathbf{p}$ *and* $\mathbf{e}_1$ *as points in the domain of* $u$*.*

3. *Realize the second curve as the intersection of a vertical plane through the first curve.*

Your solution should look something like what you see in Figure 2.

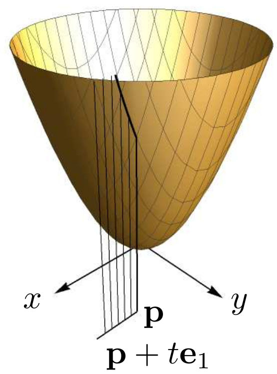


Figure 2: A curve in the plane $y = p_2$ on the graph of $u$.

The difference quotient appearing in (11) is the slope of a secant line in the plane $y = p_2$. Thus, the limit (the $x$-partial derivative of $u$) is the slope, in this plane, of the line tangent to the curve of intersection of the graph of $u$

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in U\}$$

with the vertical plane $y = p_2$. A similar interpretation applies to the other partial derivative

$$\frac{\partial u}{\partial y}(\mathbf{p}) = \lim_{h \to 0} \frac{u(\mathbf{p} + he_2) - u(\mathbf{p})}{h}.$$

More generally, we can take a curve $\alpha(t) = \mathbf{p} + t\mathbf{v}$ in any direction $\mathbf{v}$, and attempt to compute the derivative:

$$D_{\mathbf{v}} u(\mathbf{p}) = \lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{v}) - u(\mathbf{p})}{h}. \tag{12}$$

If this limit exists, it is called the **directional derivative** of $u$ at $\mathbf{p}$ in the direction $\mathbf{v}$.

**Exercise 31** *If* $\mathbf{v}$ *is a unit vector in (12), then the value of* $D_{\mathbf{v}} u(\mathbf{p})$ *is the rate of change of* $u$ *in the direction* $\mathbf{v}$. *What does* $D_{\mathbf{v}} u(\mathbf{p})$ *measure when* $\mathbf{v}$ *is a nonzero vector but does not have unit length?*

Returning to the special case where the direction of differentiation is $\mathbf{v} = \mathbf{e}_1$ or $\mathbf{e}_2$, the values

$$\frac{\partial u}{\partial x}(\mathbf{p}) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(\mathbf{p})$$

defined above are called the **first partial derivatives** of $\mathbf{u}$ at $\mathbf{p}$. There are various notations for these partial derivatives. Among them are

$$u_x \qquad \text{and} \qquad u_y,$$

$$D_1 u \qquad \text{and} \qquad D_2 u,$$

$$u_1 \qquad \text{and} \qquad u_2,$$

and

$$D^{(1,0)}u \qquad \text{and} \qquad D^{(0,1)}u. \tag{13}$$

And of course, we can interpret these partials as special directional derivatives

$$D_{\mathbf{e}_1} u \qquad \text{and} \qquad D_{\mathbf{e}_2} u.$$

The notation in (13) can be especially useful in certain contexts and we will discuss it carefully below, though it is rather cumbersome in this simple case.

Notice that the construction of taking a path in the direction of any standard unit basis vector and attempting the form the limit of a difference quotient works the same way in any dimension. More precisely, given $u : U \to \mathbb{R}$ with $U$ and open subset of $\mathbb{R}^n$ and $\mathbf{p} \in U$, the **partial derivative** of $u$ at $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is given by

$$\frac{\partial u}{\partial x_j}(bp) = \lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{e}_j) - u(\mathbf{p})}{h}$$

when this limit exists. The picture is harder to draw in higher dimensions. The construction for general **directional derivatives** (12) may also be applied with only a dimensional change.

**Exercise 32** *How do each of the notations for the first partial derivatives above change when the dimension is greater than two?*

If first partials exist at every point in an open set $U$ on which the function $u$ is defined, then they also define functions on $U$. Here is an important point:

> For functions of one variable, if the derivative of $f$ exists on an open interval $(a, b)$, then we say the function $f$ is **differentiable** and write $f \in \mathrm{Diff}(a, b)$, but...

**In more than one dimension, we do not** (usually) **identify the existence of partial derivatives with differentiability.**

Let us say that if the first partial derivatives of a function of several variables all exist on an open set $U$, then $u$ is **partially differentiable** on $U$. The following results illustrate why partially differentiable and differentiable are different things in higher dimensions.

**Theorem 8** *If $f : (a,b) \to \mathbb{R}$ is differentiable at a point $x_0 \in (a,b)$, then $f$ is continuous at $x_0$. But the first partial derivatives of $u : \mathbb{R}^2 \to \mathbb{R}$ by*

$$u(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases} \tag{14}$$

*both exist at the point $(0,0) \in \mathbb{R}^2$, but $u$ is not continuous at $(0,0)$. Therefore, partial differentiability at a point does not imply continuity.*

**Exercise 33** *Compute the first partials $u_x$ and $u_y$ at $(x,y) = (0,0)$ where $u : \mathbb{R}^2 \to \mathbb{R}$ is given by (14), and show that $u$ is not continuous at $(0,0)$.*

**Theorem 9** *If $f : (a,b) \to \mathbb{R}$ is differentiable at every point in $(a,b)$, then $f \in C^0(a,b)$. But the first partial derivatives of $u : \mathbb{R}^2 \to \mathbb{R}$ by*

$$u(x,y) = \begin{cases} xy/(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{15}$$

*both exist at all points of $\mathbb{R}^2$, but $u \notin C^0(\mathbb{R}^2)$. Therefore, partial differentiability at all points does not imply continuity.*

**Exercise 34** *Compute the first partials $u_x$ and $u_y$ at all points in the plane where $u : \mathbb{R}^2 \to \mathbb{R}$ is given by (15), and show that $u$ is not continuous at $(0,0)$.*

The following result is about differential approximation:

**Theorem 10** *If $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$, then*

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - hf'(x_0)}{h} = 0,$$

*but for either of the functions $u : \mathbb{R}^2 \to \mathbb{R}$ given by (14) or (15), one does not have*

$$\lim_{\mathbf{w}\to\mathbf{0}} \frac{u(\mathbf{p}+\mathbf{w}) - u(\mathbf{p}) - (u_x(\mathbf{p}), u_y(\mathbf{p}))\cdot\mathbf{w}}{|\mathbf{w}|} = 0 \tag{16}$$

*where* $\mathbf{p} = (0, 0)$ *and*

$$\mathbf{v} \cdot \mathbf{w} = (v_1, v_2) \cdot (w_1, w_2) = v_1 w_1 + v_2 w_2$$

*is the* **dot product**.

**Exercise 35** *Verify the assertion of this theorem.*

Notice that the limit in (16) is a somewhat different kind of limit than we have encountered before. The vector valued increment $\mathbf{w}$ can approach $\mathbf{p}$ in a much greater variety of ways than a number $x_0 + h$ can approach $x_0$ as $h \to 0$. And this makes a significant difference.

**Definition 1** *A function* $u : U \to \mathbb{R}$ *with* $U$ *an open subset of* $\mathbb{R}^n$ *and* $\mathbf{p} \in U$ *is* **differentiable** *at* $\mathbf{p}$ *if there is a linear function* $L : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lim_{\mathbf{w} \to \mathbf{0}} \frac{u(\mathbf{p} + \mathbf{w}) - u(\mathbf{p}) - L(\mathbf{w})}{|\mathbf{w}|} = 0. \tag{17}$$

*The function* $u$ *is* **differentiable** *on* $U$ *if* $u$ *is differentiable at every point* $\mathbf{p} \in U$.

**Theorem 11** *If* $u : U \to bbr$ *with* $U$ *an open subset of* $\mathbb{R}^n$ *has first order partial derivatives defined at all points in some open ball* $B_r(\mathbf{p}) \subset U$*, and* $D_j u$ *is continuous at* $\mathbf{p}$ *for* $j = 1, \ldots, n$*, then* $u$ *is differentiable at* $\mathbf{p}$.

**Exercise 36** *If* $u : U \to \mathbb{R}$ *is differentiable at* $\mathbf{p} \in U$*, then show the first partial derivatives* $D_j u(\mathbf{p})$ *exist for* $j = 1, 2, \ldots, n$*, and express the linear function* $L : \mathbb{R}^n \to \mathbb{R}$ *for which (17) holds in terms of the* **gradient vector**

$$Du(\mathbf{p}) = (D_1 u(\mathbf{p}), D_2 u(\mathbf{p}), \ldots, D_n u(\mathbf{p})).$$

*Hint: Compare (17) to (16).*

**Theorem 12** *If* $u : U \to \mathbb{R}$ *with* $U$ *an open subset of* $\mathbb{R}^n$ *has first order partial derivatives which are continuous, i.e.,* $u \in C^1(U)$*, then* $u$ *is differentiable and* $u$ *is continuous.*

We are now in a position to define the differentiability classes $C^k(U)$ for $U \subset \mathbb{R}^n$. Some aspects of these definitions formalize and extend our discussion even in $\mathbb{R}^1$. First of all, if $U$ is open,

$$C^1(U) = \{u \in C^0(U) : D_j u(\mathbf{p}) \text{ exists for every } \mathbf{p} \in U \text{ and } D_j u \in C^0(U) \text{ for } j = 1, \ldots, n \ \}.$$

For $k = 2, 3, 4, \ldots$,

$$C^k(U) = \{u : D^\beta u \in C^1(U) \text{ for } |\beta| = k - 1\}.$$

Notice that this definition implies the partial derivatives $D^\beta u(\mathbf{p})$ with $|\beta| = k - 1$ and $\mathbf{p} inU$ exist as well as the derivatives $D^\beta u$ with $|\beta| = k$.

For a general set $E \subset \mathbb{R}^n$,

$$C^k(E) = \{u \in C^0(E) : \text{ there exists an open set } U \subset \mathbb{R}^n \text{ such that } E \subset U \text{ and there exists } \overline{u} \in C^k(U) \text{ with } \overline{u}\Big|_E = u\}.$$

The spaces $C^k(U)$ and $C^k(E)$ as we have defined them, will not be normed spaces in general. Following the constructions above, we define

$$[u]_{C^1} = \sup_j \sup_{\mathbf{x}} |D_j u(\mathbf{x})|$$

which is a seminorm, called the $C^1$ seminorm, on the set of all $C^1$ functions for which the value is finite. We also define for an open set $U \subset \mathbb{R}^n$

$$C^1_b(U) = \{u \in C^1(U) : \|u\|_{C^0(U)} < \infty \text{ and } [u]_{C^1(U)} < \infty\}$$

which is a normed space with

$$\|u\|_{C^1(U)} = \|u\|_{C^0(U)} + [u]_{C^1(U)}.$$

For $k = 2, 3, \ldots$, we define the seminorm

$$[u]_{C^k} = \sup_{|\beta|=k} \sup_{\mathbf{x}} |D^\beta u(\mathbf{x})|$$

and

$$C^k_b(U) = \{u \in C^{k-1}_b(U) : [u]_{C^k(U)} < \infty\}.$$

The $C^k$ norm is given in general by

$$\|u\|_{C^k} = \sum_{j=0}^k [u]_{C^j}$$

where we take $[u]_{C^0} = \|u\|_{C^0}$.

## Taylor Expansion and Power Series

As an application of the multi-index notation for partial derivatives, we mention some useful facts about multivariable Taylor expansions and power series. This material is also covered in Assignment 2 and relates to Boas section 4.2. As in the assignment, we start with recalling how Taylor expansion works in one dimension. The **Taylor expansion** of a function

$$f \in C^\infty(\mathbb{R}) = \cap_{k=0}^\infty C^k(\mathbb{R})$$

at $x_0 \in \mathbb{R}$ is given by

$$\sum_{j=0}^\infty \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j. \tag{18}$$

Here $f^{(j)}$ denotes the $j$-th (ordinary) derivative of $f$ as usual:

$$f^{(j)} = \frac{d^j f}{dx^j}.$$

A function $f \in C^\infty(\mathbb{R})$ is said to be **real analytic** in the interval $I = (x_0 - r, x_0 + r)$ if the series in (18) converges for each $x \in I$ and

$$f(x) = \sum_{j=0}^\infty \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j.$$

The set of real analytic functions is denoted by $C^\omega$. Even for a function which is only in $C^k(a,b)$, the **Taylor approximation theorem** always holds. It is very useful and powerful.

**Theorem 13** *If $f \in C^{k+1}(a,b)$ and $x_0 \in (a,b)$, then for any $x \in (a,b)$*

$$f(x) = \sum_{j=0}^k \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + R_k(x)$$

*where $R_k = R_k(x, x^*)$ is the $k$-th order* **Taylor remainder** *given by*

$$R_k(x) = \frac{f^{(k+1)}(x^*)}{(k+1)!}(x-x_0)^{k+1}$$

*and $x^*$ is some (unknown) point between $x$ and $x_0$. Generally, one can say $x^*$ depends (in some complicated unknown way) on $x$, but it is often useful to consder $R_n$ as a function of both $x$ and $x^*$ (and $x_0$ as well) since the dependence on each of these "variables" is explicitly known.*

The question of whether a $C^\infty$ function is real analytic is precisely the question of whether or not one has

$$\lim_{k\to\infty} R_k(x) = 0.$$

This will generally be true in some open interval $B_r(x_0) = (x_0 - r, x_0 + r)$, though for actual convergence, one can only ensure $r \geq 0$. If $r = 0$, then this (open) interval is empty. You still get trivial convergence for $x = x_0$, but of course that tells you nothing. More generally, using the Taylor approximation formula effectively boils down to getting an estimate (a uniform estimate on some interval) for the $k+1$ order derivative.

The basic outline of these results holds in any dimension, though the set of convergence becomes a general ball $B_r(\mathbf{x}_0)$ and you need estimates for all partial derivatives of order $k+1$. I will not state the results in detail.

The **Taylor expansion** of a function $u \in C^\infty(U)$ at $\mathbf{x}_0 \in U \subset \mathbb{R}^n$ is given by

$$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta. \tag{19}$$

In this expansion formula $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a **multi-index**, which simply means

$$\beta \in \mathbb{N}^n = \{(m_1, \ldots, m_n) : m_1, \ldots, m_n \in \mathbb{N}\} \qquad \text{where} \qquad \mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The derivative $D^\beta u$ denotes the partial derivative taken $\beta_j$ times with respect to $x_j$ for each $j = 1, 2, \ldots, n$:

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}.$$

The "length" of a multi-index $\beta$ is defined by

$$|\beta| = \sum_{j=1}^{n} \beta_j.$$

The **factorial** of a multi-index $\beta$ is given by

$$\beta! = \beta_1! \beta_2! \cdots \beta_n!.$$

The **multi-index power** of a vector variable $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is

$$\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}.$$

A function $u \in C^\infty(\mathbb{R}^n)$ is said to be **real analytic** in the ball $B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$ if the series in (19) converges for each $\mathbf{x} \in B_r(\mathbf{p})$ and

$$u(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta.$$

The set of real analytic functions denoted by $C^\omega(U)$ where $U$ is an open subset of $\mathbb{R}^n$ consists of those functions $u$ for which $u$ is real analytic on some ball centered at each point $\mathbf{x}_0 \in U$.

# 7 Hölder Continuity and the $C^{k,\alpha}$ spaces.

We have already given the basic condition (9) for a function to be Hölder continuous. There are two main settings in which this condition can be used to construct a space of functions which is normed and lies withing a nested continuum of spaces measuring fractional differentiability. Naturally, for any function $u$ defined on any set $U$ (consisting of at least two points) it makes sense to consider the Hölder seminorm given in (10) by

$$[u]_{C^{0,\alpha}} = \sup_{\mathbf{p} \in U} \sup_{\mathbf{x} \in U \setminus \{\mathbf{p}\}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|^\alpha} = \sup_{\substack{\mathbf{p},\mathbf{x} \in U \\ \mathbf{x} \neq \mathbf{p}}} \frac{|u(\mathbf{x}) - u(\mathbf{p})|}{|\mathbf{x} - \mathbf{p}|^\alpha}.$$

This will be a seminorm on the set $\text{Höl}_b^\alpha(U)$ where the value is finite and will be a term in the $C^{0,\alpha}$ Hölder norm

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + [u]_{C^{0,\alpha}} \tag{20}$$

whenever that norm is well-defined. The first situation of primary interest is when $U$ is an open subset of $\mathbb{R}^n$ and

$$C_b^{0,\alpha}(U) = \{u \in C^0(U) : \|u\|_{C^0} < \infty \text{ and } [u]_{C^{0,\alpha}} < \infty\}.$$

In this case, $C_b^{0,\alpha}(U)$ is a normed space with norm given by (20), and we have

$$C_b^0(U) \supset C_b^{0,\alpha}(U) \supset C_b^{0,\gamma}(U) \supset C_b^1(U)$$

for $0 < \alpha \leq \gamma < 1$. We can also define for $k = 1, 2, \ldots$

$$C_b^{k,\alpha}(U) = \{u \in C_b^k(U) : [D^\beta u]_{C^{0,\alpha}} < \infty \text{ for } |\beta| = k\}.$$

This is a normed space with

$$\|u\|_{C^{k,\alpha}(U)} = \|u\|_{C^k(U)} + [u]_{C^{k,\alpha}(U)} \qquad \text{where} \qquad [u]_{C^{k,\alpha}(U)} = \sup_{|\beta|=k} [D^\beta u]_{C^{0,\alpha}(U)}. \quad (21)$$

Here we have

$$C_b^k(U) \supset C_b^{k,\alpha}(U) \supset C_b^{k,\gamma}(U) \supset C_b^{k+1}(U)$$

for $0 < \alpha \leq \gamma < 1$. Notice that these spaces admit

$$\tan^{-1} \in \cap_{k,\alpha} C^{k,\alpha}(\mathbb{R}).$$

The next situation of primary interest is when $U$ is an open subset $\mathbb{R}^n$ whose closure $\overline{U}$ is compact. In this case, we define

$$C^{k,\alpha}(\overline{U}) = \{u \in C^k(U) : D^\beta u \text{ has an extension } w_\beta \in C^0(\overline{U}) \text{ for } 0 \leq |\beta| \leq k$$
$$\text{and } w_\beta \in C^{0,\alpha}(\overline{U}) \text{ for } |\beta| = k\}.$$

This is a normed space with the same norm given in (21) and we have

$$C_b^k(\overline{U}) \supset C_b^{k,\alpha}(\overline{U}) \supset C_b^{k,\gamma}(\overline{U}) \supset C_b^{k+1}(\overline{U})$$

for $0 < \alpha \leq \gamma < 1$ as desired. Notice that by denoting the set using the closure notation, $\overline{U}$, we are implicitly requiring that the closure be compact.

There are two or three other spaces of Hölder continuous functions which are in common use. These are now easy to describe, though our primary interest is in the two above. For example, given an arbitrary subset $E \subset \mathbb{R}^n$, on may consider

$$C_b^{k,\alpha}(E) = \{u \in C_b^k(E) : \text{ there exists an open set } U \subset \mathbb{R}^n \text{ such that } E \subset U$$
$$\text{and there exists } \overline{u} \in C_b^{k,\alpha}(U) \text{ with } \overline{u}\big|_E = u\}.$$

This is a normed space with the desired continuum nesting properties, though it is not commonly used. If the set $E$ in $C_b^{k,\alpha}(E)$ is compact, then the condition of boundedness need not be explicit. That is, for a compact set $K \subset \mathbb{R}^n$, we can set

$$C^{k,\alpha}(K) = \{u \in C^k(K) : \text{ there exists an open set } U \subset \mathbb{R}^n \text{ such that } K \subset U$$
$$\text{and there exists } \overline{u} \in C^{k,\alpha}(U) \text{ with } \overline{u}\big|_E = u\}.$$

One that is used very often is

$$C_{loc}^{k,\alpha}(U) = \{u \in C^k(U) : u \in C^{k,\alpha}(K) \text{ for each compact set } K \subset U\}.$$

This is generally not a normed space, but one may generally restrict to some open subset of $U$ whose closure is compact.