

Method of Characteristics: Examples

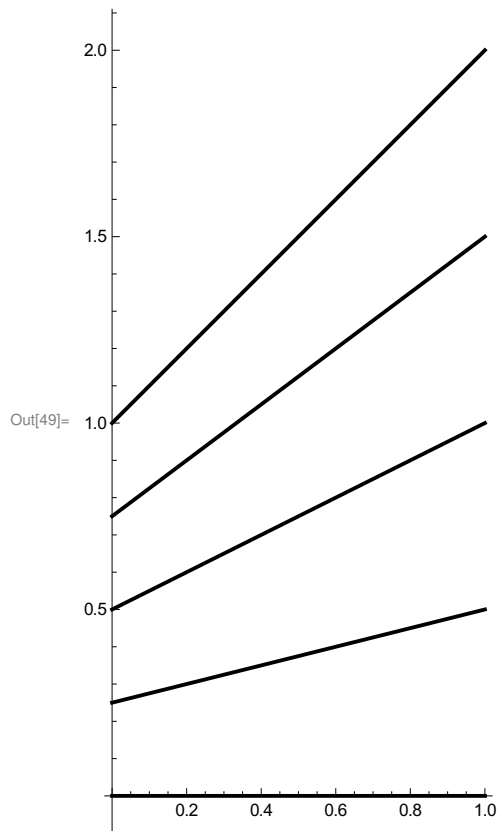
The quasilinear first order PDE $uu_x + u_t = 0$

A rarefaction wave

In[21]:= $u[x_, t_] = x / (1 + t)$

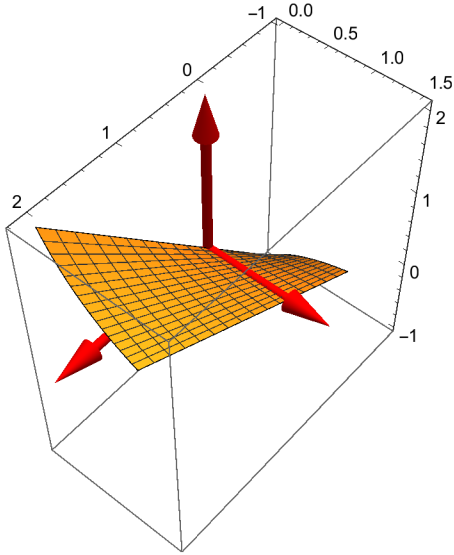
Out[21]= $\frac{x}{1 + t}$

In[49]:= `Plot[{0 + 0 t, 0.25 + 0.25 t, 0.5 + 0.5 t, 0.75 + 0.75 t, 1 + t}, {t, 0, 1},
PlotStyle -> {{Thick, Black}, {Thick, Black}, {Thick, Black}, {Thick, Black}},
AspectRatio -> Automatic]`



```
In[40]:= rare1 = Show[Plot3D[u[x, t], {x, -1, 2}, {t, 0, 1}, BoxRatios -> Automatic],  
Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{0, 0, 0}, {0, 1.5, 0}], 0.05]}],  
Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{0, 0, 0}, {2.1, 0, 0}], 0.05]}],  
Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{0, 0, 0}, {0, 0, 2}], 0.05]}],  
PlotRange -> All]
```

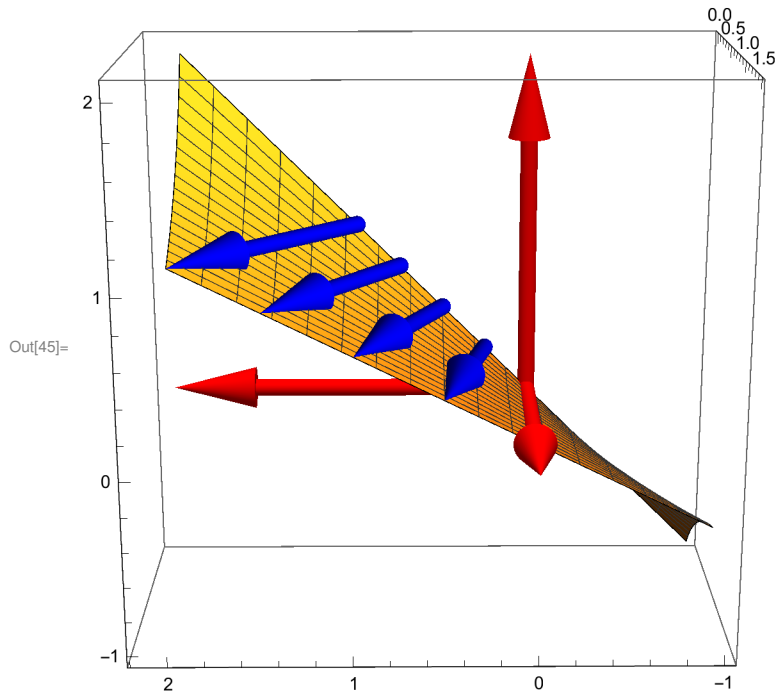
Out[40]=



```

In[45]:= Show[rare1, Graphics3D[{Blue, Arrowheads[0.1],
  Arrow[Tube[{{0.25, 0, 0.25}, {0.25 + 0.25, 1, 0.25}}, 0.05]]}], Graphics3D[
  {Blue, Arrowheads[0.1], Arrow[Tube[{{0.5, 0, 0.5}, {0.5 + 0.5, 1, 0.5}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1],
  Arrow[Tube[{{0.75, 0, 0.75}, {0.75 + 0.75, 1, 0.75}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{1, 0, 1}, {2, 1, 1}}, 0.05]]}],
PlotRange -> All]

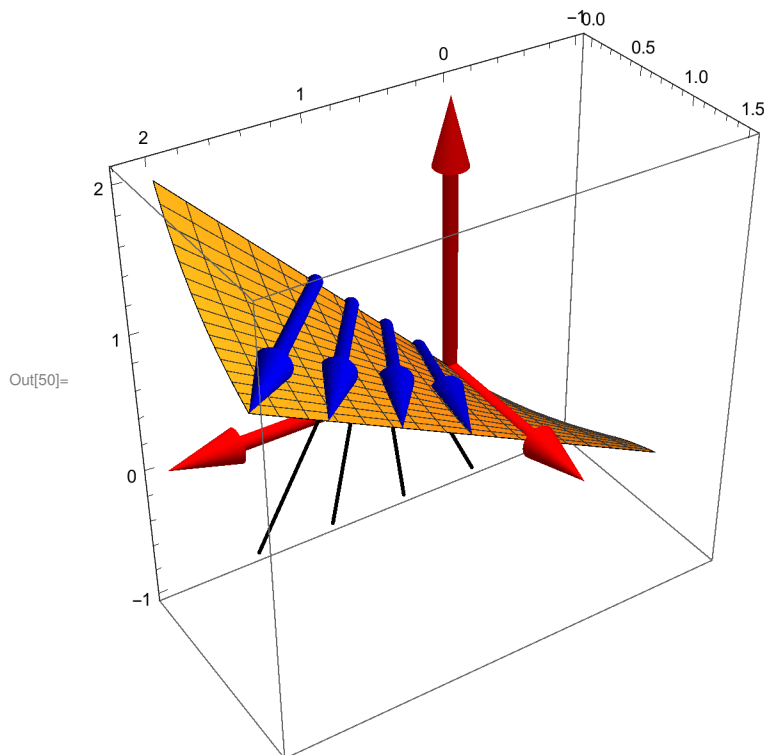
```



```

In[50]:= Show[rare1, Graphics3D[{Blue, Arrowheads[0.1],
  Arrow[Tube[{{0.25, 0, 0.25}, {0.25 + 0.25, 1, 0.25}}, 0.05]]]}, Graphics3D[
  {Blue, Arrowheads[0.1], Arrow[Tube[{{0.5, 0, 0.5}, {0.5 + 0.5, 1, 0.5}}, 0.05]]]},
Graphics3D[{Blue, Arrowheads[0.1],
  Arrow[Tube[{{0.75, 0, 0.75}, {0.75 + 0.75, 1, 0.75}}, 0.05]]]},
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{1, 0, 1}, {2, 1, 1}}, 0.05]]]},
ParametricPlot3D[{0.25 + 0.25 t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{0.5 + 0.5 t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{0.75 + 0.75 t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{1 + t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
PlotRange -> All]

```



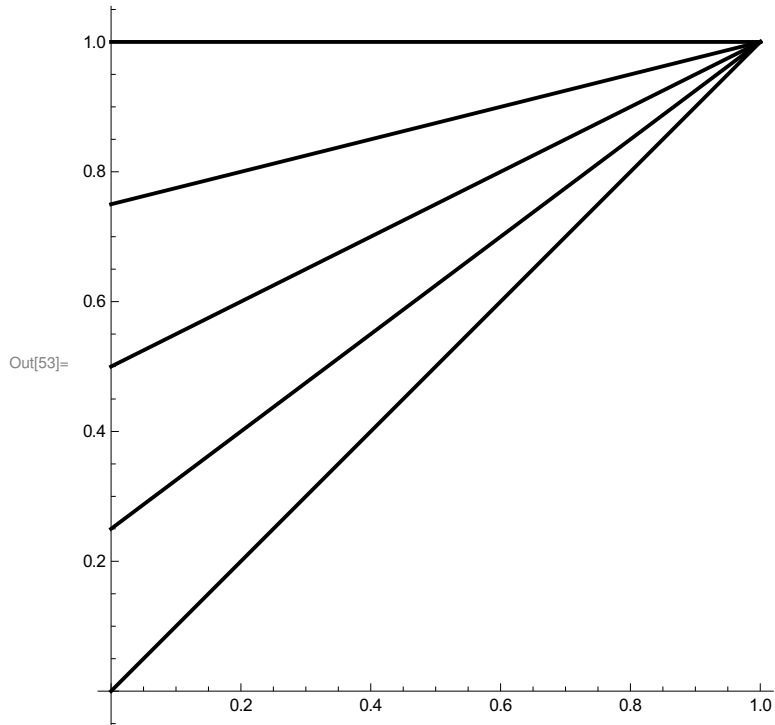
A compression wave

```
In[57]:= u[x_, t_] = (1 - x) / (1 - t)
```

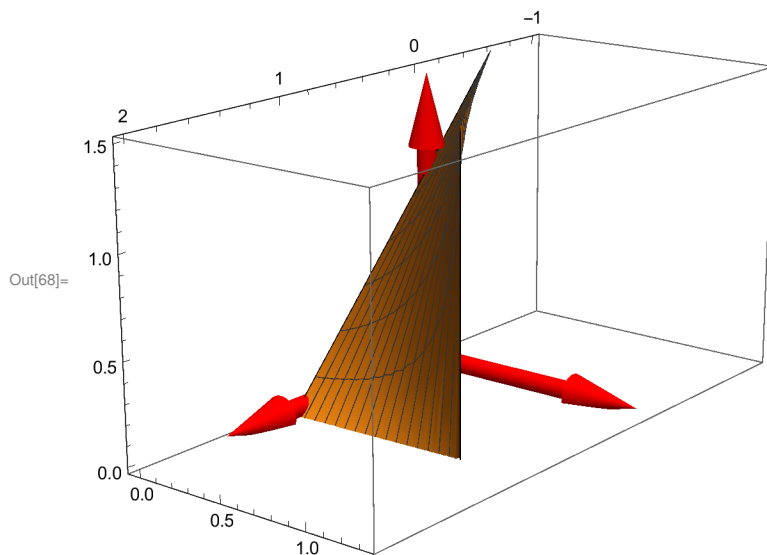
```
Out[57]= 
$$\frac{1 - x}{1 - t}$$

```

```
In[53]:= Plot[{0 + (1 - 0) t, 0.25 + (1 - 0.25) t, 0.5 + (1 - 0.5) t, 0.75 + (1 - 0.75) t, 1}, {t, 0, 1},
  PlotStyle -> {{Thick, Black}, {Thick, Black}, {Thick, Black}, {Thick, Black}},
  AspectRatio -> Automatic]
```



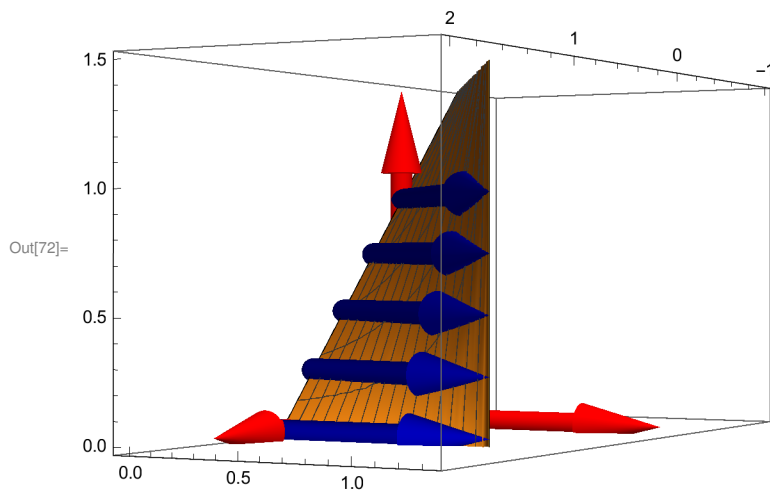
```
In[68]:= compress1 = Show[Plot3D[u[x, t], {x, -1, 2}, {t, 0, 1}, BoxRatios -> Automatic],
  Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{{0, 0, 0}, {0, 1.3, 0}}, 0.05]]}],
  Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{{0, 0, 0}, {1.5, 0, 0}}, 0.05]]}],
  Graphics3D[{Red, Arrowheads[0.1], Arrow[Tube[{{0, 0, 0}, {0, 0, 1.5}}, 0.05]]}],
  PlotRange -> {0, 1.5}]
```



```

In[72]:= Show[compress1,
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{0, 0, 1}, {1, 1, 1}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1], Arrow[
Tube[{{0.25, 0, 1 - 0.25}, {0.25 + (1 - 0.25), 1, 1 - 0.25}}, 0.05]]}], Graphics3D[
{Blue, Arrowheads[0.1], Arrow[Tube[{{0.5, 0, 0.5}, {0.5 + 0.5, 1, 0.5}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1],
Arrow[Tube[{{0.75, 0, 1 - 0.75}, {0.75 + (1 - 0.75), 1, 1 - 0.75}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{1, 0, 0}, {1, 1, 0}}, 0.05]]}]]]

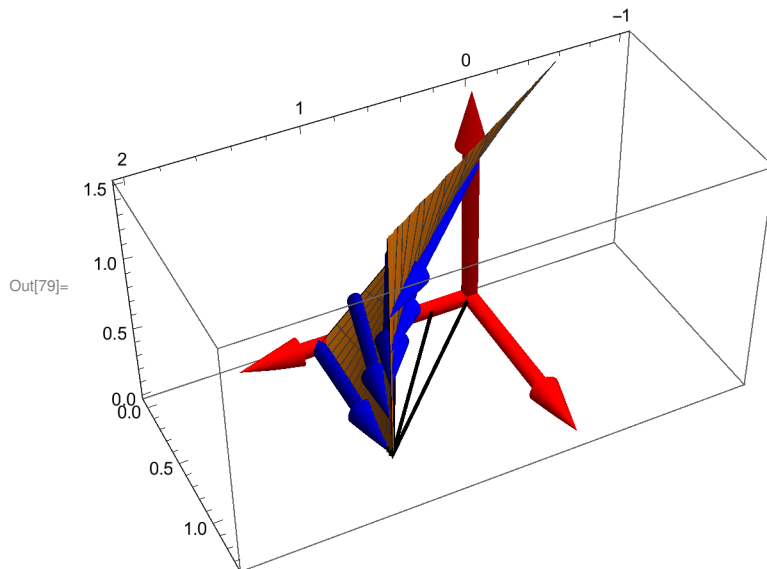
```



```

In[79]:= Show[compress1,
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{0, 0, 1}, {1, 1, 1}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1], Arrow[
Tube[{{0.25, 0, 1 - 0.25}, {0.25 + (1 - 0.25), 1, 1 - 0.25}}, 0.05]]}], Graphics3D[
{Blue, Arrowheads[0.1], Arrow[Tube[{{0.5, 0, 0.5}, {0.5 + 0.5, 1, 0.5}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1],
Arrow[Tube[{{0.75, 0, 1 - 0.75}, {0.75 + (1 - 0.75), 1, 1 - 0.75}}, 0.05]]}],
Graphics3D[{Blue, Arrowheads[0.1], Arrow[Tube[{{1, 0, 0}, {1, 1, 0}}, 0.05]]}],
ParametricPlot3D[{0 + t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{0.25 + (1 - 0.25) t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{0.5 + (1 - 0.5) t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{0.75 + (1 - 0.75) t, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}],
ParametricPlot3D[{1, t, 0}, {t, 0, 1}, PlotStyle -> {Thick, Black}]]

```



The (nonhomogeneous) linear first order PDE

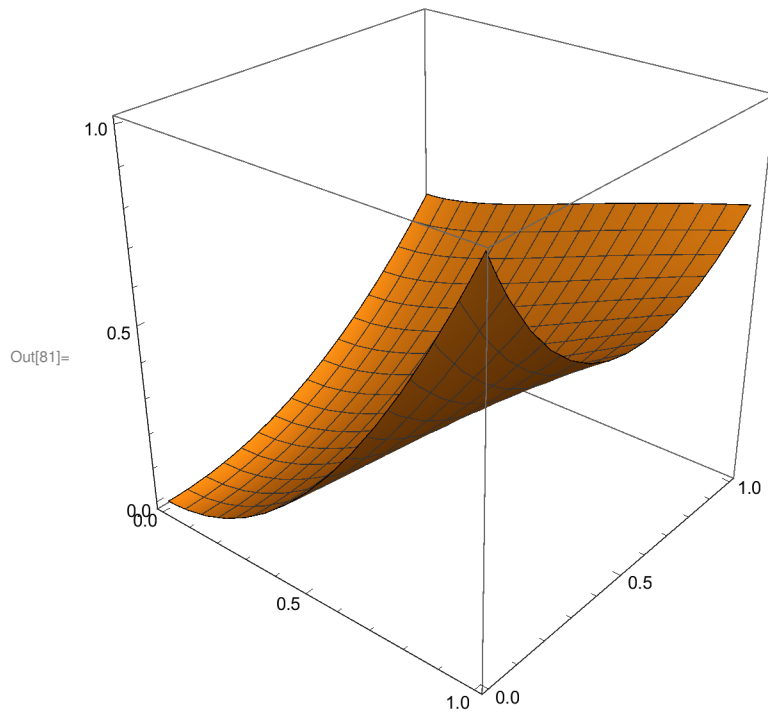
$$x^2 u_x + u_t = t$$

```
In[80]:= u[x_, t_] = (x / (x t + 1))^2 + t^2 / 2
```

```
Out[80]= 
$$\frac{t^2}{2} + \frac{x^2}{(1 + t x)^2}$$

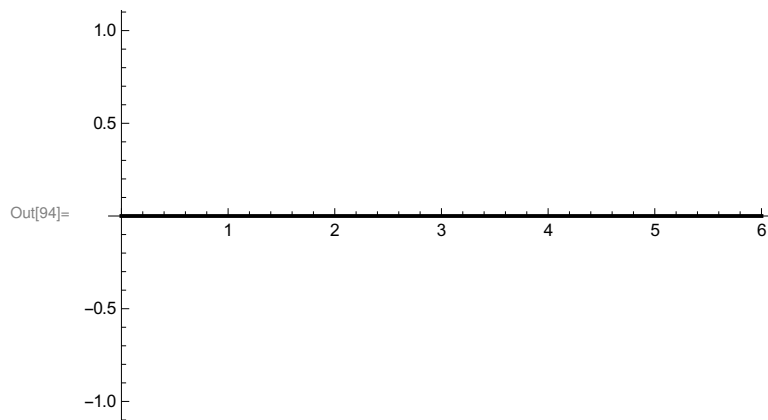
```

```
In[81]:= Plot3D[u[x, t], {x, 0, 1}, {t, 0, 1}, BoxRatios -> Automatic]
```

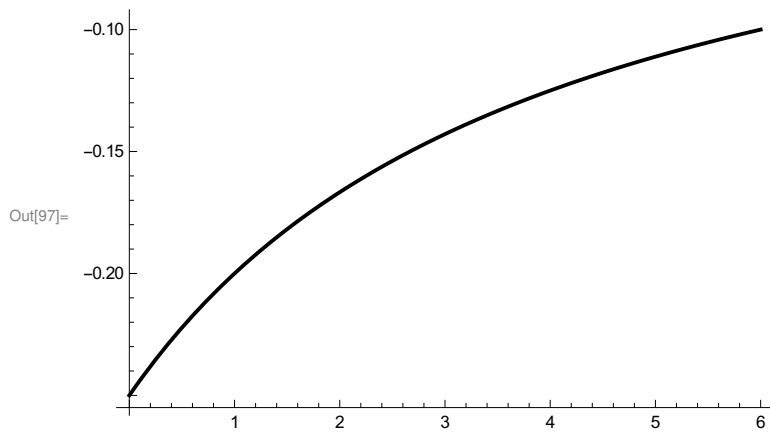


The Characteristics

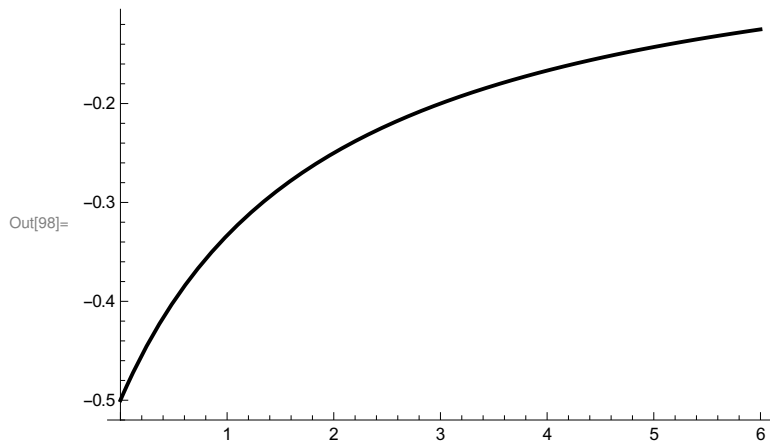
```
In[94]:= char1 = Plot[0, {t, 0, 6}, PlotStyle -> {Thick, Black}]
```



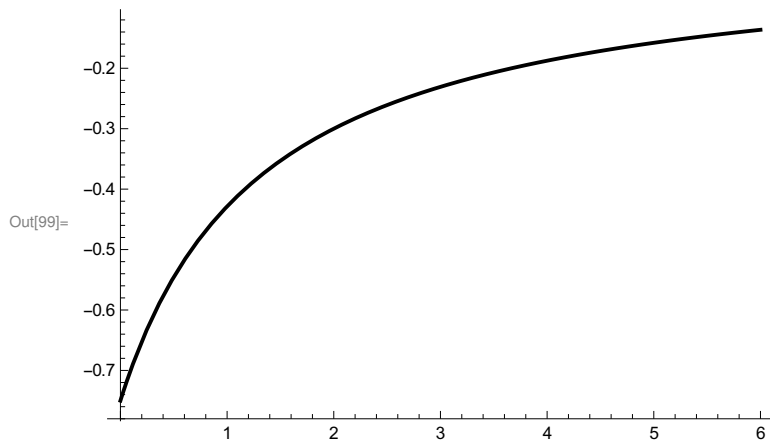

```
In[97]:= char2 = Plot[-0.25 / (1 - (-0.25) t), {t, 0, 6}, PlotStyle -> {Thick, Black}]
```



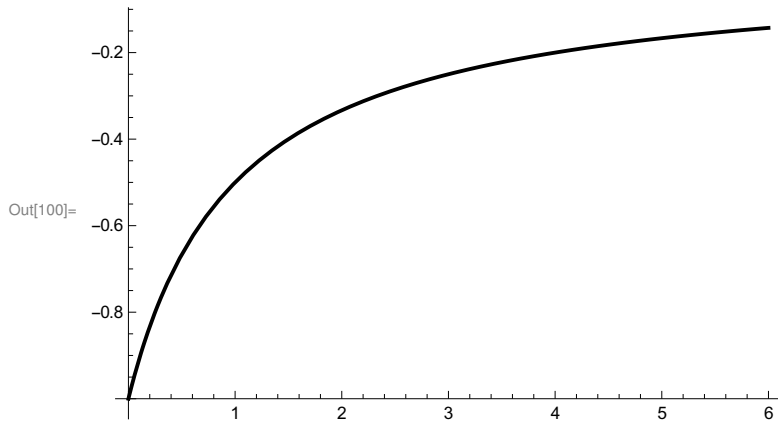
```
In[98]:= char3 = Plot[-0.5 / (1 - (-0.5) t), {t, 0, 6}, PlotStyle -> {Thick, Black}]
```



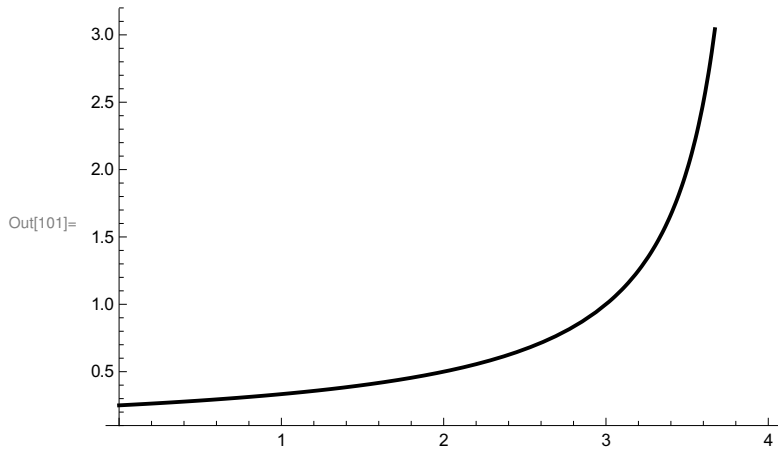
```
In[99]:= char4 = Plot[-0.75 / (1 - (-0.75) t), {t, 0, 6}, PlotStyle -> {Thick, Black}]
```



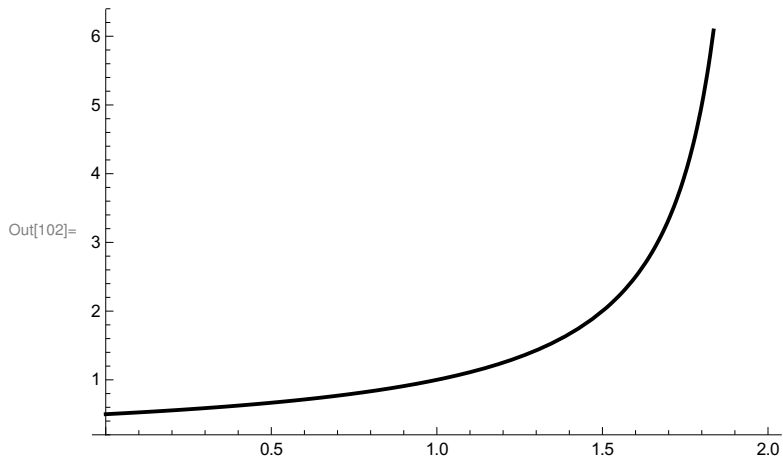
```
In[100]:= char5 = Plot[-1 / (1 - (-1) t), {t, 0, 6}, PlotStyle -> {Thick, Black}]
```



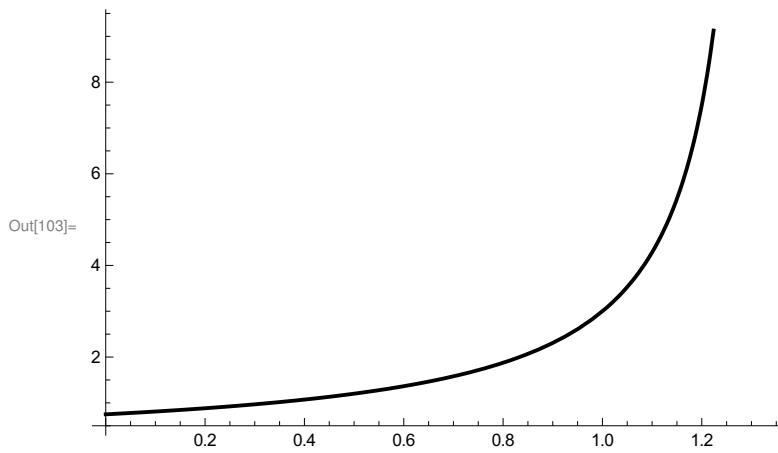
```
In[101]:= chara = Plot[0.25 / (1 - (0.25) t), {t, 0, 1 / 0.25}, PlotStyle -> {Thick, Black}]
```



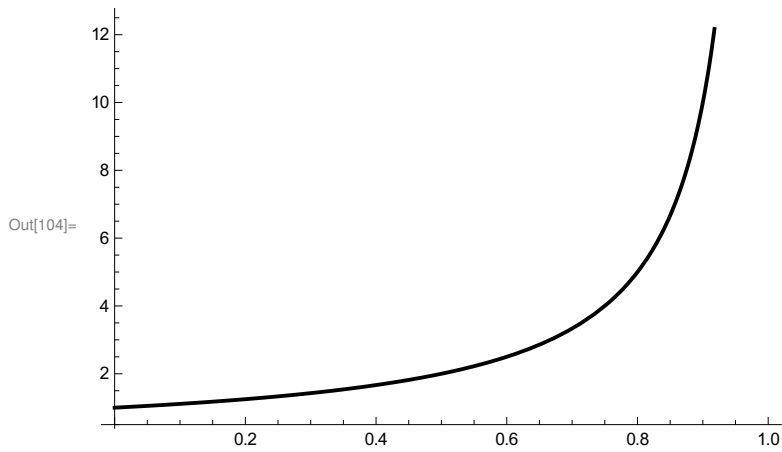
```
In[102]:= charb = Plot[0.5 / (1 - (0.5) t), {t, 0, 1 / 0.5}, PlotStyle -> {Thick, Black}]
```



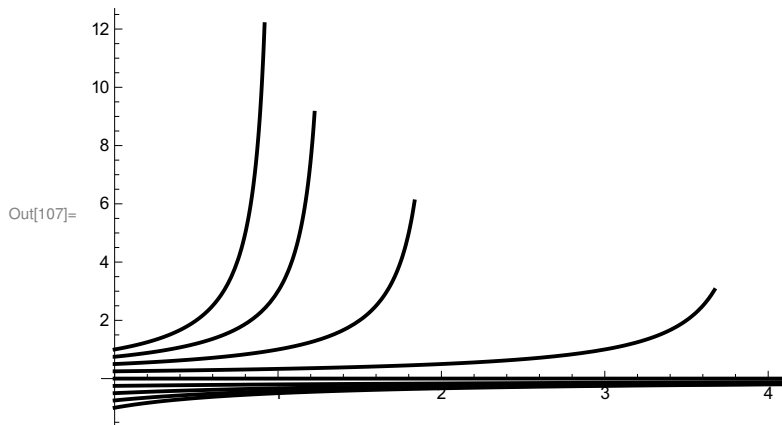
```
In[103]:= charc = Plot[0.75 / (1 - (0.75) t), {t, 0, 1 / 0.75}, PlotStyle -> {Thick, Black}]
```



```
In[104]:= chard = Plot[1 / (1 - (1) t), {t, 0, 1 / 1}, PlotStyle -> {Thick, Black}]
```



```
In[107]:= Show[char1, char2, char3, char4, char5, chara,
charb, charc, chard, PlotRange -> {{0, 4}, {-1, 12}}]
```



```
In[116]:= Show[char1, char2, char3, char4, char5, chara, charb, charc, chard,  
  ParametricPlot[{-x, 1/x}, {x, -3, 0}, PlotStyle -> {Thick, Red}],  
  PlotRange -> {{0, 4}, {-3, 12}}]
```

