Lecture 20
Weak Solutions for Poisson's Equation
Existence and Uniqueness

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I want to prove the existence and uniqueness of weak solutions for the problem

$$\begin{cases} -\Delta u = f, & \text{on } U \\ u\Big|_{\partial U} \equiv= 0. \end{cases} \tag{1}$$

Theorem 1 (existence and uniqueness) *Given any bounded open set $U \subset \mathbb{R}^n$ and any $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U) = W_0^{1,2}(U)$ of the boundary value problem (1) for Poisson's equation with homogeneous boundary values.*

Before discussing and proving this theorem, I will state a complementary related result:

Theorem 2 (regularity) *If $U \subset \mathbb{R}^n$ is an open bounded domain with ∂U a smooth (C^∞) $n-1$ dimensional submanifold*[1] *and $f \in C^\infty(\overline{U})$, then the unique weak solution from the existence and uniqueness theorem is a classical solution of (1) with $u \in C^\infty(\overline{U})$.*

Let me say at the outset that, while the existence and uniqueness theorem is (in the main) a relatively "soft" result relying fundamentally on abstract linear/functional analysis techniques (which are considered "easy" compared to "hard" analysis like that required to obtain the regularity result for example) there is still quite a lot involved, and maybe it is a little ambitious to present such a result. There is a lot to

[1] A smooth **curve** or collection of curves if $U \subset \mathbb{R}^2$, a smooth **surface** if $U \subset \mathbb{R}^3$, etc..

keep track of. We have covered most of the ingredients in some context, and there are a few new things. In any case, this is a good time and opportunity to make sure the necessary ideas are consolidated and understood.

The proof may be divided into three parts.

1. Identify a bounded linear functional on $H_0^1(U)$.
2. Identify a non-standard inner product on $H_0^1(U)$.
3. Apply the Riesz representation theorem to obtain existence and uniqueness.

In this lecture I will begin with a review of a preliminary part which could be prepended to our list above as

0. Give a weak formulation of the problem.

1 Weak Formulation

Remember that the classical formulation says something like this: Find a function $u \in C^2(U) \cap C^0(\overline{U})$ satisfying (1). The corresponding precise weak formulation is

Find $u \in H_0^1(U)$ with

$$\int_U Du \cdot D\phi = \int_U f\,\phi \qquad \text{for every } \phi \in C_c^\infty(U). \tag{2}$$

This looks like a weak formulation obtained using integration by parts on a presumed classical solution:

$$\int_U (-\Delta u)\,\phi = \int_U f\,\phi.$$

Exercise 1 *Use the divergence theorem to show that for a classical solution $u \in C^2(\overline{U})$ of (1) the condition (2) holds.*

Questions you should be asking: What is $H_0^1(U)$ and where does the homogeneous boundary condition come in with respect to the weak formulation? The answers to these questions are related.

First of all, it should be noted that we only integrated by parts once to obtain the weak formulation, and the derivatives Du appearing on the left in (2) are weak first partial derivatives. That is Du is a **weak gradient** or a vector of weak first partials.

You should know about these and know the space in which they may be found is the space of weakly differentiable functions $W^1(U)$ or $W^1_{loc}(U)$. In this particular case, we are starting out with $H^1(U) = W^{1,2}(U)$ which is the space of weakly differentiable functions which are **square integrable**

$$\int_U |u|^2 < \infty$$

and with weak first partial derivatives (i.e., "one" weak derivative) in $L^2(U)$. The point of this is that $L^2(U)$ and consequently $W^{1,2}(U)$ is an **inner product space** with inner product

$$\langle u, v \rangle_{W^{1,2}} = \int_U u\, v + \sum_{j=1}^n \int_U Du_j \cdot Dv_j. \tag{3}$$

In fact, $H^1(U) = W^{1,2}(U)$ is a **complete** inner product space or **Hilbert space**.

This much you should, more or less, know. And you should pause and make sure you know it. In particular, (and I repeat) you should be aware of and appreciate the fact that the derivatives appearing in (3) are weak derivatives.

The space $H^1_0(U) = W^{1,2}_0(U)$ may be somewhat new to you, though I think I mentioned it before. Now, let's make sure we understand it. We have an inner product on $H^1(U)$. This means we have a norm and a distance. This set of functions is a **metric space**, and we know about open sets, closed sets, convergence of sequences and all the notions that make sense in a metric space. In particular, the intersection of all closed sets containing a particular subset $A \subset H^1(U)$ is the **closure** of that set and it is well defined as the smallest closed set $\overline{A}$ containing the set A. This is topology which should be familiar at least for sets in $\mathbb{R}^n$, but here we're applying it in a vector space of functions.

To get $H^1_0(U)$ we take $A = C^\infty_c(U)$, which is a very familiar vector space. It is also quite clear that every function $\phi \in C^\infty_c(U)$ satisfies $\phi \in L^2(U)$ and the classical partials $D_j\phi$ are weak partials with $D_j\phi \in L^2(U)$ as well.

$$H^1_0(U) = \overline{C^\infty_c(U)}$$

where the closure is taken with respect to the $W^{1,2}$ norm

$$\|u\|_{W^{1,2}} = \sqrt{\int_U |u|^2 + \int_U |Du|^2}.$$

You know $A = C_c^\infty(U)$ is not only a subset of $H^1(U)$, but it is a **subspace**. It should be no surprise that $H_0^1(U)$, therefore, is a **closed subspace**. It is a standard theorem from functional analysis, furthermore, that a closed subspace in a Hilbert space is also metrically complete with respect to the same inner product norm (restricted to the subspace). Thus, $H_0^1(U)$ is a Hilbert space with respect to (3) as well.

Finally, if you think about it, looking for a (weak) solution in $H_0^1(U) = \overline{C_c^\infty(U)}$ is a pretty reasonable way to weakly formulate the boundary condition

$$u\Big|_{\partial U} \equiv 0. \tag{4}$$

Exercise 2 *If $U \subset \mathbb{R}^2$ has ∂U a C^2 curve and $u \in C^2(\overline{U})$ satisfies (4) classically then show $u \in H_0^1(U)$. Hint: Consider $U_\delta = \{\mathbf{x} \in U : \operatorname{dist}(\mathbf{x}, \partial U) > \delta\}$ and mollify $u\chi_{U_\delta}$.*

The crucial converse also holds: If $u \in H_0^1(U) \cap C^0(\overline{U})$, then (4) holds.

I think we're in a position to start the proof.

2 Preliminaries

The weak formulation can be written as

$$B[u, \phi] = \langle f, \phi \rangle \qquad \text{for all } \phi \in C_c^\infty(U) \tag{5}$$

where $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ by

$$B[u, v] = \int_U Du \cdot Dv.$$

This bilinear form B will immediately be recognized, on the one hand, as consisting of terms from the H^1 inner product. It is also directly related to the standard $W^{1,2}$ seminorm:

$$[u]_{W^{1,2}} = \sqrt{\int_U |Du|^2}.$$

We will observe in the second step that, in fact, B is (due to the restriction to $H_0^1(U)$) unexpectedly a bona fide inner product, different from but equivalent to the standard inner product (3). But we're getting a little ahead of ourselves.

The first step, as listed above, is to "identify a bounded linear functional." The bounded linear functional is on the right side in (2) and (5): Consider $\ell : H_0^1(U) \to \mathbb{R}$ by

$$\ell(v) = \int_U f\, v.$$

When we use "bounded" here it is in the sense of operators, as in $\sqsupset^0(H_0^1(U) \to \mathbb{R}) = \sqsupset^0(H_0^1(U)) = [H_0^1(U)]^*$. If you don't remember what that means, now is a good time to think about it. It means there is a constant C for which

$$|\ell(v)| \le C\|v\|_{H^1} = C\|v\|_{W^{1,2}} \qquad \text{for all } v \in H_0^1(U). \tag{6}$$

Notice that this is something like being Lipschitz, except that the domain here is some infinite dimensional function space instead of a subset of $\mathbb{R}^n$.

Exercise 3 *Show that if ℓ satisfies (6), then $\ell \in C^0(H_0^1(U))$.*

To see that (6) holds, we can simply use the Cauchy-Schwarz inequality in the Hilbert space $L^2(U)$:

$$|\ell(v)| = \left|\int_U f\, v\right| \le \int_U |f|\,|v| \le \|f\|_{L^2(U)}\|v\|_{L^2(U)} \le \|f\|_{L^2(U)}\|v\|_{H^1(U)}.$$

Thus, we take $C = \|f\|_{L^2(U)}$ which is a fixed constant, and we've got (6).

Step 1 is now complete. This is probably adequate for one lecture. The next thing I'm going to do is give an extended discussion of preliminaries for the third step. That is, the representation of bounded linear functionals on a Hilbert space and the Riesz representation theorem.