MATH 6702 Assignment 8 = Challenge Problems
Due Friday May 7, 2021

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There are two challenge problems below. The first is quite involved to a large extent due to technical issues (and has some difficult points as well), and the second is much easier and less technical but still definitely challenging for most students at your level. They are in this order to correspond to the order of material presented on partial differential equations in the course, namely, the first problem is about gradient flow in infinite dimensions (we presented the heat equation as a gradient flow on a subset/subspace of $L^2(\mathcal{U})$), and the second problem concerns the derivation of the wave equation (this derivation was not presented in the lecture, but we discussed some aspects of the wave equation at the end of the course). I wanted to put a problem like the first one on the final exam, but I didn't because I thought the exam was already long enough and would probably turn out to be long, involved, and difficult if I put such a problem. (I was correct about all three of those things.) I had prepared most of the material for the second problem for the last/extra lecture, but we didn't get to it. Both variational constructions are quite beautiful I think.

Infinite Dimensional Gradient Flow

Problem 1 *We will denote by $\mathbb{S}^2$ the two-dimensional unit sphere in $\mathbb{R}^3$. That is,*

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

It should be clear that we can take $\mathbb{S}^2$ both as a domain of integration and as a domain for real valued functions $f : \mathbb{S}^2 \to \mathbb{R}$. In fact, the set of continuously differentiable functions $f \in C^1(\mathbb{S}^2)$ makes good sense as does $C^2(\mathbb{S}^2)$.

Calculus/Differentiability on a Sphere

(a) *Give two different ways to define what it means for a function $f : \mathbb{S}^2 \to \mathbb{R}$ to be continuously differentiable and show they are equivalent. Here are some hints if you need them*

- **(i)** *Consider extensions $\overline{f} : \mathcal{V} \to \mathbb{R}$ of f to an open subset $\mathcal{V} \subset \mathbb{R}^3$ with $\mathbb{S}^2 \subset \mathcal{V}$. In order to decide if $f \in C^1(\mathbb{S}^2)$ consider whether or not there exists an extension $\overline{f} \in C^1(\mathcal{V})$.*
- **(ii)** *Consider compositions $f \circ X : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U} \subset \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $X \in C^1(\mathcal{U} \to \mathbb{S}^2)$. The definition of C^1 you get should only use values of f that are given on $\mathbb{S}^2$.*

(b) *Given a unit vector $\mathbf{v} \in T_{\mathbf{p}}\mathbb{S}^2$, define the directional derivative of $f \in C^1(\mathbb{S}^2)$ by*

$$\nabla_{\mathbf{v}} f(\mathbf{p}) = D_{\mathbf{v}}\overline{f}(\mathbf{p}) = D\overline{f}(\mathbf{p}) \cdot \mathbf{v}$$

where $\overline{f} \in C^1(\mathcal{U})$ is an extension of f to an open set $\mathcal{U} \subset \mathbb{R}^3$. Show that this definition does not depend on the choice of extension $\overline{f}$. Hint: Express $\nabla_{\mathbf{v}} f(\mathbf{p})$ in terms of composition with a curve.

(c) *Define $C^2(\mathbb{S}^2)$.*

(d) *Note that the* **unit sphere map** *given by* $\mathbf{u} : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ *by*

$$\mathbf{u}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

parameterizes most of the sphere. Calculate the scaling factor $\sigma_{\mathbf{u}}$ *for area with respect to this parameterization. That is, given* $f \in C^0(\mathbb{S}^2)$ *and* $\mathcal{U}$ *a domain of integration in* $(0, \pi) \times (0, 2\pi)$, *we have*

$$\int_{\mathbf{u}(\mathcal{U})} f = \int_{\mathcal{U}} f \circ \mathbf{u}\ \sigma_{\mathbf{u}}.$$

(e) *Consider*

$$\mathcal{A} = \{f \in C^1(\mathbb{S}^2) : f > 0\}$$

as an **admissible class** *and the associated functions* $\mathbf{g} : \mathbb{S}^2 \to \mathbb{R}^3$ *given by*

$$\mathbf{g}(\mathbf{p}) = f(\mathbf{p})\mathbf{p}.$$

Find the area scaling factor $\sigma : \mathbb{S}^2 \to \mathbb{R}$ *on* $\mathbb{S}^2$ *giving the area of the* **radial graph**

$$\mathcal{G} = \{f(\mathbf{p})\mathbf{p} : \mathbf{p} \in \mathbb{S}^2\}$$

so that the area functional is given by $\mathfrak{A} : \mathcal{A} \to \mathbb{R}$ *by*

$$\mathfrak{A}[f] = \int_{\mathcal{G}} 1 = \int_{\mathbb{S}^2} \sigma.$$

Hints: Calculate the area scaling factor σ_X *associated with the parameterization* $X : \mathcal{U} \to \mathbb{R}^3$ *by* $X = \mathbf{g} \circ \mathbf{u}$. *Then*

$$\sigma = \frac{\sigma_X}{\sigma_{\mathbf{u}}}.$$

(i) *First obtain an expression involving the quantities* $D\overline{f} \cdot \mathbf{u}_\phi$ *and* $D\overline{f} \cdot \mathbf{u}_\theta$.

(ii) *Simplify your expression using the* **surface gradient** *on* $\mathbb{S}^2$. *Remember this is a function* $\operatorname{grad} f : \mathbb{S}^2 \to \mathbb{R}^3$ *having the properties*

$$\operatorname{grad} f(\mathbf{p}) \in T_{\mathbf{p}}\mathbb{S}^2 \qquad \text{and} \qquad \operatorname{grad} f(\mathbf{p}) \cdot \mathbf{w} = \nabla_{\mathbf{w}} f(\mathbf{p}) \quad \textit{for all } \mathbf{w} \in T_{\mathbf{p}}\mathbb{S}^2.$$

(f) *Calculate the first variation* $\delta\mathfrak{A}_f[\phi]$ *of the area functional.*

Gradient Flow

(g) *Write down the equation of gradient flow on* $\mathcal{A} \subset L^2(\mathbb{S}^2)$*. Here, of course, we mean the infinite dimensional gradient flow with respect to the* L^2 *inner product*

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{S}^2} fg$$

on $\mathcal{A}$*. That is, the gradient of* $\mathfrak{A}$ *is defined to be the element* $\operatorname{grad}\mathfrak{A}[f]$ *of* $C^0(\mathbb{S}^2)$ *such that*

$$\delta\mathfrak{A}_f[\phi] = \int_{\mathbb{S}^2} \operatorname{grad}\mathfrak{A}[f]\ \phi \qquad \text{for all } \phi \in C_c^\infty(\mathbb{S}^2).$$

As a bit of an aside, note that since $\mathbb{S}^2$ *is compact, we know* $C_c^\infty(\mathbb{S}^2) = C^\infty(\mathbb{S}^2)$*. Hint: You can define a* **surface divergence** *using the usual limit of flux integrals and the usual proof will also give you a divergence theorem for domains in* $\mathbb{S}^2$*. You'll want to assume (or prove) that a* **product rule for the surface divergence** *of a scaled field has the usual form:*

$$\operatorname{div}(f\mathbf{w}) = \nabla f \cdot \mathbf{w} + f \operatorname{div}\mathbf{w}.$$

(h) *Find an explicit solution for the evolution of spheres under this gradient flow. Hint: A sphere is given by the radial graph associated to* $f \equiv$ constant*. This means you can look for solutions* $f = f(\mathbf{p}, t)$ *having the form* $f = f(t)$*.*

Two Derivations of the Wave Equation

Problem 2 *The one-dimensional wave equation can be written as*

$$u_{tt} = u_{xx}.$$

This equation is usually derived as a **small amplitude approximation** *of the equation for the vertical displacement of a horizontal one-dimensional elastic continuum as indicated in Figure 1.*

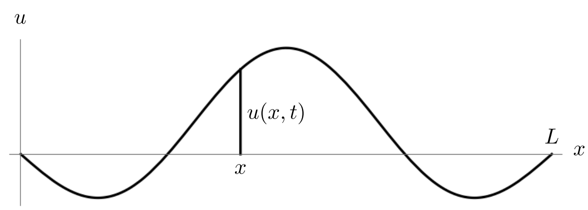

Figure 1: A "vibrating string." The one-dimensional continuum or "string" is assumed to be elastic and have equilibrium corresponding to $u \equiv 0$. The value of u represents an approximation of the vertical displacement above the horizontal position x.

In this context the function u typically has domain $[0, L] \times [0, T)$ for some $L > 0$ and $T > 0$ and satisfies $u(0, t) = u(L, t) \equiv 0$. You can look up the derivation of the 1-D wave equation from this point of view in many textbooks on partial differential equations and in many other places (e.g., on the internet) as well. I'm first going to walk you through a derivation of the 1-D wave equation which I view as much superior to the usual one. In particular, no approximation is required. I have not seen this/my derivation elsewhere. My derivation requires one to assign a different physical meaning to the value of the function u.

Horizontal Displacements

Let $u : [0, L] \times [0, T) \to \mathbb{R}$ *represent the horizontal displacement of a one-dimensional elastic continuum with fixed endpoints at* $x = 0$ *and* $x = L$*; see Figure 2.*

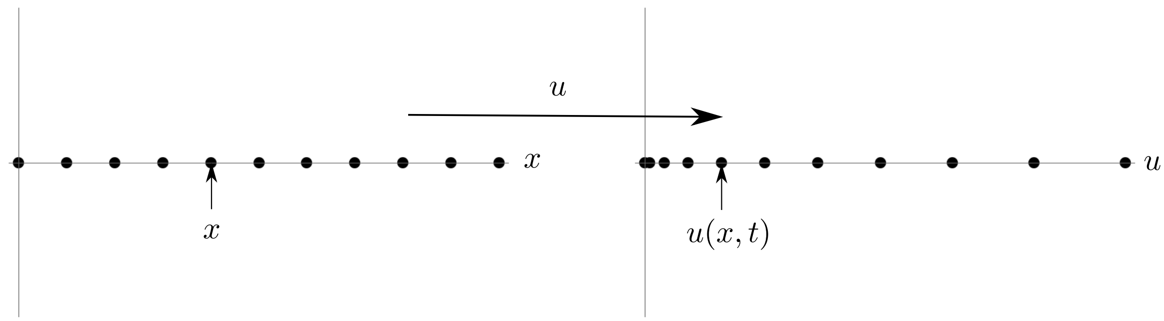

Figure 2: A horizontally displaced one-dimensional continuum with fixed endpoints. Here we also assume elasticity and an equilibrium corresponding to no displacement or $u(x) \equiv x$*. In the illustrated displacement each point (except for the two endpoints) is displaced to the left. One can imagine this also as an initial displacement* $u_0(x) = u(x, 0)$ *which corresponds to a restoring motion/force to the right. Naturally there may also be an initial velocity distribution along the continuum.*

This model for horizontal displacements with fixed endpoints is naturally suited to the boundary conditions:

$$\begin{cases} u(0, t) \equiv 0 & t \geq 0, \\ u(L, t) \equiv L & t \geq 0, \end{cases}$$

and the constraint

$$u_x(x, t) > 0.$$

The constraint corresponds to keeping the continuum ordered, so that there is no folding or overlap. Thus, horizontal displacements are naturally associated with the admissible class

$$\mathcal{A} = \{u \in C^2([0, L] \times [0, T)) : u_x(x, t) > 0,\ u(0, t) \equiv 0,\ u(L, t) \equiv L,\ t \geq 0\}.$$

(a) *The displacement illustrated in Figure 2 corresponds to* $u(x, t) = x^2$ *on the spatial interval* $[0, L] = [0, 1]$*. Find and plot a (horizontal) displacement* $u_0 \in C^0[0, 1]$ *defined by the following*

(i) *$u_0(1/2) = 3/4$,*

(ii) *u_0 is linear on the interval $[0, 1/2]$, and*

(iii) *u_0 agrees with an affine function on the interval $[1/2, 1]$.*

You can plot u_0 in two different ways, once in the style of Figure 2 and also simply as a graph in the x, u-plane.

(iv) *Use the method of characteristics to determine the solution of*

$$\begin{cases} u_{tt} = u_{xx} & \text{on } [0,1] \times [0, \infty) \\ u(x,0) = u_0, & \\ u_t(x,0) = 0, & \\ u(0,t) \equiv 0, & \\ u(1,t) \equiv 1 & \end{cases}$$

and make an animation of the image of u as a function of time represented by image dots as on the right in Figure 2.

Constitutive Relation for Elasticity

We assume our continuum has the elastic properties of an **inhomogeneously compressed/extended linear spring**. *Given this assumption, we need to determine how forces are determined locally in terms of the displacement u. Recall that in the elementary modeling of a spring compression and extension are assumed to be homogeneous. Specifically, if the spring is at equilibrium and of length L it is assumed there is a constant k, called Hooke's constant, such that the spring exerts a force $F = -k(X - L)$ on any object attached to the end located at $x = X$ for $X > 0$. Clearly this simple model needs to be generalized or otherwise modified for our application. The following parts suggest one way to do this based on the assumption that local forces exerted under inhomogeneous displacement should be related to* **density**.

(b) *Assume a spring/string (one-dimensional elastic continuum) has one endpoint fixed at $x = 0$ and one free end. Assume also an equilibrium length L corresponding to a linear density ρ_0. Determine the horizontal displacement function $u_0 : [0, L] \to \mathbb{R}$ corresponding to a* **homogeneous horizontal displacement** *to the interval $[0, X]$. Express the density ρ is the displaced spring and the resulting force associated with the displacement u_0.*

(c) *Use the previous part to explain/justify the elastic assumption*

$$\tau = \alpha(u_x - 1)$$

for the local tension is an inhomogeneously displaced spring where $u : [0, L] \times [0, T) \to \mathbb{R}$ *describes the displacement,* α *is an appropriate constant, and* τ *is positive for extension beyond the equilibrium density and negative for compression.*

Newton's Second Law and the Continuum Assumption

Let x_1 *and* x_2 *be two points with* $0 \leq x_1 < x_2 < L$ *and images* $u(x_1, t)$ *and* $u(x_2, t)$ *at time* t *so that the image interval is*

$$I = \{u(x, t) : x_1 \leq x \leq x_2\}.$$

(d) *Show the center of mass of the image interval is*

$$\frac{1}{\rho_0(x_2 - x_1)} \int_I u(x, t)\rho(x, t)\, dx = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} u(x, t)\, dx.$$

The **continuum assumption** *for motion is that the sum of the forces acting on* I*, expressed with respect to the center of mass of* I*, is given by the resultant tension forces on the endpoints of* I*.*

(e) *Under the continuum assumption, show that Newton's second law gives*

$$\rho_0(x_2 - x_1) \frac{d^2}{dt^2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} u(x, t)\, dx = \alpha[u_x(x_2, t) - 1] - \alpha[u_x(x_1, t) - 1].$$

(f) *Simplify, manipulate, and use the expression from part (e) along with the fundamental lemma of the calculus of variations to finish the derivation of the one-dimensional wave equation in the form*

$$\rho_0 u_{tt} = \alpha u_{xx}.$$

Hamilton's Principle

Remember that Hamilton's principle says that any particle motion determined by Newton's second law in a potential field can be obtained as an extremal for the action functional

$$H[x] = \frac{1}{2}\int_0^T \dot{x}(\tau)^2\,d\tau - \int_0^T \Phi(x(\tau),\tau)\,d\tau \tag{1}$$

on the admissible class of motions with determinant outcomes

$$\mathcal{A} = \{x \in C^2[0,T] : x(0) = x_0 \text{ and } x(T) = x_1\}$$

where Φ *is the (possibly time varying) potential function for the field satisfying* $F(x,t) = -\Phi_x(x,t)$. *In view of the derivation above (using Newton's second law) this rather strongly suggests there should be a variational derivation of the wave equation using some kind of Hamilton's principle. In fact, it is true that such a derivation is possible, and what is moreover true is that this derivation applies in any dimension to give the wave equation in arbitrary spatial dimensions*

$$u_{tt} = \Delta u.$$

(g) *Obtain Newton's second law for particle motion in a potential force field (once more) using the Hamiltonian action functional defined in (1).*

(h) *Define an appropriate admissible class and an appropriate generalization of Hamilton's action functional on that admissible class so that the wave equation is given as the Euler-Lagrange equation of* C^2 *extremals.*