MATH 6702 Assignment 8 = Challenge Problems
Due Friday May 7, 2021
Solutions and Notes

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There are two challenge problems below. The first is quite involved to a large extent due to technical issues (and has some difficult points as well), and the second is much easier and less technical but still definitely challenging for most students at your level. They are in this order to correspond to the order of material presented on partial differential equations in the course, namely, the first problem is about gradient flow in infinite dimensions (we presented the heat equation as a gradient flow on a subset/subspace of $L^2(\mathcal{U})$), and the second problem concerns the derivation of the wave equation (this derivation was not presented in the lecture, but we discussed some aspects of the wave equation at the end of the course). I wanted to put a problem like the first one on the final exam, but I didn't because I thought the exam was already long enough and would probably turn out to be long, involved, and difficult if I put such a problem. (I was correct about all three of those things.) I had prepared most of the material for the second problem for the last/extra lecture, but we didn't get to it. Both variational constructions are quite beautiful I think.

Infinite Dimensional Gradient Flow

Problem 1 *We will denote by $\mathbb{S}^2$ the two-dimensional unit sphere in $\mathbb{R}^3$. That is,*

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

It should be clear that we can take $\mathbb{S}^2$ both as a domain of integration and as a domain for real valued functions $f : \mathbb{S}^2 \to \mathbb{R}$. In fact, the set of continuously differentiable functions $f \in C^1(\mathbb{S}^2)$ makes good sense as does $C^2(\mathbb{S}^2)$.

Calculus/Differentiability on a Sphere

(a) *Give two different ways to define what it means for a function $f : \mathbb{S}^2 \to \mathbb{R}$ to be continuously differentiable and show they are equivalent. Here are some hints if you need them*

(i) *Consider extensions $\overline{f} : \mathcal{V} \to \mathbb{R}$ of f to an open subset $\mathcal{V} \subset \mathbb{R}^3$ with $\mathbb{S}^2 \subset \mathcal{V}$. In order to decide if $f \in C^1(\mathbb{S}^2)$ consider whether or not there exists an extension $\overline{f} \in C^1(\mathcal{V})$.*

Solution: A function $f : \mathbb{S}^2 \to \mathbb{R}$ is in $C^1(\mathbb{S}^2)$ if there exists an open subset $\mathcal{V} \subset \mathbb{R}^3$ with $\mathbb{S}^2 \subset \mathcal{V}$ and an extension $\overline{f} : \mathcal{V} \to \mathbb{R}$ satisfying

$$\overline{f} \in C^1(\mathcal{V}) \qquad \text{and} \qquad \overline{f}\big|_{\mathbb{S}^2} \equiv f.$$

(ii) *Consider compositions $f \circ X : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U} \subset \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $X \in C^1(\mathcal{U} \to \mathbb{S}^2)$. The definition of C^1 you get should only use values of f that are given on $\mathbb{S}^2$.*

Solution: A function $f : \mathbb{S}^2 \to \mathbb{R}$ is in $C^1(\mathbb{S}^2)$ if $f \circ X \in C^1(\mathcal{U})$ for any $X \in C^1(\mathcal{U} \to \mathbb{S}^2)$ defined on an an open set $\mathcal{U} \subset \mathbb{R}^2$.

To show the two definitions above are equivalent, note that it is clear the existence of a C^1 extension $\overline{f}$ gives the regularity of $f \circ X$ because $f \circ X = \overline{f} \circ X$ (and a composition of C^1 functions on open subsets of Euclidean spaces is C^1. To see that the second definition implies the first, consider the extension

$$\overline{f}(\mathbf{x}) = f(\mathbf{x}/|\mathbf{x}|) \qquad \text{for } 1/2 < |\mathbf{x}| < 3/2.$$

In order to show $\overline{f} \in C^1(\mathcal{V})$ where

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 : 1/2 < |\mathbf{x}| < 3/2\},$$

it is enough to show that for each $\mathbf{p} \in \mathcal{V}$, there is some open set $\mathcal{V}_0$ with $\mathbf{p} \in \mathcal{V}_0$ and $\overline{f} \in C^1(\mathcal{V}_0)$.

Take, for example, the situation where $\mathbf{p} = (p_1, p_2, p_3) \in \mathcal{V}$ and $p_1 > 0$. That is,

$$\mathbf{p} \in \mathcal{V}_1 = \{\mathbf{x} \in \mathbb{R}^3 : 1/2 < |\mathbf{x}| < 3/2, \ x_1 > 0\}.$$

Note that $\mathcal{V}_1$ is an open set and there is a C^1 function $\mathbf{w} : \mathcal{V}_1 \to \mathcal{U} = (0, \pi) \times (-\pi/2, \pi/2)$ given by

$$\mathbf{w}(\mathbf{x}) = \left(\cos^{-1}\left(\frac{x_3}{|\mathbf{x}|}\right), \sin^{-1}\left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) \right).$$

In the definition of $\mathbf{w}$ we are using the principal arcsine and arccosine, and $\mathbf{w}$ gives the spherical angles ϕ and θ (used in spherical coordinates) associated with the point $\mathbf{x}$. It is easily checked that

$$\mathbf{u} \circ \mathbf{w}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad \text{for } \mathbf{x} \in \mathcal{V}_1$$

where $\mathbf{u} : \mathcal{U} = (0, \pi) \times (-\pi/2, \pi/2) \to \mathbb{S}^2$ by

$$\mathbf{u}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Consequently,

$$\bar{f}(\mathbf{x}) = f\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = f \circ \mathbf{u} \circ \mathbf{w}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathcal{V} \cap \mathcal{V}_1.$$

Since $\mathbf{u} \in C^1(\mathcal{U} \to \mathbb{S}^2)$, we know, from the second definition, $f \circ \mathbf{u} \in C^1(\mathcal{U})$. This then gives $\overline{f} \in C^1(\mathcal{V} \cap \mathcal{V}_1)$. In particular, $\overline{f}$ is C^1 locally near $\mathbf{p} \in \mathcal{V} \cap \mathcal{V}_1$.

Similar constructions can be used for each of the open sets $\mathcal{V}_j$, $j = 2, 3, \cdots, 6$ associated with the six coordinate hemispheres of $\mathbb{S}^2$. That is,

$$\begin{aligned}
\mathcal{V}_2 &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0\} \\
\mathcal{V}_3 &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \\
\mathcal{V}_4 &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0\} \\
\mathcal{V}_5 &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < 0\} \\
\mathcal{V}_6 &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}.
\end{aligned}$$

Since $\mathcal{V} \subset \cup_{j=1}^{6} \mathcal{V}_j$ and $\overline{f} \in C^1(\mathcal{V} \cap \mathcal{V}_j)$ for $j = 1, 2, \ldots, 6$, we have (or at least can) establish the equivalence of the two definitions.

(b) *Given a unit vector* $\mathbf{v} \in T_{\mathbf{p}}\mathbb{S}^2$*, define the directional derivative of* $f \in C^1(\mathbb{S}^2)$ *by*

$$\nabla_{\mathbf{v}} f(\mathbf{p}) = D_{\mathbf{v}} \overline{f}(\mathbf{p}) = D\overline{f}(\mathbf{p}) \cdot \mathbf{v}$$

where $\overline{f} \in C^1(\mathcal{U})$ *is an extension of* f *to an open set* $\mathcal{U} \subset \mathbb{R}^3$*. Show that this definition does not depend on the choice of extension* $\overline{f}$*. Hint: Express* $\nabla_{\mathbf{v}} f(\mathbf{p})$ *in terms of composition with a curve.*

Solution: If $\epsilon > 0$ and $\gamma \in C^1((-\epsilon, \epsilon) \to \mathbb{S}^2)$ with $\gamma(0) = \mathbf{p}$ and $\gamma'(0) = \mathbf{v}$, then

$$\frac{d}{dt} f \circ \gamma(t)\Big|_{t=0} = \frac{d}{dt} \overline{f} \circ \gamma(t)\Big|_{t=0} = D\overline{f}(\mathbf{p}) \cdot \mathbf{v} = \nabla_{\mathbf{v}} f(\mathbf{p}).$$

Notice the first expression does not depend on the extension. One may also ask about the existence of such a curve γ. Letting $\gamma_0 : \mathbb{R} \to \mathbb{R}^3$ by $\gamma_0(t) = \mathbf{p} + t\mathbf{v}$, we can set $\gamma(t) = \gamma_0(t)/|\gamma_0(t)|$. It is relatively easy to see that $\gamma \in C^\infty(\mathbb{R} \to \mathbb{S}^2)$ and

$$\gamma(0) = \mathbf{p}/|\mathbf{p}| = \mathbf{p}$$

with

$$\gamma'(t) = \frac{\mathbf{v}}{|\gamma_0(t)|} - \frac{\gamma_0(t) \cdot \mathbf{v}}{|\gamma_0(t)|^3}(\mathbf{p} + t\mathbf{v}) \qquad \text{and} \qquad \gamma'(0) = \mathbf{v} - \mathbf{p} \cdot \mathbf{v} = \mathbf{v}$$

since $N = \mathbf{p}$ is a normal to $\mathbb{S}^2$.

(c) *Define* $C^2(\mathbb{S}^2)$.

Solution: We can do this in the two ways suggested above. That is, we can say $f \in C^2(\mathbb{S}^2)$ if there is an extension $\overline{f} : \mathcal{V} \to \mathbb{R}$ where $\mathcal{V}$ is some open set in $\mathbb{R}^3$ with $\mathbb{S}^2 \subset \mathcal{V}$, and

$$\overline{f} \in C^2(\mathcal{V}).$$

On the other hand, we can say $f \in C^2(\mathbb{S}^2)$ if whenever $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ and $X \in C^2(\mathcal{U} \to \mathbb{S}^2)$, then

$$f \circ X \in C^2(\mathcal{U}).$$

These two definitions of $C^2(\mathbb{S}^2)$ may also be shown to be equivalent.

(d) *Note that the* **unit sphere map** *given by* $\mathbf{u} : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ *by*

$$\mathbf{u}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

parameterizes most of the sphere. Calculate the scaling factor $\sigma_{\mathbf{u}}$ *for area with respect to this parameterization. That is, given* $f \in C^0(\mathbb{S}^2)$ *and* $\mathcal{U}$ *a domain of integration in* $(0, \pi) \times (0, 2\pi)$*, we have*

$$\int_{\mathbf{u}(\mathcal{U})} f = \int_{\mathcal{U}} f \circ \mathbf{u}\ \sigma_{\mathbf{u}}.$$

Solution: $\sigma_{\mathbf{u}} = \sqrt{\det(D\mathbf{u}^T D\mathbf{u})}$ and

$$D\mathbf{u} = \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{pmatrix}$$

Thus,

$$\sigma_{\mathbf{u}} = \sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}} = \sin\phi.$$

(e) *Consider*

$$\mathcal{A} = \{f \in C^1(\mathbb{S}^2) : f > 0\}$$

as an **admissible class** *and the associated functions* $\mathbf{g} : \mathbb{S}^2 \to \mathbb{R}^3$ *given by*

$$\mathbf{g}(\mathbf{p}) = f(\mathbf{p})\mathbf{p}.$$

Find the area scaling factor $\sigma : \mathbb{S}^2 \to \mathbb{R}$ *on* $\mathbb{S}^2$ *giving the area of the* **radial graph**

$$\mathcal{G} = \{f(\mathbf{p})\mathbf{p} : \mathbf{p} \in \mathbb{S}^2\}$$

so that the area functional is given by $\mathfrak{A} : \mathcal{A} \to \mathbb{R}$ *by*

$$\mathfrak{A}[f] = \int_{\mathcal{G}} 1 = \int_{\mathbb{S}^2} \sigma.$$

Hints: Calculate the area scaling factor σ_X associated with the parameterization $X : \mathcal{U} \to \mathbb{R}^3$ by $X = \mathbf{g} \circ \mathbf{u}$. Then

$$\sigma = \frac{\sigma_X}{\sigma_{\mathbf{u}}}.$$

Solution: There are various ways to approach this calculation, and I will try to give several possibilities. The basic dimensions under consideration in this problem are 1, 2 and 3, though it will be convenient at times to make calculations as if they apply to a general dimension n. When this happens $n = 3$. It should be kept in mind throughout that one may use extensions to allow computation with the usual chain and product rules, this requires some care with regard to the multiplication of various total derivative matrices that arise, and some complication/inconvenience is to be expected. In particular the conventions about certain vectors being row vectors or column vectors and the order of various multiplications require some attention. Perhaps the easiest way to get started is by recognizing/writing

$$DX = (X_\phi \ X_\theta).$$

Here the total derivative is viewed as consisting of two column vectors of length 3. Thus, it is natural here to view X as a column vector as well with three components X_1, X_2, and X_3 so that each row of DX is a gradient of one of these components. The column vectors X_ϕ and X_θ are tangent vectors to the surface and

$$\begin{aligned} \sigma_X &= \sqrt{\det(DX^T DX)} \\ &= \sqrt{\det \begin{pmatrix} |X_\phi|^2 & X_\phi \cdot X_\theta \\ X_\phi \cdot X_\theta & |X_\theta|^2 \end{pmatrix}} \\ &= \sqrt{|X_\phi|^2 |X_\theta|^2 - (X_\phi \cdot X_\theta)^2}. \end{aligned}$$

We can then write $X = \overline{\mathbf{g}} \circ \mathbf{u}$ where $\overline{\mathbf{g}} : \mathcal{V} \to \mathbb{R}^3$ is an extension of $\mathbf{g}$ obtained as $\overline{\mathbf{g}}(\mathbf{p}) = \overline{f}(\mathbf{p})\ \mathbf{p}$ with $\overline{f} : \mathcal{V} \to \mathbb{R}$ an extension of $f : \mathbb{S}^2 \to \mathbb{R}$ to an open set $\mathcal{V} \subset \mathbb{R}^3$ with $\mathbb{S}^3 \subset \mathcal{V}$ (as above). With these extensions we find using the chain rule and the product rule that, for example,

$$X_\phi = (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)\ \mathbf{u} + \overline{f}(\mathbf{u})\ \mathbf{u}_\phi. \tag{1}$$

Similarly,

$$X_\theta = D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta \; \mathbf{u} + \overline{f}(\mathbf{u}) \; \mathbf{u}_\theta, \tag{2}$$

so that

$$|X_\phi|^2 = (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 + (\overline{f}(\mathbf{u}))^2,$$

$$X_\phi \cdot X_\theta = (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta),$$

and

$$|X_\theta|^2 = (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 + (\overline{f}(\mathbf{u}))^2 \sin^2 \phi.$$

We have used here that $|\mathbf{u}| = 1$, $|\mathbf{u}_\phi| = 1$, $\mathbf{u} \cdot \mathbf{u}_\phi = 0 = \mathbf{u} \cdot \mathbf{u}_\theta$, $\mathbf{u}_\phi \cdot \mathbf{u}_\theta = 0$, and $|\mathbf{u}_\theta|^2 = \sin^2 \phi$.

From here, it is not difficult to complete the task suggested in the hint (i) below. We give the details below under that heading. The route we have taken, however, depends to a certain extent on the recognition that σ_X can be expressed in terms of the tangent vectors X_ϕ and X_θ. Though it is a natural thing to recognize, you might not have recognized this possibility. Also, it may simply be of general interest to compute the total derivative DX more directly. Therefore, I will begin again and describe how this may be done.

In terms of the extensions of $\mathbf{g}$ and f we can use the usual product and chain rules to write

$$DX = D(\mathbf{g} \circ \mathbf{u}) = D\overline{\mathbf{g}} \; D\mathbf{u} \tag{3}$$

where $D\overline{\mathbf{g}} = D\overline{\mathbf{g}}(\mathbf{u})$ is an appropriate 3×3 matrix determined by the product rule and having a form something like

$$D\overline{\mathbf{g}} = D\overline{f}(\mathbf{p}) * \mathbf{p} + \overline{f}(\mathbf{p}) \; I \tag{4}$$

since $\overline{\mathbf{g}}(\mathbf{p}) = \overline{f}(\mathbf{p}) \; \mathbf{p}$. The nature of the product denoted $D\overline{f}(\mathbf{p}) * \mathbf{p}$ is not intended to be entirely clear in (4), but it is imagined that there should be some kind of product rule according to which this product gives the correct term in the total derivative matrix $D\overline{\mathbf{g}}$. Now I will attempt to explain this product and make it entirely clear. First of all, note generally that we expect the derivatives in $D\mathbf{u}$ to be with respect to ϕ and θ. In particular, the tangent vectors X_ϕ and X_θ discussed above should appear here, and there should be no derivatives with respect to ϕ and θ in the factor $D\overline{\mathbf{g}}$ in (3), though there is a composition with $\mathbf{u}$, so that $D\overline{\mathbf{g}} = D\overline{\mathbf{g}}(\mathbf{u})$. The question then is how to compute this matrix.

Recall that $\overline{f}$ is scalar valued, so if we write $\overline{\mathbf{g}}$ out as a column vector in components, we get something like

$$\overline{\mathbf{g}}(\mathbf{p}) = \begin{pmatrix} \overline{f}(\mathbf{p})\ p_1 \\ \overline{f}(\mathbf{p})\ p_2 \\ \vdots \\ \overline{f}(\mathbf{p})\ p_n \end{pmatrix}.$$

Therefore, taking Euclidean gradients with respect to the variable $\mathbf{p}$, we can write

$$\begin{aligned} D\overline{\mathbf{g}}(\mathbf{p}) &= \begin{pmatrix} D\overline{f}(\mathbf{p})\ p_1 + \overline{f}(\mathbf{p})\ Dp_1 \\ D\overline{f}(\mathbf{p})\ p_2 + \overline{f}(\mathbf{p})\ Dp_2 \\ \vdots \\ D\overline{f}(\mathbf{p})\ p_n + \overline{f}(\mathbf{p})\ Dp_n \end{pmatrix} \\ &= \begin{pmatrix} D\overline{f}(\mathbf{p})\ p_1 + \overline{f}(\mathbf{p})\ \mathbf{e}_1 \\ D\overline{f}(\mathbf{p})\ p_2 + \overline{f}(\mathbf{p})\ \mathbf{e}_2 \\ \vdots \\ D\overline{f}(\mathbf{p})\ p_n + \overline{f}(\mathbf{p})\ \mathbf{e}_n \end{pmatrix} \\ &= \begin{pmatrix} p_1\ D\overline{f}(\mathbf{p}) \\ p_2\ D\overline{f}(\mathbf{p}) \\ \vdots \\ p_n\ D\overline{f}(\mathbf{p}) \end{pmatrix} + \overline{f}(\mathbf{p})\ I. \end{aligned}$$

Comparing this expression to (4), we conclude

$$D\overline{f}(\mathbf{p}) * \mathbf{p} = \begin{pmatrix} p_1 \frac{\partial \overline{f}}{\partial p_1} & p_1 \frac{\partial \overline{f}}{\partial p_2} & \cdots & p_1 \frac{\partial \overline{f}}{\partial p_n} \\ p_2 \frac{\partial \overline{f}}{\partial p_1} & p_2 \frac{\partial \overline{f}}{\partial p_2} & \cdots & p_2 \frac{\partial \overline{f}}{\partial p_n} \\ \vdots & & & \\ p_n \frac{\partial \overline{f}}{\partial p_1} & p_n \frac{\partial \overline{f}}{\partial p_2} & \cdots & p_n \frac{\partial \overline{f}}{\partial p_n} \end{pmatrix}.$$

There are two, more or less, obvious ways to obtain this matrix as a product of the gradient $D\overline{f}(\mathbf{p})$ and the vector $\mathbf{p}$. Let us agree to consider $D\overline{f}(\mathbf{p})$ as a row vector and $\mathbf{p}$ as a column vector, then

$$D\overline{f}(\mathbf{p}) * \mathbf{p} = \mathbf{p}\ D\overline{f}(\mathbf{p}).$$

Under this interpretation (4) becomes

$$D\overline{\mathbf{g}}(\mathbf{p}) = D(\overline{f}(\mathbf{p})\ \mathbf{p}) = \mathbf{p}\ D\overline{f}(\mathbf{p}) + \overline{f}(\mathbf{p})\ I$$

which, in turn, gains some unity and follows the usual form of the chain rule if the usual convention for scaling vectors is reversed: $\mathbf{g}(\mathbf{p}) = \mathbf{p}\ \overline{f}(\mathbf{p})$ and

$$D\overline{\mathbf{g}} = D(\mathbf{p}\ \overline{f}) = I\ \overline{f}(\mathbf{p}) + \mathbf{p}\ D\overline{f}(\mathbf{p}).$$

Alternatively, since the matrix is symmetric, one may keep the order and introduce the transpose:

$$D\overline{\mathbf{g}}(\mathbf{p}) = D(\overline{f}(\mathbf{p})\ \mathbf{p}) = D\overline{f}(\mathbf{p})^T\ \mathbf{p}^T + D\overline{f}(\mathbf{p})\ I$$

so that

$$D\overline{f}(\mathbf{p}) * \mathbf{p} = D\overline{f}(\mathbf{p})^T\ \mathbf{p}^T$$

which is the same matrix. What one cannot do (and should be careful not to do) is to interpret the matrix product $D\overline{f}(\mathbf{p}) * \mathbf{p}$ in $D(f\mathbf{p})$ as a dot product of two vectors $Df \cdot \mathbf{p} + f\ I$; that would be incorrect.

Returning to (3), we have obtained

$$DX = [\mathbf{u}\ D\overline{f}(\mathbf{u}) + \overline{f}(\mathbf{u})\ I]D\mathbf{u} = \mathbf{u}\ D\overline{f}(\mathbf{u})\ D\mathbf{u} + \overline{f}(\mathbf{u})\ D\mathbf{u}$$

or

$$DX = [D\overline{f}(\mathbf{u})^T\ \mathbf{u}^T + \overline{f}(\mathbf{u})\ I]D\mathbf{u} = D\overline{f}(\mathbf{u})^T\ \mathbf{u}^T D\mathbf{u} + \overline{f}(\mathbf{u})\ D\mathbf{u}.$$

In order to complete task (i) suggested below we compute the product(s)

$$\begin{aligned} DX^T DX &= [D\mathbf{u}^T\ D\overline{f}(\mathbf{u})^T\ \mathbf{u}^T + \overline{f}(\mathbf{u})\ D\mathbf{u}^T][\mathbf{u}\ D\overline{f}(\mathbf{u})\ D\mathbf{u} + \overline{f}(\mathbf{u})\ D\mathbf{u}] \\ &= [D\mathbf{u}^T\ \mathbf{u}\ D\overline{f}(\mathbf{u}) + \overline{f}(\mathbf{u})\ D\mathbf{u}^T][D\overline{f}(\mathbf{u})^T\ \mathbf{u}^T D\mathbf{u} + \overline{f}(\mathbf{u})\ D\mathbf{u}] \end{aligned}$$

where we interpret $D\mathbf{u}$ as the 3×2 matrix consisting of columns

$$D\mathbf{u} = \left(\mathbf{u}_\phi \quad \mathbf{u}_\theta\right).$$

Noting that $\mathbf{u}^T\mathbf{u} = 1$ and $\mathbf{u}^T D\mathbf{u} = 0$, we can write

$$\begin{aligned}
DX^T DX &= D\mathbf{u}^T \ D\overline{f}(\mathbf{u})^T \ D\overline{f}(\mathbf{u}) \ D\mathbf{u} + \overline{f}(\mathbf{u})^2 \ D\mathbf{u}^T \ D\mathbf{u} \\
&= \begin{pmatrix} D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi \\ D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta \end{pmatrix} (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi \quad D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta) \\
&\qquad + \overline{f}(\mathbf{u})^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix} \\
&= \begin{pmatrix} (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 & (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta) \\ (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta) & (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 \end{pmatrix} \\
&\qquad + \overline{f}(\mathbf{u})^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix} \\
&= \begin{pmatrix} (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 + \overline{f}(\mathbf{u})^2 & (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta) \\ (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta) & (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 + \overline{f}(\mathbf{u})^2 \ \sin^2\phi \end{pmatrix}.
\end{aligned}$$

This evidently leads to the same value of σ_X and of σ calculated below.

(i) *First obtain an expression involving the quantities $D\overline{f} \cdot \mathbf{u}_\phi$ and $D\overline{f} \cdot \mathbf{u}_\theta$.*

Solution: Based on the computation of the tangent vectors X_ϕ and X_θ above we have

$$\begin{aligned}
\sigma_X &= \sqrt{f(\mathbf{u})^2[(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 \sin^2\phi + (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 + \overline{f}(\mathbf{u})^2 \sin^2\phi]} \\
&= f(\mathbf{u})\sqrt{(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 \sin^2\phi + (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 + f(\mathbf{u})^2 \sin^2\phi}.
\end{aligned}$$

Notice that wherever we have the value $\overline{f}$ in the result of this computation we can replace $\overline{f}$ with f. When we have the Euclidean gradient $D\overline{f}$ however, we cannot write Df.

Recalling that $\sigma = \sigma_X/\sigma_\mathbf{u}$, we can now write (at least for $0 < \phi < \pi$)

$$\begin{aligned}
\sigma &= \frac{f(\mathbf{u})}{\sin\phi}\sqrt{(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 \sin^2\phi + (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta)^2 + f(\mathbf{u})^2 \sin^2\phi} \\
&= f(\mathbf{u})\sqrt{(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 + (D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\theta/\sin\phi)^2 + f(\mathbf{u})^2} \\
&= f(\mathbf{u})\sqrt{(D\overline{f}(\mathbf{u}) \cdot \mathbf{u}_\phi)^2 + (D\overline{f}(\mathbf{u}) \cdot (-\sin\theta, \cos\theta, 0))^2 + f(\mathbf{u})^2}.
\end{aligned}$$

This essentially completes the task suggested by hint (i). The fundamental problem with what we have obtained is that we have a function of ϕ and θ, but we are supposed to integrate the scaling factor σ over the surface $\mathbb{S}^2$ rather than the parameter domain $\mathcal{U}$. One way to deal with this "defect," at least partially, is to introduce an inverse $\mathbf{u}^{-1} : \mathbb{S}^2 \to \mathcal{U}$ and, wherever we see (ϕ, θ) in this expression, replace ϕ with first component of the inverse of $\mathbf{u}$ and replace θ with the second component of the inverse of $\mathbf{u}$. In this way, we can obtain a function on some subset $\mathbf{u}(\mathcal{U})$ of $\mathbb{S}^2$, and it would make sense to compute an integral

$$\int_{\mathbf{u}(\mathcal{U})} \sigma.$$

This still doesn't give us the entire integral

$$\int_{\mathbb{S}^2} \sigma$$

since $\mathbf{u}$ does not give a one-to-one and onto (bijective) map onto the sphere.

(ii) *Simplify your expression using the* **surface gradient** *on* $\mathbb{S}^2$. *Remember this is a function* $\operatorname{grad} f : \mathbb{S}^2 \to \mathbb{R}^3$ *having the properties*

$$\operatorname{grad} f(\mathbf{p}) \in T_{\mathbf{p}}\mathbb{S}^2 \qquad \text{and} \qquad \operatorname{grad} f(\mathbf{p}) \cdot \mathbf{w} = \nabla_{\mathbf{w}} f(\mathbf{p}) \quad \textit{for all } \mathbf{w} \in T_{\mathbf{p}}\mathbb{S}^2.$$

Solution: We begin with a local computation in coordinates based on the observation that $\{\mathbf{u}_\phi, \mathbf{u}_\theta\}$ is a basis for the tangent space $T_{\mathbf{u}}\mathbb{S}^2$ at least at points where the spherical map is well-defined. In this case, any vector $\mathbf{w}$ can be written as $\mathbf{w} = a\mathbf{u}_\phi + b\mathbf{u}_\theta$ for some a and b. Taking inner products we find

$$\mathbf{w} \cdot \mathbf{u}_\phi = a \qquad \text{and} \qquad \mathbf{w} \cdot \mathbf{u}_\theta = b \sin^2 \phi.$$

Consequently,

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{u}_\phi)\ \mathbf{u}_\phi + \frac{\mathbf{w} \cdot \mathbf{u}_\theta}{\sin^2 \phi}\ \mathbf{u}_\theta$$

and

$$\begin{aligned} \nabla_{\mathbf{w}} f &= D\overline{f} \cdot \mathbf{w} \\ &= (\mathbf{w} \cdot \mathbf{u}_\phi)\ (D\overline{f} \cdot \mathbf{u}_\phi) + \frac{\mathbf{w} \cdot \mathbf{u}_\theta}{\sin^2 \phi}\ (D\overline{f} \cdot \mathbf{u}_\theta) \\ &= \left[(D\overline{f} \cdot \mathbf{u}_\phi)\ \mathbf{u}_\phi + \left(D\overline{f} \cdot \frac{\mathbf{u}_\theta}{\sin \phi} \right) \frac{\mathbf{u}_\theta}{\sin \phi} \right] \cdot \mathbf{w}. \end{aligned}$$

Thus, considering the definition of the surface gradient, we have

$$\operatorname{grad} f = (D\overline{f} \cdot \mathbf{u}_\phi)\, \mathbf{u}_\phi + \left(D\overline{f} \cdot \frac{\mathbf{u}_\theta}{\sin\phi} \right) \frac{\mathbf{u}_\theta}{\sin\phi},$$

and comparison with our expression for σ according to hint (i) gives

$$\sigma = f\sqrt{|\operatorname{grad} f|^2 + f^2}.$$

This function is entirely and unambiguously defined on $\mathbb{S}^2$ in terms of f, so that

$$\mathfrak{A}[f] = \int_{\mathbb{S}^2} \sigma = \int_{\mathbb{S}^2} f\sqrt{|\operatorname{grad} f|^2 + f^2}.$$

(f) *Calculate the first variation $\delta\mathfrak{A}_f[\phi]$ of the area functional.*

Solution: Though the surface gradient is a somewhat new object for us, it can be checked (and it is entirely believable) that linearity holds so that, for example,

$$\operatorname{grad}(f+\epsilon\phi) = \operatorname{grad} f + \epsilon \operatorname{grad}\phi$$

where $\phi \in C^1(\mathbb{S}^2)$ and $\operatorname{grad}\phi \in T_{\mathbf{p}}\mathbb{S}^2$. Consequently, the first variation is relatively easy to compute in the form

$$\begin{aligned}
\delta\mathfrak{A}_f[\phi] &= \frac{d}{d\epsilon}\int_{\mathbb{S}^2}(f+\epsilon\phi)\sqrt{|\operatorname{grad} f + \epsilon\operatorname{grad}\phi|^2 + (f+\epsilon\phi)^2}\,\Big|_{\epsilon=0} \\
&= \int_{\mathbb{S}^2}\left[\phi\sqrt{|\operatorname{grad} f|^2+f^2} + f\;\frac{\operatorname{grad} f\cdot\operatorname{grad}\phi + f\phi}{\sqrt{|\operatorname{grad} f|^2+f^2}}\right] \\
&= \int_{\mathbb{S}^2}\phi\left[\sqrt{|\operatorname{grad} f|^2+f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2+f^2}}\right] \\
&\qquad + \int_{\mathbb{S}^2}\frac{f\;\operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}}\cdot\operatorname{grad}\phi.
\end{aligned}$$

It should be clear at this point that we should look for some version of the **divergence theorem on a surface** and an associated **product rule** applied to the scaled field

$$\frac{\phi\; f\;\operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}} \tag{5}$$

when $f,\phi \in C^2(\mathbb{S}^2)$. This, of course, will require a **surface divergence** for vector fields on $\mathbb{S}^2$. This suggestion is given as a hint for the next part below, but it is natural to discuss these topics in the context of the calculus of variations, so I will go ahead and fill in the details here.

The treatment of the divergence theorem is quite easy, as our discussion from Euclidean space applies without change. To be precise, we define for $\mathbf{v} \in C^1(\mathbb{S}^2 \to \mathbb{R}^3)$ and $\mathbf{p} \in \mathbb{S}^2$

$$\operatorname{div}^{\mathbb{S}^2}\mathbf{v}(\mathbf{p}) = \lim_{\mathcal{W}\to\{\mathbf{p}\}}\frac{1}{\mathcal{H}^2(\mathcal{W})}\int_{\partial\mathcal{W}}\mathbf{v}\cdot\mathbf{n} \tag{6}$$

where $\mathcal{W}$ is a family of "nice" domains in the surface $\mathbb{S}^2$ shrinking to the point $\mathbf{p}$, $\mathcal{H}^2$ is two-dimensional Hausdorff (surface) measure on subsets of $\mathbb{R}^3$, and $\mathbf{n}$ is a unit vector tangent to $\mathbb{S}^2$, normal to $\partial\mathcal{W}$, and pointing out of $\mathcal{W}$. The vector $\mathbf{n}$ is called the **unit co-normal** to $\partial\mathcal{W}$ pointing out of $\mathcal{W}$.

With this definition, and some assumed convergence, the divergence theorem is easy. If $\mathcal{W}$ is a domain in $\mathbb{S}^2$ partitioned into small pieces $\mathcal{W} = \cup_{j=1}^k \mathcal{W}_j$, then when the norm of the partition $\mathcal{P} = \{\mathcal{W}_j\}_{j=1}^k$ is small (and the pieces are reasonably well-behaved) we have

$$\begin{aligned}\int_{\partial\mathcal{W}} \mathbf{v}\cdot\mathbf{n} &= \sum_{j=1}^k \int_{\partial\mathcal{W}_j} \mathbf{v}\cdot\mathbf{n} \\ &= \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^k \int_{\partial\mathcal{W}_j} \mathbf{v}\cdot\mathbf{n} \\ &= \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^k \operatorname{div}\mathbf{v}(\mathbf{p}_j^*)\mathcal{H}^2(\mathcal{W}_j) \\ &= \int_{\mathcal{W}} \operatorname{div}\mathbf{v},\end{aligned}$$

just as in the Euclidean case. In the first equality, the integrals over boundary curves of adjacent partitition pieces $\mathcal{W}_j$ (having opposite co-normals) cancel. The third equality follows from the definition of the divergence on the surface (6) and uniformity. The last equality is the definition of integration on $\mathcal{W}$. Incidentally, if the field $\mathbf{v}$ is defined globally on the entire sphere $\mathbf{w} = \mathbb{S}^2$, then

$$\int_{\mathbb{S}^2} \operatorname{div}\mathbf{v} = \sum \int_{\mathcal{W}_j} \operatorname{div}\mathbf{v} = \sum \int_{\partial\mathcal{W}_j} \mathbf{v}\cdot\mathbf{n} = 0$$

because every boundary component of a partition piece $\mathcal{W}_j$ is adjacent to another, and all the boundary integrals cancel one another.

Next, we need a product rule for this divergence. You may recall that I do not know a nice proof without using coordinates giving the product rule. Thus, as we did for the product rule for the divergence of a scaled field in Euclidean space, I will will use coordinates. Given that ϕ and θ are the natural (spherical) coordinates on $\mathbb{S}^2$ used in the definition of $\mathbf{u} = \mathbf{u}(\phi,\theta)$ above, and ϕ is appearing as a test function both above and below, I will change notation slightly. Also, I denoted the surface gradient by ∇ below, but I will use "grad" here for the gradient so that the rule I want to justify is

$$\mathrm{div}^{\mathbb{S}^2}(\psi\mathbf{v})(\mathbf{p}) = \mathrm{grad}\,\psi(\mathbf{p})\cdot\mathbf{v}(\mathbf{p}) + \psi(\mathbf{p})\ \mathrm{div}^{\mathbb{S}^2}\mathbf{v}(\mathbf{p}) \tag{7}$$

for $\psi \in C^1(\mathbb{S}^2)$ a scalar function and $\mathbf{v} \in C^2(\mathbb{S}^2 \to \mathbb{R}^3)$ a vector field tangent to $\mathbb{S}^2$ so that $\mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}\mathbb{S}^2$. The approach is similar to the standard derivation of the formula

$$\mathrm{div}\,\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$$

for a vector field on $\mathbb{R}^2$ using shrinking rectangles. We start with a rectangle $[\phi_0-\epsilon,\phi_0+\epsilon]\times[\theta_0-\delta,\theta_0+\delta]$ as a domain $\underline{\mathcal{W}}$ for $\mathbf{u}$ and write the image as

$$\begin{aligned}\mathcal{W} &= \mathbf{u}([\phi_0-\epsilon,\phi_0+\epsilon]\times[\theta_0-\delta,\theta_0+\delta]) \\ &= \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) : |\phi-\phi_0| \le \epsilon, |\theta-\theta_0| \le \delta\}.\end{aligned}$$

We also write $\mathbf{p} = \mathbf{u}(\phi_0,\theta_0)$. I find it very helpful to draw pictures of all these sets and each of the sets considered below as I go through the calculation. We wish to calculate the limit

$$\mathrm{div}\,\mathbf{v}(\mathbf{p}) = \lim_{\epsilon,\delta\to 0}\frac{1}{\mathcal{H}^2(\mathcal{W})}\int_{\partial\mathcal{W}}\mathbf{v}\cdot\mathbf{n}.$$

Let us denote by $\Gamma = \Gamma_1$ the image of $\underline{\Gamma} = \{\phi_0+\epsilon\}\times[\theta_0-\delta,\theta_0+\delta]$. That is,

$$\Gamma_1 = \{(\sin(\phi_0+\epsilon)\cos\theta, \sin(\phi_0+\epsilon)\sin\theta, \cos(\phi_0+\epsilon)) : |\theta-\theta_0| \le \delta\}$$

is the image of the right side of the rectangle $\underline{\mathcal{W}}$ and is the lower longitudinal boundary of $\mathcal{W}$ on $\mathbb{S}^2$. Note that the tangent vector $\mathbf{v} = \mathbf{v}\circ\mathbf{u}(\theta,\phi_0+\epsilon)$ along Γ can be written as

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_\phi)\ \mathbf{u}_\phi + \frac{\mathbf{v}\cdot\mathbf{u}_\theta}{\sin^2(\phi_0+\epsilon)}\mathbf{u}_\theta$$

where the tangent vectors $\mathbf{u}_\phi = \mathbf{u}_\phi(\phi_0+\epsilon,\theta)$ and $\mathbf{u}_\theta = \mathbf{u}_\theta(\phi_0+\epsilon,\theta)$ are given above. On the other hand, if we take any point $(\phi_0+\epsilon,\theta)$ along $\underline{\Gamma}$, we can consider a vector $\underline{\mathbf{w}} = (\underline{w}_1, \underline{w}_2) \in \mathbb{R}^2$ and write

$$\alpha(t) = (\phi_0+\epsilon,\theta) + t\underline{\mathbf{w}}$$

where we think of θ as fixed. Observe that the tangent vector to this curve at $\mathbf{p}$ is

$$\frac{d}{dt}\mathbf{u} \circ \alpha(t)\Big|_{t=0} = \underline{w}_1 \mathbf{u}_\phi + \underline{w}_2 \mathbf{u}_\theta.$$

Comparing this to the expression for $\mathbf{v}$ above, we obtain a vector

$$\underline{\mathbf{v}} = \left(\mathbf{v} \cdot \mathbf{u}_\phi, \frac{\mathbf{v} \cdot \mathbf{u}_\theta}{\sin^2(\phi_0 + \epsilon)}\right) \in T_{(\phi_0+\epsilon,\theta)}\mathbb{R}^2.$$

Evidently, the same construction may be used to determine a vector field $\underline{\mathbf{v}}$ on the portion of the ϕ, θ-plane corresponding to $\mathcal{W}$, namely $\underline{\mathcal{W}}$.

Notice also that we have the unit co-normal $\mathbf{n}$ along $\Gamma = \Gamma_1$ given by $\mathbf{n} = \mathbf{u}_\phi$. Thus the corresponding vector field on $\underline{\mathcal{W}}$ is $\underline{\mathbf{n}} = (1, 0) = \mathbf{e}_1$ (which happens to be a unit field and the unit co-normal to $\partial\underline{\mathcal{W}}$ along $\underline{\Gamma}$).

It is natural to ask at this point what is the relation between $\mathbf{v} \cdot \mathbf{n}$ and $\underline{\mathbf{v}} \cdot \underline{\mathbf{n}}$, and to note that these are equal. That is to say

$$\mathbf{v} \circ \mathbf{u}(\phi_0 + \epsilon, \theta) \cdot \mathbf{n} \circ \mathbf{u}(\phi_0 + \epsilon, \theta) = \underline{\mathbf{v}}(\phi_0 + \epsilon, \theta) \cdot \underline{\mathbf{n}}(\phi_0 + \epsilon, \theta).$$

Finally, we can parameterize $\Gamma = \Gamma_1$ on $\underline{\Gamma}$ by

$$\gamma(\theta) = \mathbf{u}(\phi_0 + \epsilon, \theta) = (\sin(\phi_0 + \epsilon)\cos\theta, \sin(\phi_0 + \epsilon)\sin\theta, \cos(\phi_0 + \epsilon))$$

for $\theta_0 - \delta \leq \theta \leq \theta_0 + \delta$. Changing variables, we have

$$\int_{\Gamma_1} \mathbf{v} \cdot \mathbf{n} = \int_{\theta_0-\delta}^{\theta_0+\delta} \underline{\mathbf{v}} \cdot \underline{\mathbf{n}} \ \sin(\phi_0 + \epsilon)\, d\theta = \sin(\phi_0 + \epsilon) \int_{\underline{\Gamma}} \underline{\mathbf{v}} \cdot \underline{\mathbf{n}}$$

because $|\gamma'(\theta)| = |\mathbf{u}_\theta(\phi_0 + \epsilon, \theta)| = \sin(\phi_0 + \epsilon)$.

Proceeding counterclockwise around $\partial\underline{\mathcal{W}}$, or equivalently around $\partial\mathcal{W}$, we write

$$\Gamma_2 = \{(\sin\phi\cos(\theta_0 + \epsilon), \sin\phi\sin(\theta_0 + epsilon), \cos\phi) : |\phi - \phi_0| < \epsilon\}$$
$$\Gamma_3 = \{(\sin(\phi_0 - \epsilon)\cos\theta, \sin(\phi_0 - \epsilon)\sin\theta, \cos(\phi_0 + \epsilon)) : |\theta - \theta_0| \leq \delta\}, \quad \text{and}$$
$$\Gamma_4 = \{(\sin\phi\cos(\theta_0 - \epsilon), \sin\phi\sin(\theta_0 - \epsilon), \cos\phi) : |\phi - \phi_0| < \epsilon\}$$

Let us consider the integral $\int_{\Gamma_2} \mathbf{v} \cdot \mathbf{n}$ in detail. The co-normal field along Γ_2 is

$$\mathbf{n} = \frac{1}{\sin\phi}\ \mathbf{u}_\theta(\phi, \theta + \delta)$$

which induces the field

$$\underline{\mathbf{n}} = \left(\mathbf{n} \cdot \mathbf{u}_\phi, \frac{\mathbf{n} \cdot \mathbf{u}_\theta}{\sin^2 \phi}\right) = \left(0, \frac{1}{\sin \phi}\right)$$

which is not a unit field but is a co-normal. Furthermore, using the expression for $\mathbf{v}$ above, we see

$$\mathbf{v} \cdot \mathbf{n} = \frac{\mathbf{v} \cdot \mathbf{u}_\theta}{\sin \phi} \qquad \text{while} \qquad \underline{\mathbf{v}} \cdot \underline{\mathbf{n}} = \frac{\mathbf{v} \cdot \mathbf{u}_\theta}{\sin^3 \phi},$$

so along $\underline{\Gamma}_2$

$$\mathbf{v} \circ \mathbf{u}(\phi, \theta_0 + \delta) \cdot \mathbf{n} \circ \mathbf{u}(\phi, \theta_0 + \delta) = \sin^2 \phi \underline{\mathbf{v}}(\phi, \theta_0 + \delta) \cdot \underline{\mathbf{n}}(\phi, \theta_0 + \delta).$$

Taking $\gamma(\phi) = \mathbf{u}(\phi, \theta_0 + \delta)$, we have $|\gamma'(\phi)| = 1$, so

$$\int_{\Gamma_2} \mathbf{v} \cdot \mathbf{n} = \int_{\phi_0 - \epsilon}^{\phi_0 + \epsilon} \frac{\mathbf{v} \circ \mathbf{u}(\phi, \theta_0 + \delta) \cdot \mathbf{u}_\theta(\phi, \theta_0 + \delta)}{\sin \phi} \, d\phi = \int_{\underline{\Gamma}_2} \sin^2 \phi \, (\underline{\mathbf{v}} \cdot \underline{\mathbf{n}}).$$

The last expression is significant in that it illustrates the difficulty of expressing a boundary integral on a surface like $\int_\Gamma \mathbf{v} \cdot \mathbf{n}$ in terms of a simple Euclidean dot product on the primage curve $\underline{\Gamma}$ in coordinates.

Exercise 1 *Let $X : \mathcal{U} \to \mathbb{R}^3$ be a parameterization of a surface which may be considered to be C^1 on the closure of the domain $\mathcal{U}$ in the u, v-plane. (This notation is not uncommon for parameterized surfaces.) In particular, you may also assume X is one-to-one and onto (and continuously invertible) onto its image surface $\mathcal{S} = X(\mathcal{U})$. Define an* **inner product** $\langle \cdot \, , \, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{U} \to \mathbb{R}$ *so that for each $(u, v) \in \mathcal{U}$, the function $\langle \cdot \, , \, \cdot \rangle_{(u,v)} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an abstract inner product in the usual sense and*

$$\int_\Gamma \mathbf{v} \cdot \mathbf{w} = \int_{(u,v) \in \underline{\Gamma}} \langle \underline{\mathbf{v}}, \underline{\mathbf{w}} \rangle_{(u,v)} \; \underline{\sigma}$$

where $\underline{\mathbf{v}}$ and $\underline{\mathbf{w}}$ are defined by the relations

$$\frac{d}{dt} X \circ \alpha(t)\Big|_{t=0} = \mathbf{v} = \mathbf{p} \qquad \text{and} \qquad \frac{d}{dt} X \circ \beta(t)\Big|_{t=0} = \mathbf{w}$$

where $\alpha(0) = X^{-1}(\mathbf{p}) = \beta(0)$, $\alpha'(0) = \underline{\mathbf{v}}$, $\beta'(0) = \underline{\mathbf{w}}$, and $\underline{\sigma}$ is the scaling factor associated with the change of variables from Γ to $\underline{\Gamma} = X^{-1}(\Gamma)$.

Similarly, we find

$$\begin{aligned}\int_{\Gamma_3} \mathbf{v}\cdot\mathbf{n} &= -\sin(\phi_0-\epsilon)\int_{\theta_0-\delta}^{\theta_0+\delta} \underline{\mathbf{v}}\cdot\underline{\mathbf{n}}\,d\theta \\ &= -\sin(\phi_0-\epsilon)\int_{\underline{\Gamma}} \underline{\mathbf{v}}\cdot\underline{\mathbf{n}} \\ &= -\sin(\phi_0-\epsilon)\int_{\theta_0-\delta}^{\theta_0+\delta} \mathbf{v}\circ\mathbf{u}(\phi_0-\epsilon,\theta)\cdot\mathbf{u}_\phi(\phi_0-\epsilon,\theta)\,d\theta\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_4} \mathbf{v}\cdot\mathbf{n} &= -\int_{\phi_0-\epsilon}^{\phi_0+\epsilon} \frac{\mathbf{v}\circ\mathbf{u}(\phi,\theta_0-\delta)\cdot\mathbf{u}_\theta(\phi,\theta_0-\delta)}{\sin\phi}\,d\phi \\ &= -\int_{\underline{\Gamma}_4} \sin^2\phi\ (\underline{\mathbf{v}}\cdot\underline{\mathbf{n}}).\end{aligned}$$

Note that we may write

$$\begin{aligned}\int_{\Gamma_1\cup\Gamma_3} \mathbf{v}\cdot\mathbf{n} = \int_{\theta_0-\delta}^{\theta_0+\delta} &\left[\sin(\phi_0+\epsilon)\ \mathbf{v}\circ\mathbf{u}(\phi_0+\epsilon,\theta)\cdot\mathbf{u}_\phi(\phi_0+\epsilon,\theta)\right. \\ &\left.-\sin(\phi_0-\epsilon)\ \mathbf{v}\circ\mathbf{u}(\phi_0-\epsilon,\theta)\cdot\mathbf{u}_\phi(\phi_0-\epsilon,\theta)\right]\ d\theta \\ = 2\epsilon\int_{\theta_0-\delta}^{\theta_0+\delta} &\frac{\partial}{\partial\phi}\left[\sin\phi\ \mathbf{v}\circ\mathbf{u}(\phi,\theta)\cdot\mathbf{u}_\phi(\phi,\theta)\right]\Big|_{\phi=\phi_*}\ d\theta\end{aligned}$$

for some ϕ_* with $|\phi_*-\phi_0|<\epsilon$.

Turning to the area of $\mathcal{W}$, we can write

$$\begin{aligned}\mathcal{H}^2(\mathcal{W}) &= \int_{\phi_0-\epsilon}^{\phi_0+\epsilon}\int_{\theta_0-\delta}^{\theta_0+\delta} \sin\phi\,d\theta\,d\phi \\ &= (2\delta)[\cos(\phi_0-\epsilon)-\cos(\phi_0+\epsilon)] \\ &= (2\delta)(2\epsilon)\sin\phi_{**}\end{aligned}$$

for some ϕ_{**} with $|\phi_{**}-\phi_0|<\epsilon$. We have then

$$\begin{aligned}\lim_{\epsilon,\delta\searrow 0}\frac{1}{\mathcal{H}^2(\mathcal{W})}\int_{\Gamma_1\cup\Gamma_3}\mathbf{v}\cdot\mathbf{n} &= \frac{1}{\sin\phi_0}\frac{\partial}{\partial\phi}\left[\sin\phi\ \mathbf{v}\circ\mathbf{u}(\phi,\theta_0)\cdot\mathbf{u}_\phi(\phi,\theta_0)\right]\Big|_{\phi=\phi_0} \\ &= \cot\phi_0\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi(\phi_0,\theta_0)+\frac{\partial}{\partial\phi}\left[\mathbf{v}\circ\mathbf{u}(\phi,\theta_0)\cdot\mathbf{u}_\phi(\phi,\theta_0)\right]\Big|_{\phi=\phi_0}.\end{aligned}$$

On the other hand, we can also write

$$\begin{aligned}\int_{\Gamma_2\cup\Gamma_4} \mathbf{v}\cdot\mathbf{n} &= \int_{\phi_0-\epsilon}^{\phi_0+\epsilon} \frac{1}{\sin\phi}\left[\mathbf{v}\circ\mathbf{u}(\phi,\theta_0+\delta)\cdot\mathbf{u}_\theta(\phi,\theta_0+\delta)\right.\\ &\qquad\qquad \left. -\ \mathbf{v}\circ\mathbf{u}(\phi,\theta_0-\delta)\cdot\mathbf{u}_\theta(\phi,\theta_0-\delta)\right]\, d\phi\\ &= 2\delta\int_{\phi_0-\epsilon}^{\phi_0+\epsilon} \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\left[\mathbf{v}\circ\mathbf{u}(\phi,\theta)\cdot\mathbf{u}_\theta(\phi,\theta)\right]\Big|_{\theta=\theta_*} d\phi\end{aligned}$$

for some θ_* with $|\theta_*-\theta_0|<\delta$. Hence,

$$\lim_{\epsilon,\delta\searrow 0}\frac{1}{\mathcal{H}^2(\mathcal{W})}\int_{\Gamma_2\cup\Gamma_4}\mathbf{v}\cdot\mathbf{n} = \frac{1}{\sin^2\phi_0}\frac{\partial}{\partial\theta}\left[\mathbf{v}\circ\mathbf{u}(\phi_0,\theta)\cdot\mathbf{u}_\theta(\phi_0,\theta)\right]\Big|_{\theta=\theta_0}.$$

We have obtained the following coordinate formula for the surface divergence:

$$\begin{aligned}\operatorname{div}^{\mathbb{S}^2}\mathbf{v}(\mathbf{p}) &= \lim_{\epsilon,\delta\searrow 0}\frac{1}{\mathcal{H}^2(\mathcal{W})}\int_{\partial\mathcal{W}}\mathbf{v}\cdot\mathbf{n}\\ &= \cot\phi_0\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi(\phi_0,\theta_0) + \frac{\partial}{\partial\phi}\left[\mathbf{v}\circ\mathbf{u}(\phi,\theta_0)\cdot\mathbf{u}_\phi(\phi,\theta_0)\right]\Big|_{\phi=\phi_0}\\ &\qquad + \frac{1}{\sin^2\phi_0}\frac{\partial}{\partial\theta}\left[\mathbf{v}\circ\mathbf{u}(\phi_0,\theta)\cdot\mathbf{u}_\theta(\phi_0,\theta)\right]\Big|_{\theta=\theta_0}.\end{aligned}$$

Now we apply this formula to the product $\psi\mathbf{v}$. Replacing $\mathbf{v}$ with $\psi\mathbf{v}$, we find

$$\begin{aligned}\operatorname{div}^{\mathbb{S}^2}(\psi\mathbf{v})(\mathbf{p}) &= \cot\phi_0\psi(\mathbf{p})\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi(\phi_0,\theta_0)\\ &\qquad + \frac{\partial}{\partial\phi}\left[\psi\circ\mathbf{u}(\phi,\theta_0)\ \mathbf{v}\circ\mathbf{u}(\phi,\theta_0)\cdot\mathbf{u}_\phi(\phi,\theta_0)\right]\Big|_{\phi=\phi_0}\\ &\qquad + \frac{1}{\sin^2\phi_0}\frac{\partial}{\partial\theta}\left[\psi\circ\mathbf{u}(\phi_0,\theta)\ \mathbf{v}\circ\mathbf{u}(\phi_0,\theta)\cdot\mathbf{u}_\theta(\phi_0,\theta)\right]\Big|_{\theta=\theta_0}\\ &= \frac{\partial}{\partial\phi}\psi\circ\mathbf{u}(\phi,\theta_0)\Big|_{\phi=\phi_0}\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi(\phi_0,\theta_0)\\ &\qquad + \frac{1}{\sin^2\phi_0}\frac{\partial}{\partial\theta}\psi\circ\mathbf{u}(\phi_0,\theta)\Big|_{\theta=\theta_0}\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\theta(\phi_0,\theta_0)\\ &\qquad + \psi(\mathbf{p})\operatorname{div}^{\mathbb{S}^2}\mathbf{v}(\mathbf{p}).\end{aligned}$$

Comparison with (7) indicates that it remains to show

$$\operatorname{grad}\psi(\mathbf{p})\cdot\mathbf{v} = \frac{\partial}{\partial\phi}\psi\circ\mathbf{u}(\phi,\theta_0)\Big|_{\phi=\phi_0}\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi(\phi_0,\theta_0) + \frac{1}{\sin^2\phi_0}\frac{\partial}{\partial\theta}\psi\circ\mathbf{u}(\phi_0,\theta)\Big|_{\theta=\theta_0}\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\theta(\phi_0,\theta_0).$$

Recalling the definition of the surface gradient as considered in part (ii), we can use an extension $\overline{\psi}:\mathcal{V}\to\mathbb{R}$ and write

$$\operatorname{grad}\psi(\mathbf{p}) = (D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\phi)\mathbf{u}_\phi + \frac{D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\theta}{\sin^2\phi_0}\mathbf{u}_\theta.$$

Since

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_\phi)\mathbf{u}_\phi + \frac{\mathbf{v}\cdot\mathbf{u}_\theta}{\sin^2\phi_0}\mathbf{u}_\theta$$

as well, we have

$$\operatorname{grad}\psi(\mathbf{p})\cdot\mathbf{v}(\mathbf{p}) = (D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\phi)(\mathbf{v}(\mathbf{p})\cdot\mathbf{u}_\phi) + \frac{(D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\theta)(\mathbf{v}\cdot\mathbf{u}_\theta)}{\sin^2\phi_0}.$$

The chain rule gives

$$\frac{\partial}{\partial\phi}\psi\circ\mathbf{u}(\phi,\theta_0)\Big|_{\phi=\phi_0} = D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\phi \qquad\text{and}\qquad \frac{\partial}{\partial\theta}\psi\circ\mathbf{u}(\phi_0,\theta)\Big|_{\theta=\theta_0} = D\overline{\psi}(\mathbf{p})\cdot\mathbf{u}_\theta,$$

so we have established the product rule (7) for the divergence of a scaled vector field on a surface.

In particular, we can apply this to the scaled field

$$\frac{\phi\ f\ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}}$$

appearing in (5) which rose in relation to our first variation formula. If $f\in C^2(\mathbb{S}^2)$, then we can conclude

$$\operatorname{div}^{\mathbb{S}^2}\left(\frac{\phi\ f\ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}}\right) = \operatorname{grad}\phi\ \cdot\left(\frac{f\ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}}\right) + \phi\ \operatorname{div}^{\mathbb{S}^2}\left(\frac{f\ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2+f^2}}\right).$$

Substituting this into our first variation formula, we find that for $f \in C^2(\mathbb{S}^2)$

$$\begin{aligned}
\delta\mathfrak{A}_f[\phi] &= \int_{\mathbb{S}^2} \phi \left[\sqrt{|\operatorname{grad} f|^2 + f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right] \\
&\qquad + \int_{\mathbb{S}^2} \frac{f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}} \cdot \operatorname{grad} \phi \\
&= \int_{\mathbb{S}^2} \phi \left[\sqrt{|\operatorname{grad} f|^2 + f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}} - \operatorname{div}^{\mathbb{S}^2}\left(\frac{f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right)\right] \\
&\qquad + \int_{\mathbb{S}^2} \operatorname{div}^{\mathbb{S}^2}\left(\frac{\phi \ f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right) \\
&= \int_{\mathbb{S}^2} \phi \left[\sqrt{|\operatorname{grad} f|^2 + f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}} - \operatorname{div}^{\mathbb{S}^2}\left(\frac{f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right)\right].
\end{aligned}$$

Note that the last expression is

$$\left\langle \sqrt{|\operatorname{grad} f|^2 + f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}} - \operatorname{div}^{\mathbb{S}^2}\left(\frac{f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right), \phi \right\rangle_{L^2(\mathbb{S}^2)}.$$

Therefore, we have identified $\operatorname{grad} \mathfrak{A}$:

$$\operatorname{grad} \mathfrak{A}[f] = \sqrt{|\operatorname{grad} f|^2 + f^2} + \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}} - \operatorname{div}^{\mathbb{S}^2}\left(\frac{f \ \operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right).$$

Gradient Flow

(g) *Write down the equation of gradient flow on $\mathcal{A} \subset L^2(\mathbb{S}^2)$. Here, of course, we mean the infinite dimensional gradient flow with respect to the L^2 inner product*

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{S}^2} fg$$

on $\mathcal{A}$. That is, the gradient of $\mathfrak{A}$ is defined to be the element $\operatorname{grad} \mathfrak{A}[f]$ of $C^0(\mathbb{S}^2)$ such that

$$\delta\mathfrak{A}_f[\phi] = \int_{\mathbb{S}^2} \operatorname{grad} \mathfrak{A}[f] \ \phi \qquad \textit{for all } \phi \in C_c^\infty(\mathbb{S}^2).$$

As a bit of an aside, note that since $\mathbb{S}^2$ *is compact, we know* $C_c^\infty(\mathbb{S}^2) = C^\infty(\mathbb{S}^2)$. *Hint: You can define a* **surface divergence** *using the usual limit of flux integrals and the usual proof will also give you a divergence theorem for domains in* $\mathbb{S}^2$. *You'll want to assume (or prove) that a* **product rule for the surface divergence** *of a scaled field has the usual form:*

$$\operatorname{div}(f\mathbf{w}) = \nabla f \cdot \mathbf{w} + f \operatorname{div} \mathbf{w}.$$

Solution: We identified the gradient of the area functional in the last part. The gradient flow equation is for a function $f = f(\mathbf{p}, t) : \mathbb{S}^2 \times [0, T) \to \mathbb{R}$ and is given by

$$\frac{\partial f}{\partial t} = -\operatorname{grad} \mathfrak{A}[f].$$

That is, the gradient flow equation in this case is

$$\frac{\partial f}{\partial t} = \operatorname{div}^{\mathbb{S}^2}\left(\frac{f \;\operatorname{grad} f}{\sqrt{|\operatorname{grad} f|^2 + f^2}}\right) - \sqrt{|\operatorname{grad} f|^2 + f^2} - \frac{f^2}{\sqrt{|\operatorname{grad} f|^2 + f^2}}.$$

(h) *Find an explicit solution for the evolution of spheres under this gradient flow. Hint: A sphere is given by the radial graph associated to* $f \equiv \text{constant}$. *This means you can look for solutions* $f = f(\mathbf{p}, t)$ *having the form* $f = f(t)$.

Solution: If we have a graph which is a sphere concentric with $\mathbb{S}^2$, then we have $f = c(t) > 0$ is spatially constant on the sphere and of course the spatial gradient vanishes. The equation for f becomes the ODE

$$c' = -2c.$$

Thus, $c(t) = r_0 e^{-2t}$. The radius starts at r_0 and shrinks to $r = 0$ exponentially. This seems strange, as I would expect something like **mean curvature flow** according to which the radius should shrink to zero in finite time according to the ODE

$$r' = -\frac{1}{r} \qquad \text{(the mean curvature with respect to the outward normal)}$$

For this equation $r(t) = \sqrt{r_0^2 - 2t}$ which vanishes in finite time ($T = r_0^2/2$).

Another interesting possibility to consider here is the evolution of a sphere which is a graph of a function $f > 0$ over the unit sphere $\mathbb{S}^2$ centered at the origin but is not concentric with the unit sphere.

Two Derivations of the Wave Equation

Problem 2 *The one-dimensional wave equation can be written as*

$$u_{tt} = u_{xx}.$$

This equation is usually derived as a **small amplitude approximation** *of the equation for the vertical displacement of a horizontal one-dimensional elastic continuum as indicated in Figure 1.*

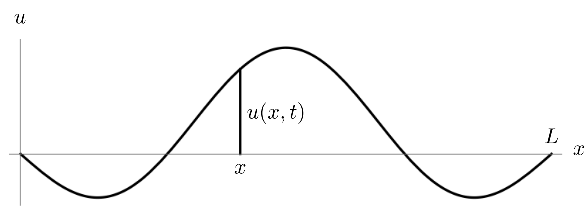

Figure 1: A "vibrating string." The one-dimensional continuum or "string" is assumed to be elastic and have equilibrium corresponding to $u \equiv 0$*. The value of* u *represents an approximation of the vertical displacement above the horizontal position* x*.*

In this context the function u *typically has domain* $[0, L] \times [0, T)$ *for some* $L > 0$ *and* $T > 0$ *and satisfies* $u(0, t) = u(L, t) \equiv 0$*. You can look up the derivation of the 1-D wave equation from this point of view in many textbooks on partial differential equations and in many other places (e.g., on the internet) as well. I'm first going to walk you through a derivation of the 1-D wave equation which I view as much superior to the usual one. In particular, no approximation is required. I have not seen this/my derivation elsewhere. My derivation requires one to assign a different physical meaning to the value of the function* u*.*

Let $u : [0, L] \times [0, T) \to \mathbb{R}$ *represent the horizontal displacement of a one-dimensional elastic continuum with fixed endpoints at* $x = 0$ *and* $x = L$*; see Figure 2.*

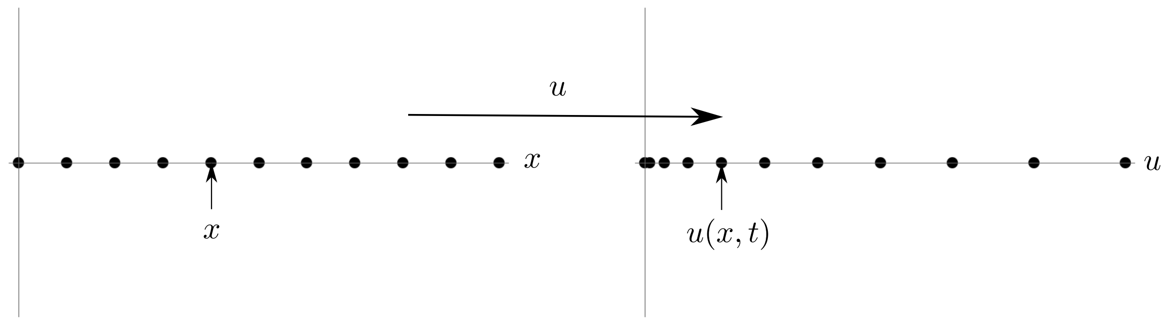

Figure 2: A horizontally displaced one-dimensional continuum with fixed endpoints. Here we also assume elasticity and an equilibrium corresponding to no displacement or $u(x) \equiv x$*. In the illustrated displacement each point (except for the two endpoints) is displaced to the left. One can imagine this also as an initial displacement* $u_0(x) = u(x, 0)$ *which corresponds to a restoring motion/force to the right. Naturally there may also be an initial velocity distribution along the continuum.*

This model for horizontal displacements with fixed endpoints is naturally suited to the boundary conditions:

$$\begin{cases} u(0, t) \equiv 0 & t \geq 0, \\ u(L, t) \equiv L & t \geq 0, \end{cases}$$

and the constraint

$$u_x(x, t) > 0.$$

The constraint corresponds to keeping the continuum ordered, so that there is no folding or overlap. Thus, horizontal displacements are naturally associated with the admissible class

$$\mathcal{A} = \{u \in C^2([0, L] \times [0, T)) : u_x(x, t) > 0,\ u(0, t) \equiv 0,\ u(L, t) \equiv L,\ t \geq 0\}.$$

(a) *The displacement illustrated in Figure 2 corresponds to* $u(x, t) = x^2$ *on the spatial interval* $[0, L] = [0, 1]$*. Find and plot a (horizontal) displacement* $u_0 \in C^0[0, 1]$ *defined by the following*

(i) $u_0(1/2) = 3/4$,

(ii) u_0 *is linear on the interval* $[0, 1/2]$, *and*

(iii) u_0 *agrees with an affine function on the interval* $[1/2, 1]$.

You can plot u_0 *in two different ways, once in the style of Figure 2 and also simply as a graph in the* x, u*-plane.*

Solution:

$$u_0(x) = \begin{cases} 3x/2, & x \leq 1/2 \\ (1+x)/2, & x \geq 1/2. \end{cases}$$

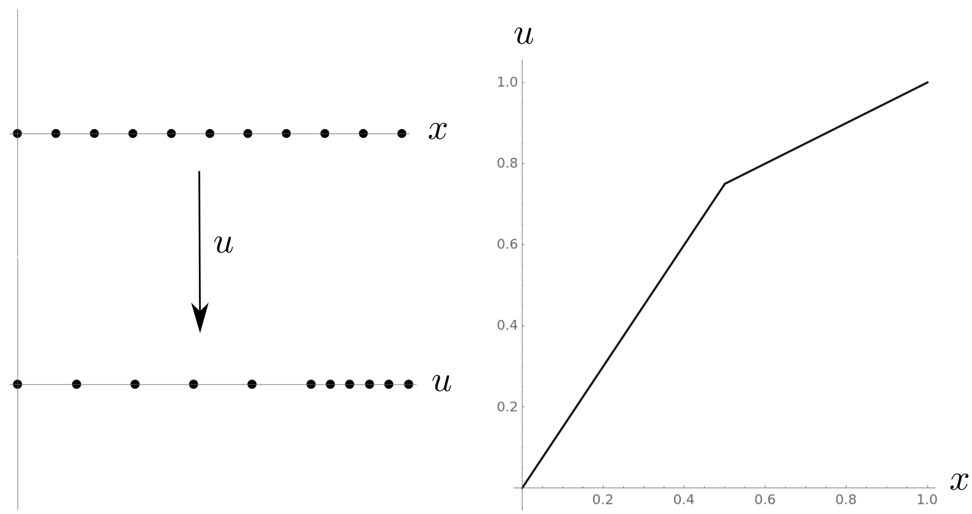

Figure 3: The piecewise affine displacement function u_0 illustrated in two different ways.

(iv) *Use the method of characteristics to determine the solution of*

$$\begin{cases} u_{tt} = u_{xx} & \text{on } [0,1] \times [0, \infty) \\ u(x,0) = u_0, & \\ u_t(x,0) = 0, & \\ u(0,t) \equiv 0, & \\ u(1,t) \equiv 1 & \end{cases}$$

and make an animation of the image of u as a function of time represented by image dots as on the right in Figure 2.

Solution: Let us begin with d'Alembert's solution for the PDE which in the case of zero velocity reads:

$$u(x,t) = \frac{1}{2}[u_0(x-t) + u_0(x+t)].$$

On the face of it, this solution will not apply in general for this problem as $x-t$ and $x+t$ are not generally in the domain of u_0. Considering the domain of independence and the characteristics used to derive this solution, however, we note that it should apply in the triangle $\{(x,t) : t < 1/2 - |x - 1/2|,\ 0 < x < 1\}$ indicated in Figure 4. Due to the piecewise nature of the initial values u_0, we also must consider three different cases within the initial triangle corresponding determined by the conditions/lines $x - t = 1/2$ and $x + t = 1/2$. These division lines are also indicated as dashed lines in Figure 4.

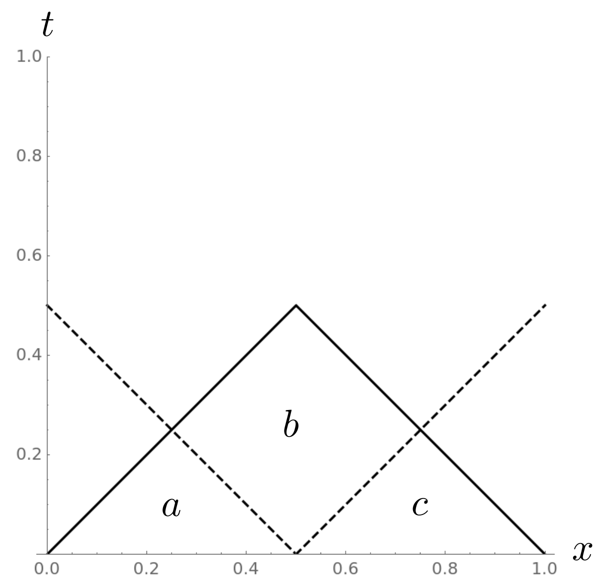

Figure 4: Domain for the solution of the 1-D wave equation.

For this initial triangular domain, we obtain

$$u(x,t) = \begin{cases} \frac{1}{2}[3(x-t)/2 + 3(x+t)/2], & \text{for (a) } x-t \le 1/2 \text{ and } x+t \le 1/2 \\ \frac{1}{2}[3(x-t)/2 + (1+x+t)/2], & \text{for (b) } x-t \le 1/2 \text{ and } x+t \ge 1/2 \\ \frac{1}{2}[(1+x-t)/2 + (1+x+t)/2], & \text{for (c) } x-t \ge 1/2 \text{ and } x+t \le 1/2 \end{cases}$$

$$= \begin{cases} 3x/2, & \text{for (a) } x-t \le 1/2 \text{ and } x+t \le 1/2 \\ 1/4 + x - t/2, & \text{for (b) } x-t \le 1/2 \text{ and } x+t \ge 1/2 \\ (1+x)/2, & \text{for (c) } x-t \ge 1/2 \text{ and } x+t \le 1/2. \end{cases}$$

If we move to a point outside the initial triangle, say to a point $p = (p_1, p_2) = (x, t)$ with $x + t < 1/2$, as indicated in Figure 5, then one of the characteristics $t = x - p_1 + p_2$ passing through p intersects the lateral boundary $x = 0$ instead of the initial boundary $t = 0$. There are several possibilities for how to proceed in this case, one of which is to solve the wave equation again from scratch using the method of characteristics. There is another process according to which one can extend the initial data u_0 to the rest of $\mathbb{R}$ and just use d'Alembert's formula. My favorite method is to introduce a rectangular region Ω as indicated in Figure 5 and apply the divergence theorem.

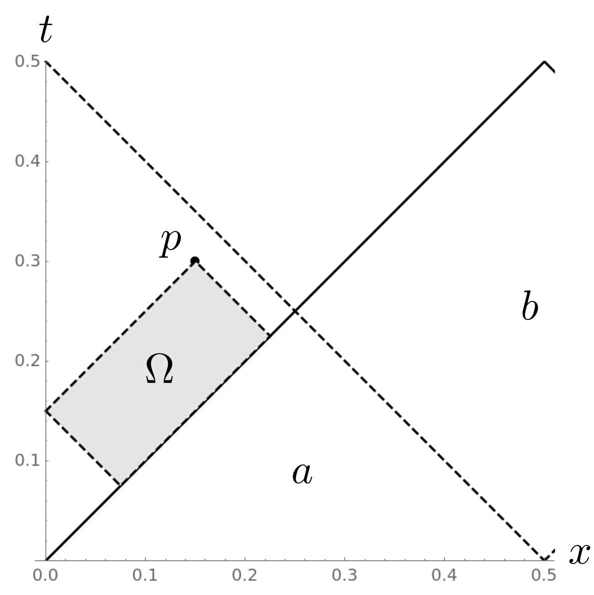

Figure 5: Using the method of characteristics outside the initial trianglular domain.

Notice that the wave equation in one dimension $(u_x - u_t)_x + (u_x - u_t)_t = 0$ can be written as $\operatorname{div} \mathbf{v} = 0$ where div denotes the plane divergence on the x, t-plane and $\mathbf{v}$ is the "diagonal" vector field $(u_x - u_t, u_x, u_t)$ with the same entry in both coordinates. As you might guess, this method will apply to other divergence form equations associated with other (more complicated) vector fields. But let me carry out the details for the point p under consideration. By the divergence theorem we know

$$\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the outward co-normal to the rectangle. Let us decompose the boundary of Ω into four side segments Γ_1, Γ_2, Γ_3, and Γ_4 going around counterclockwise with

$$\Gamma_1 = \left\{ (\xi, \xi) : \frac{p_2 - p_1}{2} \leq \xi \leq \frac{p_2 + p_1}{2} \right\}$$

is the segment on the initial triangle where we already know $u(x, t)$. The limits on ξ in this set description were obtained by considering parameterized charactericts. For example, the expression $(\xi(t), \eta(t)) = (p_1 + t, p_2 - t)$ has equal components when $t = (p_2 - p_1)/2$ corresponding to $\xi = p_1 + t = (p_1 + p_2)/2$. This is the upper limit for ξ. We can calculate the integral over Γ_1 as follows:

$$\int_{\Gamma_1} \mathbf{v} \cdot \mathbf{n} = \int_{(p_2 - p_1)/2}^{(p_1 + p_2)/2} \mathbf{v}(\xi, \xi) \cdot \frac{(1, -1)}{\sqrt{2}} \sqrt{2}\, d\xi = 0,$$

since $\mathbf{n} = (1, -1)/\sqrt{2}$ along this segment and $\mathbf{v} \cdot \mathbf{n} = 0$. The segment Γ_2 is parameterized by $\gamma(t) = (p_1 + t, p_2 - t)$ with $0 let \leq (p_2 - p_1)/2$. Therefore,

$$\begin{aligned}
\int_{\Gamma_2} \mathbf{v} \cdot \mathbf{n} &= \int_0^{(p_2 - p_1)/2} \mathbf{v} \cdot (1, 1)\, dt \\
&= \int_0^{(p_2 - p_1)/2} 2[u_x(p_1 + t, p_2 - t) - u_t(p_1 + t, p_2 - t)]\, dt \\
&= \int_0^{(p_2 - p_1)/2} 2\frac{d}{dt}[u(p_1 + t, p_2 - t)]\, dt \\
&= 2u((p_1 + p_2)/2, (p_1 + p_2)/2) - 2u(p).
\end{aligned}$$

Notice that $u(p) = u(x, y)$ appearing here is the value we would like to determine, and

$$u\left(\frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}\right) = \frac{3(p_1+p_2)}{4} = \frac{3(x+t)}{4}$$

is a value we know. Since $\mathbf{n} = (-1, 1)/\sqrt{2}$ along Γ_3, we will have

$$\int_{\Gamma_3} \mathbf{v} \cdot \mathbf{n} = 0,$$

and we have only to consider the flux integral along the segment Γ_4 connecting $(0, p_2 - p_1)$ to $((p_2 - p_1)/2, (p_2 - p_1)/2)$.

$$\begin{aligned}
\int_{\Gamma_4} \mathbf{v} \cdot \mathbf{n} &= \int_0^{(p_2-p_1)/2} \mathbf{v} \cdot (-1, -1)\, dt \\
&= -\int_0^{(p_2-p_1)/2} 2[u_x(t, p_2 - p_1 - t) - u_t(t, p_2 - p_1 - t)]\, dt \\
&= -\int_0^{(p_2-p_1)/2} 2\frac{d}{dt}[u(t, p_2 - p_1 - t)]\, dt \\
&= 2u(0, p_2 - p_1) - 2u((p_2 - p_1)/2, (p_2 - p_1)/2).
\end{aligned}$$

We know both of these values with $u(0, p_2 - p_1) = 0$ from the (lateral) boundary condition and

$$u\left(\frac{p_2-p_1}{2}, \frac{p_2-p_1}{2}\right) = \frac{3(p_2-p_1)}{4} = \frac{3(t-x)}{4}.$$

Putting these integrals together we find

$$0 = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} = \frac{3(x+t)}{2} - 2u(x, t) - \frac{3(t-x)}{2},$$

or

$$u(x, t) = \frac{3x}{2} \qquad \text{for } x + t \leq 1/2.$$

I will leave it to you to apply whatever method you prefer to find the value of $u(x, t)$ on the rest of the strip $\mathcal{U} = \{(x, t) : 0 \leq t \leq 1,\ t \geq 0\}$. Having a little experience, I can now "see" the entire solution, so I will explain to you what you should get. You can also find this solution animated in the Mathematica notebook posted on the course webpage. Notice first that the expression $u(x,t) = 3x/2$ prevails across the transition from the lateral triangle considered above to the initial d'Alembert triangle. You will find that the same thing will hold for the transition to the lateral triangle $\{(x,t) : t \le x - 1/2,\ 1/2 \le x \le 1\}$ on the right. That is, any fixed time $t_0 < 1/4$ determines a piecewise C^1 function $u(x,t_0)$ of x having two corners corresponding to $x = x_1 = 1/2 - t_0$ and $x = x_2 = 1/2 + t_0$; these are the x values corresponding to the intersections of the dashed characteristics $t = |x - 1/2|$ with the line $t = t_0$. The graph of $u(x,t_0)$ is shown for $t_0 = 1/8$ in Figure 6.

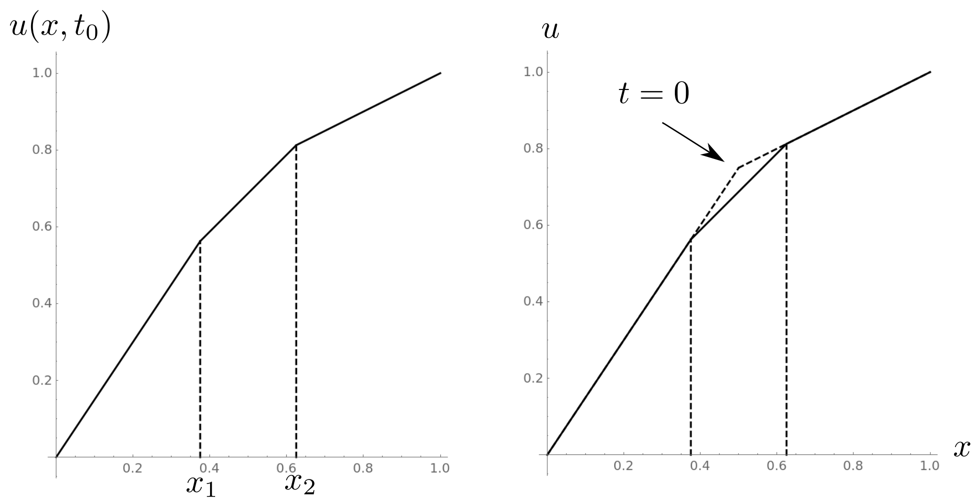

Figure 6: The displacement of the continuum for fixed small time $t < 1/4$. In this case, $t = t_0 = 1/8$.

Notice that this displacement matches the original displacement identically on the intervals $0 \le x \le x_1$ and $x_2 \le x \le 1$ in accord with the finite propogation speed associated with the wave equation. The slope $u_x(x,t_0)$ in the middle interval is the average of the slopes 1/2 and 3/4 on the lateral intervals. That is, $u_x(x,t) = 1$ for $x_1 < x < x_2$, and we can see this is the case from the formula given in the initial triangle by d'Alembert's formula. As time progresses, the initial peak illustrated on the right in Figure 3 and on the right in Figure 6 is replaced on an expanding central interval $[x_1, x_2] = [x_1(t), x_2(t)]$ by a displacement of slope identically 1. This continues until time $t = 1/2$ at which point $u(x, 1/2) \equiv x$ is the identity/"equilibrium" displacement. Of course, the resulting configuration is not an actual equilibrium because the velocity $u_t(x, 1/2)$ is not identically zero. So for times $t > 1/2$ two corner points will appear again creating a shrinking central interval on which slope $u_x(x,t) = 1$ is preserved but the displacement is "reversed" with

$$u(x,t) = \frac{x}{2} \qquad \text{for } 0 \le x \le t - 1/2$$

and

$$u(x,t) = \frac{3x-1}{2} \qquad \text{for } 3/2 - t \le x \le 1.$$

In this way a new zero velocity displacement will be attained at time $t = 1$ given by

$$u_1(x) = \begin{cases} x/2, & x \le 1/2 \\ (3x-1)/2, & x \ge 1/2. \end{cases}$$

After this, the oscillation reverses in like manner and repeats for all time.

The description above allows one to write down the solution $u(x,t)$ for all time in terms of a linear oscillation between the inital piecewise affine right displacement u_0 and the piecewise affine left displacement given by u_1. It is also perhaps easier to "see" the solution in terms of the function $w(x,t) = u(x,t) - x$ which satisfies the wave equation with homogeneous boundary values. Again, w oscillates linearly in time between two zero velocity ($w_t \equiv 0$) wave forms $w_0(x) = (1/2 - |x - 1/2|)/4$ and $w_1(x) = (|x - 1/2| - 1/2)/4$.

The animation, as mentioned above, is given in the posted Mathematica notebook in terms of w.

We assume our continuum has the elastic properties of an **inhomogeneously compressed/extended linear spring**. *Given this assumption, we need to determine how forces are determined locally in terms of the displacement u. Recall that in the elementary modeling of a spring compression and extension are assumed to be homogeneous. Specifically, if the spring is at equilibrium and of length L it is assumed there is a constant k, called Hooke's constant, such that the spring exerts a force $F = -k(X - L)$ on any object attached to the end located at $x = X$ for $X > 0$. Clearly this simple model needs to be generalized or otherwise modified for our application. The following parts suggest one way to do this based on the assumption that local forces exerted under inhomogeneous displacement should be related to* **density**.

(b) *Assume a spring/string (one-dimensional elastic continuum) has one endpoint fixed at $x = 0$ and one free end. Assume also an equilibrium length L corresponding to a linear density ρ_0. Determine the horizontal displacement function $u_0 : [0, L] \to \mathbb{R}$ corresponding to a* **homogeneous horizontal displacement** *to the interval $[0, X]$. Express the density ρ is the displaced spring and the resulting force associated with the displacement u_0.*

Solution: The homogeneous stretch u_0 described is given by

$$u_0(x) = \frac{X}{L}x.$$

The density under this stretch is

$$\rho = \frac{\text{total mass}}{\text{total length}} = \frac{\rho_0 L}{X} = \frac{\rho_0}{u_0'}. \tag{8}$$

The force on an object attached at $X > 0$ should be

$$F = -k(X - L) = -kL\,(u_0' - 1) = -kL\left(\frac{\rho_0}{\rho} - 1\right). \tag{9}$$

(c) *Use the previous part to explain/justify the elastic assumption*

$$\tau = \alpha(u_x - 1)$$

for the local tension in an inhomogeneously displaced spring where $u : [0, L] \times [0, T) \to \mathbb{R}$ *describes the displacement,* α *is an appropriate constant, and* τ *is positive for extension beyond the equilibrium density and negative for compression.*

Solution: Assuming the tension is related to the local density by the continuous version of (8)

$$\rho = \rho(x, t) = \frac{\rho_0}{u_x} \tag{10}$$

and the local density is related to the tension according to the local version of (9)

$$\tau = kL\left(\frac{\rho_0}{\rho(x,t)} - 1\right)$$

with $\tau = -F$ because the tension is opposite the force, we have

$$\tau = \alpha(u_x - 1)$$

with $\alpha = kL$.

Let x_1 and x_2 be two points with $0 \le x_1 < x_2 < L$ and images $u(x_1, t)$ and $u(x_2, t)$ at time t so that the image interval is

$$I = \{u(x,t) : x_1 \le x \le x_2\}.$$

(d) *Show the center of mass of the image interval is*

$$\frac{1}{\rho_0(x_2 - x_1)} \int_I u(x,t)\rho(x,t)\,dx = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} u(x,t)\,dx.$$

Solution: By definition, the center of mass is $x_* = x_*(t)$ with

$$\begin{aligned}
x_* &= \frac{1}{\rho_0(x_2 - x_2)} \int_{u \in I} u\rho \\
&= \frac{1}{\rho_0(x_2 - x_2)} \int_{u \in I} u(x,t)\rho(x,t) \\
&= \frac{1}{\rho_0(x_2 - x_2)} \int_{x_1}^{x_2} u(x,t)\rho(x,t)\ u_x(x,t)\,dx \\
&= \frac{1}{\rho_0(x_2 - x_2)} \int_{x_1}^{x_2} u(x,t)\rho_0\,dx \\
&= \frac{1}{x_2 - x_2} \int_{x_1}^{x_2} u(x,t)\,dx.
\end{aligned}$$

The key is the third equality where the scaling factor for the parameterization $\gamma(x) = u(x,t)$ is $\sigma = |\gamma'(x)| = |u_x(x,t)| = u_x(x,t)$. The fourth equality follows from (10).

The **continuum assumption** *for motion is that the sum of the forces acting on I, expressed with respect to the center of mass of I, is given by the resultant tension forces on the endpoints of I.*

(e) *Under the continuum assumption, show that Newton's second law gives*

$$\rho_0(x_2 - x_1)\,\frac{d^2}{dt^2}\,\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} u(x,t)\,dx = \alpha[u_x(x_2,t) - 1] - \alpha[u_x(x_1,t) - 1].$$

Solution: Newton's second law (under the continuum assumption) gives

$$\rho_0(x_2 - x_1)\frac{d^2}{dt^2}x_* = \tau(x_2,t) - \tau(x_1,t).$$

Replacing the center of mass x_* with the expression from part (d) and the tension with the expression from part (c), we have

$$\begin{aligned}\rho_0(x_2 - x_1)\frac{d^2}{dt^2}\frac{1}{x_2 - x_2}\int_{x_1}^{x_2} u(x,t)\,dx &= \alpha[u_x(x_2,t) - 1] - \alpha[u_x(x_1,t) - 1]\\ &= \alpha[u_x(x_2,t) - u_x(x_1,t)].\end{aligned}$$

(f) *Simplify, manipulate, and use the expression from part (e) along with the fundamental lemma of the calculus of variations to finish the derivation of the one-dimensional wave equation in the form*

$$\rho_0 u_{tt} = \alpha u_{xx}.$$

Solution: Continuing from part (e), we have

$$\begin{aligned}\rho_0\frac{d^2}{dt^2}\int_{x_1}^{x_2} u(x,t)\,dx &= \alpha[u_x(x_2,t) - u_x(x_1,t)]\\ &= \alpha\int_{x_1}^{x_2}\frac{\partial}{\partial x}u_x(x,t)\,dx\\ &= \alpha\int_{x_1}^{x_2} u_{xx}(x,t)\,dx.\end{aligned}$$

Differentiating under the integral sign on the left and collecting both sides in a single integral we get

$$\int_{x_1}^{x_2} [\rho_0 \; u_{tt}(x,t) - \alpha \; u_{xx}(x,t)] \; dx = 0.$$

The argument from the fundamental lemma tells us that we get a contradiction unless

$$\rho_0 \; u_{tt}(x,t) - \alpha \; u_{xx}(x,t) \equiv 0.$$

This completes this derivation of the 1-D wave equation.

Exercise 2 *Generalize the derivation above to higher dimensions. Hint: Start at the end with the application of the fundamental lemma of the calculus of variations and deduce the appropriate form of the elastic assumption for a higher-dimensional continuum/membrane. (I think the step of justifying this assumption using homogeneous expansions is rather complicated.)*

Hamilton's Principle

Remember that Hamilton's principle says that any particle motion determined by Newton's second law in a potential field can be obtained as an extremal for the action functional

$$H[x] = \frac{1}{2}\int_0^T m\dot{x}(\tau)^2 \, d\tau - \int_0^T \Phi(x(\tau),\tau) \, d\tau \tag{11}$$

on the admissible class of motions with determinant outcomes

$$\mathcal{A} = \{x \in C^2[0,T] : x(0) = x_0 \text{ and } x(T) = x_1\}$$

where Φ *is the (possibly time varying) potential function for the field satisfying* $F(x,t) = -\Phi_x(x,t)$*. In view of the derivation above (using Newton's second law) this rather strongly suggests there should be a variational derivation of the wave equation using some kind of Hamilton's principle. In fact, it is true that such a derivation is possible, and what is moreover true is that this derivation applies in any dimension to give the wave equation in arbitrary spatial dimensions*

$$u_{tt} = \Delta u.$$

(g) *Obtain Newton's second law for particle motion in a potential force field (once more) using the Hamiltonian action functional defined in (11).*

Solution: Taking the first variation of the Hamiltonian give above we get

$$\begin{aligned}\delta H_x[\phi] &= \int_0^T \Big[m\dot{x}(\tau)\ \dot{\phi}(\tau) - \Phi_x(x(\tau),\tau)\ \phi(\tau)\Big]\, d\tau \\ &= \int_0^T \left[-m\frac{d}{dt}\dot{x}(\tau) - \Phi_x(x(\tau),\tau)\right]\ \phi(\tau)\, d\tau \\ &= \int_0^T [-m\ddot{x}(\tau) - \Phi_x(x(\tau),\tau)]\ \phi(\tau)\, d\tau.\end{aligned}$$

Thus, the Euler-Lagrange equation (for C^2 extremals $x \in \mathcal{A}$) is

$$m\ddot{x} = -\Phi(x,t).$$

Since the potential gives the force according to $F = -\Phi$, this is $F = ma$.

(h) *Define an appropriate admissible class and an appropriate generalization of Hamilton's action functional on that admissible class so that the wave equation is given as the Euler-Lagrange equation of C^2 extremals.*

Solution: Let us consider

$$\mathcal{A} = C^2(\mathcal{U} \times (0,T))$$

and the functional $H : \mathcal{A} \to \mathbb{R}$ by

$$\begin{aligned}H[u] &= \frac{1}{2}\int_{\mathcal{U}} \left[\int_0^T u_t(\mathbf{x},\tau)^2\, d\tau - \int_0^T |Du(\mathbf{x},\tau)|^2\, d\tau\right] \\ &= \frac{1}{2}\int_{\mathcal{U}\times(0,T)} \left[u_t(\mathbf{x},\tau)^2 - |Du(\mathbf{x},\tau)|^2\right].\end{aligned}$$

With a full variation $\phi \in C_c^\infty(\mathcal{U} \times (0,T))$ we have

$$(u+\epsilon\phi)_t = u_t + \epsilon\phi_t \qquad \text{and} \qquad D(u+\epsilon\phi) = Du + \epsilon D\phi,$$

so

$$\delta H_u[\phi] = \int_{\mathcal{U}\times(0,T)} [u_t\phi_t - Du \cdot D\phi].$$

In the first integral, we separate out the time dependence (Fubini's theorem) and integrate by parts:

$$\int_{\mathcal{U}} \left(\int_0^T u_t(\mathbf{x}, \tau)\ \phi_t(\mathbf{x}, \tau)\, d\tau \right) = \int_{\mathcal{U}} \left(- \int_0^T u_{tt}(\mathbf{x}, \tau)\ \phi(\mathbf{x}, \tau)\, d\tau \right).$$

For the second integral, we reverse the order of integration and apply the divergence theorem/product rule:

$$\int_{(0,T)} \left(\int_{\mathcal{U}} Du \cdot D\phi \right) = \int_{(0,T)} \left(- \int_{\mathcal{U}} \Delta u \phi \right).$$

Recombining the integrals we find

$$\delta H_u[\phi] = \int_{\mathcal{U} \times (0,T)} [\Delta u - u_{tt}]\, \phi.$$

Thus, the Euler-Lagrange equation is the wave equation

$$u_{tt} = \Delta u.$$