Remarks on Awning Design
(MATH 6702 Final Exam Problem 4)

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Abstract

This is a report for Kendra and James Riddle of the Sonshine Awning Company, Pheonix Arizona, concerning their proposed design of an awning satisfying a certain Dirichlet boundary value problem for the PDE $u_{xx} - 4u_{xy} + u_{yy} = 0$ on a rectangle. We introduce the problem, give some preliminary discussion of techniques which can be applied to understand the possibilities and properties of solutions, consider the consequences for this particular design, and offer some conclusions and design alternatives.

1 Introduction

The proposed design suggests finding an awning with shape determined by the boundary value problem

$$\begin{cases} u_{xx} - 4u_{xy} + u_{yy} = 0 \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x,0) = u_0(x),\ u(\pm a/2, y) = u_\pm(y),\ u(x,b) = u_1(x) \end{cases} \tag{1}$$

where we are allowed to specify/suggest the boundary values.

1.1 Change of Variables

On the face of it, we don't know any properties of this PDE, so let's try to change variables to see if we can get something (i.e., a PDE) we have seen before. For this, we assume u is a solution, and define a function

$$w(\xi, \eta) = u(a_{11}\xi + a_{12}\eta, a_{21}\xi + a_{22}\eta)$$

on a domain

$$U = \left\{ A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in R \right\}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is assumed to be an invertible matrix. We will determine the actual form/geometry of U and the appropriate boundary values after we figure out which explicit change of variables matrix A gives us a PDE we recognize. If

$$B = A^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then we can also write

$$u(x, y) = w(b_{11}x + b_{12}y, b_{21}x + b_{22}y)$$

and calculate the expression $u_{xx} - 4u_{xy} + u_{yy}$ in terms of the derivatives of w (to get a PDE for $w = w(\xi, \eta)$). The details of this calculation are as follows:

$$u_x = b_{11}w_\xi + b_{21}w_\eta, \qquad u_y = b_{12}w_\xi + b_{22}w_\eta.$$

$$\begin{aligned} u_{xx} &= b_{11}^2 w_{\xi\xi} + 2b_{11}b_{21}w_{\xi\eta} + b_{21}^2 w_{\eta\eta}, \\ u_{xy} &= b_{11}b_{12}w_{\xi\xi} + b_{11}b_{22}w_{\xi\eta} + b_{12}b_{21}w_{\xi\eta} + b_{21}b_{22}w_{\eta\eta}, \\ u_{xx} &= b_{12}^2 w_{\xi\xi} + 2b_{12}b_{22}w_{\xi\eta} + b_{22}^2 w_{\eta\eta}. \end{aligned}$$

Thus, the PDE $u_{xx} - 4u_{xy} + u_{yy} = 0$ in the ξ, η coordinates becomes

$$\begin{aligned} &[b_{11}^2 - 4b_{11}b_{12} + b_{12}^2]w_{\xi\xi} \\ &\qquad + 2[b_{11}b_{21} - 2b_{11}b_{22} - 2b_{12}b_{21} + b_{12}b_{22}]w_{\xi\eta} \\ &\qquad\quad + [b_{21}^2 - 4b_{21}b_{22} + b_{22}^2]w_{\xi\xi} = 0. \end{aligned}$$

1.2 Equation Types

When we think about the equations we know and that remarkable fact we have been faithfully told by our professor(s):

> **every second order linear constant coefficient PDE has a well-defined type**

so that after a change of variables it will be equivalent to either Laplace's equation $w_{\xi\xi} + w_{\eta\eta} = 0$, the heat equation $w_t = w_{\xi\xi}$, or the wave equation $w_{tt} = w_{\xi\xi}$, then if this is true, we should at least try to take the matrix B so that

$$b_{11}b_{21} - 2b_{11}b_{22} - 2b_{12}b_{21} + b_{12}b_{22} = 0. \tag{2}$$

Notice: None of the three standard second order linear constant coefficient PDEs have a mixed partial derivative.

Also, we expect that either one of the coefficients

$$b_{11}^2 - 4b_{11}b_{12} + b_{12}^2 \qquad \text{or} \qquad b_{21}^2 - 4b_{21}b_{22} + b_{22}^2 \tag{3}$$

should vanish (for a parabolic PDE) or else

$$b_{11}^2 - 4b_{11}b_{12} + b_{12}^2 = \pm[b_{21}^2 - 4b_{21}b_{22} + b_{22}^2]. \tag{4}$$

So we have four unknowns and (basically) two nonlinear equations. Presumably, this is underdetermined in some sense. Let's just try something. How about...

$$b_{11} = 1.$$

Then (2) becomes

$$b_{21} - 2b_{22} - 2b_{12}b_{21} + b_{12}b_{22} = 0. \tag{5}$$

Notice that we're not going to get a parabolic equation. (There are no first order terms.) Thus, we can look at (4) which becomes

$$1 - 4b_{12} + b_{12}^2 = \pm[b_{21}^2 - 4b_{21}b_{22} + b_{22}^2].$$

Looking at this, we are tempted to at least try $b_{12} = 0$. Then the coefficient of $w_{\xi\xi}$ becomes 1 which may not be definitive, but would (at least) be nice. Under this assumption (5) becomes

$$b_{21} - 2b_{22} = 0 \qquad \text{so that} \qquad b_{21} = 2b_{22}.$$

The remaining equation now reads

$$\pm[b_{21}^2 - 4b_{21}b_{22} + b_{22}^2] = \mp 3b_{22}^2 = 1.$$

Obviously, $b_{22}^2 \neq -1/3$ so we have a hyperbolic equation. In fact taking $b_{22} = 1/\sqrt{3}$ we are led to

$$B = \begin{pmatrix} 1 & 0 \\ 2/\sqrt{3} & 1/\sqrt{3} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 \\ -2 & \sqrt{3} \end{pmatrix},$$

and

$$w_{\xi\xi} = w_{\eta\eta}.$$

We record also that

$$w_\xi = a_{11}u_x + a_{21}u_y = u_x - 2u_y \quad \text{and} \quad w_\eta = a_{12}u_x + a_{22}u_y = \sqrt{3}\,u_y. \tag{6}$$

Thus, we have a PDE we know (the wave equation), and we are now in a position to consider the domain U which is a parallelogram as indicated in Figure 1.

2 Preliminary Constructions

For the following discussion, we will take $a = 1$. The new domain has also

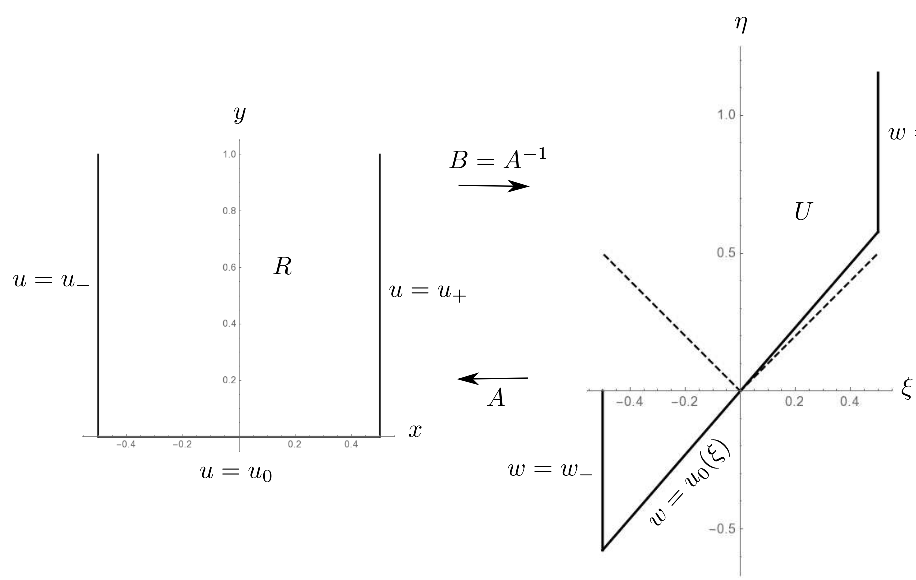

Figure 1: The rectangular footprint R of the awning, and the parallelogram region U.

what may be considered a "bottom edge" and right and left "lateral edges." The bottom edge corresponding to $y = 0$ is given by

$$\{(\xi, 2\xi/\sqrt{3}) : -1/2 \le \xi \le 1/2\}$$

with corresponding boundary values $w(\xi, 2\xi/\sqrt{3}) = u_0(\xi)$. The lateral edges also correspond to $x = \xi = \pm 1/2$, and there we have $w(\pm 1/2, \eta) = u_\pm(\eta\sqrt{3} \mp 1)$. We have also indicated on the right in Figure 1 the "characteristic" directions $\xi = \pm\eta$ for the new PDE emanating from the origin.

2.1 Initial Portion: First Approach

We will obtain an expression for $w = w(\xi, \eta)$ based on these boundary values using integration of the identity

$$\operatorname{div} Qw = 0 \qquad \text{where} \qquad Qw = (w_\xi, -w_\eta)$$

over several domains as follows. We use the characteristic directions to partition U into three initial domains

$$U_0 = \left\{ (\xi, \eta) \in U : \eta < \xi + \frac{1}{\sqrt{3}} - \frac{1}{2} \right\} = \left\{ (\xi, \eta) \in U : \frac{2}{\sqrt{3}}\xi < \eta < \xi + \frac{1}{\sqrt{3}} - \frac{1}{2} \right\},$$

$$U_1 = \left\{ (\xi, \eta) \in U : \xi + \frac{1}{\sqrt{3}} - \frac{1}{2} < \eta < \frac{1}{\sqrt{3}} + \frac{1}{2} - \xi \right\},$$

and

$$U_2 = \left\{ (\xi, \eta) \in U : \frac{1}{\sqrt{3}} + \frac{1}{2} - \xi < \eta < \xi + \frac{1}{\sqrt{3}} + \frac{3}{2} \right\}.$$

For points $(\xi, \eta) \in U_0$, we integrate the PDE over a triangular region T_0 with vertices

$$\begin{aligned}
&(\xi, \eta), \\
&\xi_0(1, 2/\sqrt{3}) = (2\sqrt{3} - 3)(\xi + \eta)(1, 2/\sqrt{3}), \quad \text{and} \\
&\xi_1(1, 2/\sqrt{3}) = -(2\sqrt{3} + 3)(\xi - \eta)(1, 2/\sqrt{3})
\end{aligned}$$

as indicated in Figure 2. Using the divergence theorem we can write

$$0 = \int_{T_0} \operatorname{div} Qw = \int_{\partial T_0} Qw \cdot n.$$

Consider two of the sides of T_0 and the corresponding flux integrals. The side connecting $\xi_0(1, 2/\sqrt{3})$ to (ξ, η) may be parameterized by

$$t \mapsto (\xi_0 - t, 2\xi_0/\sqrt{3} + t) \qquad \text{for} \qquad 0 \le t \le t_0.$$

The values of ξ_0 and t_0 are given as solutions of the system

$$\left\{ \begin{array}{rcl} \xi_0 - t_0 & = & \xi \\ 2\xi_0/\sqrt{3} + t_0 & = & \eta. \end{array} \right.$$

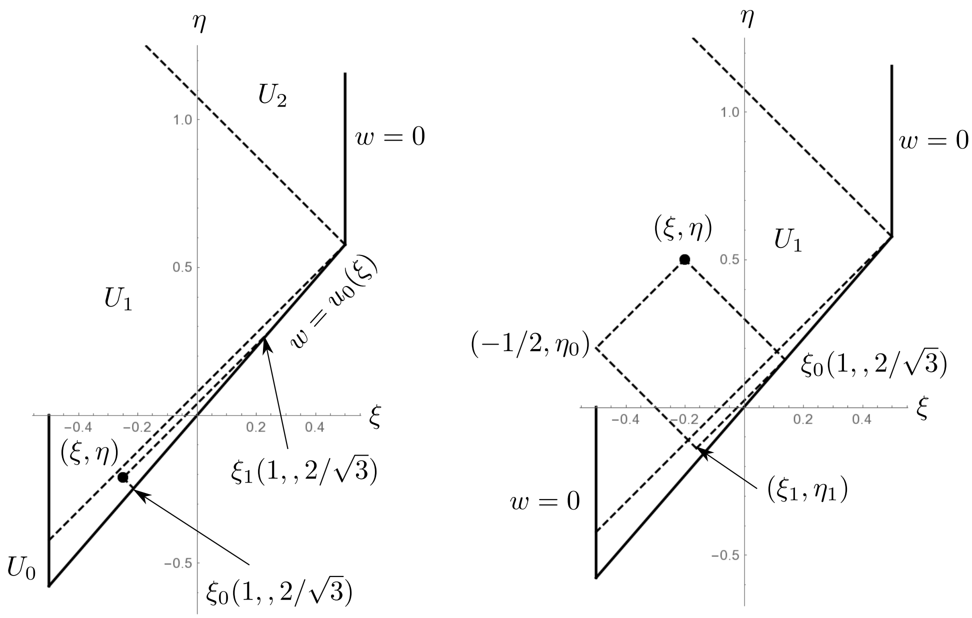

Figure 2: Partitioning of the parallelogram region U.

The value $\xi_0 = (2\sqrt{3}-3)(\xi+\eta)$ is given above, and as we shall see presently, we do not really need the value t_0. The scaling factor associated with this parameterization is $\sigma = \sqrt{2}$ and the outward unit normal along this side is $n = (-1,-1)/\sigma$. Consequently the flux integral along this side is

$$\begin{aligned}\int_0^{t_0} Qw(\xi_0 - t, 2\xi_0/\sqrt{3}+t)\cdot(-1,-1)\,dt \\ &= \int_0^{t_0} [-w_\xi(\xi_0-t, 2\xi_0/\sqrt{3}+t) + w_\eta(\xi_0-t, 2\xi_0/\sqrt{3}+t)]\,dt \\ &= \int_0^{t_0} \frac{d}{dt} w(\xi_0-t, 2\xi_0/\sqrt{3}+t)\,dt \\ &= w(\xi,\eta) - w(\xi_0, 2\xi_0/\sqrt{3}) \\ &= w(\xi,\eta) - u(\xi_0, 0) \\ &= w(\xi,\eta) - u_0(\xi_0).\end{aligned}$$

The side of T_0 along the bottom boundary has parameterization

$$\xi \mapsto \xi(1, 2/\sqrt{3}) \qquad \text{for} \qquad \xi_0 \le \xi \le \xi_1,$$

and the normal is $(2/\sqrt{3}, -1)/\sigma$ where σ is the associated scaling. Consequently, the flux integral along this side takes the form

$$\begin{aligned}
&\int_{\xi_0}^{\xi_1} Qw(\xi, 2\xi/\sqrt{3}) \cdot (2/\sqrt{3}, -1)\, d\xi \\
&\qquad = \int_{\xi_0}^{\xi_1} \left(\frac{2}{\sqrt{3}}\, w_\xi(\xi, 2\xi/\sqrt{3}) + w_\eta(\xi, 2\xi/\sqrt{3}) \right) d\xi \\
&\qquad = \int_{\xi_0}^{\xi_1} \left(\frac{2}{\sqrt{3}}\, [u_x(\xi, 0) - 2u_y(\xi, 0)] + \sqrt{3}\, u_y(\xi, 0) \right) d\xi \\
&\qquad = \frac{2}{\sqrt{3}} \int_{\xi_0}^{\xi_1} u_0'(\xi)\, d\xi - \frac{1}{\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi.
\end{aligned}$$

For the second equality we have used (6) and for the third, we have used the boundary values and denoted $u_y(x, 0)$ for $|x| \le 1/2$ by $v_0(x)$.

Using the same method, the flux integral along the third side of T_0 is $w(\xi, \eta) - u_0(\xi_1)$, so summing gives

$$\begin{aligned}
0 &= w(\xi, \eta) - u_0(\xi_0) + \frac{2}{\sqrt{3}} \int_{\xi_0}^{\xi_1} u_0'(\xi)\, d\xi - \frac{1}{\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi + w(\xi, \eta) - u_0(\xi_1) \\
&= 2w(\xi, \eta) - u_0(\xi_0) - u_0(\xi_1) + \frac{2}{\sqrt{3}} [u_0(\xi_1) - u_0(\xi_0)] - \frac{1}{\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi \\
&= 2w(\xi, \eta) - \left(\frac{2}{\sqrt{3}} + 1 \right) u_0(\xi_0) + \left(\frac{2}{\sqrt{3}} - 1 \right) u_0(\xi_1) - \frac{1}{\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi.
\end{aligned}$$

Thus,

$$w(\xi, \eta) = \left(\frac{1}{\sqrt{3}} + \frac{1}{2} \right) u_0(\xi_0) - \left(\frac{1}{\sqrt{3}} - \frac{1}{2} \right) u_0(\xi_1) + \frac{1}{2\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi.$$

where

$$\begin{cases} \xi_0 &= \quad (2\sqrt{3} - 3)(\xi + \eta) \\ \xi_1 &= -(2\sqrt{3} + 3)(\xi - \eta) \end{cases}$$

and $v_0(x) = u_y(x, 0)$. We denote this formula for the solution on U_0 by $W_0 = W_0(\xi, \eta)$:

$$\begin{aligned}
W_0(\xi, \eta) = &\left(\frac{1}{\sqrt{3}} + \frac{1}{2} \right) u_0\left((2\sqrt{3} - 3)(\xi + \eta) \right) \\
&\quad - \left(\frac{1}{\sqrt{3}} - \frac{1}{2} \right) u_0\left(-\left(2\sqrt{3} + 3 \right) (\xi - \eta) \right) \\
&\qquad + \frac{1}{2\sqrt{3}} \int_{(2\sqrt{3}-3)(\xi+\eta)}^{-(2\sqrt{3}+3)(\xi-\eta)} v_0(\xi)\, d\xi.
\end{aligned}$$

Transforming back to the rectangular domain R, we obtain a formula for the solution u valid on the region

$$R_0 = \left\{ (x,y) \in R : 0 < y < 1 - \frac{\sqrt{3}}{2} - (2-\sqrt{3})x \right\}.$$

indicated in Figure 3. In terms of x and y, we find

$$\begin{cases} \xi_0 &= x + (2-\sqrt{3})y \\ \xi_1 &= x + (2+\sqrt{3})y. \end{cases}$$

Thus, the formula determined by $u(x,y) = W_0(B(x,y))$ is

$$\begin{aligned} u(x,y) &= \left(\frac{1}{\sqrt{3}} + \frac{1}{2}\right) u_0(\xi_0) - \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) u_0(\xi_1) + \frac{1}{2\sqrt{3}} \int_{\xi_0}^{\xi_1} v_0(\xi)\, d\xi. \\ &= \left(\frac{1}{\sqrt{3}} + \frac{1}{2}\right) u_0\left(x + (2-\sqrt{3})y\right) \\ &\qquad - \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) u_0\left(x + (2+\sqrt{3})y\right) \\ &\qquad\qquad + \frac{1}{2\sqrt{3}} \int_{x+(2-\sqrt{3})y}^{x+(2+\sqrt{3})y} v_0(\xi)\, d\xi. \end{aligned}$$

2.2 Second Portion

Proceeding to the second region U_1 and the corresponding region R_1, we integrate the PDE for w over a rectangle with vertices

$$\begin{aligned} &(\xi,\eta) \in U_1, \\ &\xi_0(1, 2/\sqrt{3}) = (2\sqrt{3}-3)(\xi+\eta)(1, 2/\sqrt{3}), \\ &(-1/2, \eta_0) = (-1/2, -(\xi-\eta) - 1/2), \quad \text{and} \\ &(\xi_1, \eta_1) = 2(2-\sqrt{3})(\xi+\eta)(-1,1) + (\eta, -\xi) - (1/2, 1/2) \in U_0 \end{aligned}$$

as indicated on the right in Figure 2. Integrating $Qw \cdot n$ around this rectangle gives

$$\begin{aligned} w(\xi,\eta) &= w(-1/2, \eta_0) - w(\xi_1, \eta_1) + w(\xi_0, 2\xi_0/\sqrt{3}) \\ &= w_-(\eta_0) - W_0(\xi_1, \eta_1) + u_0(\xi_0). \end{aligned}$$

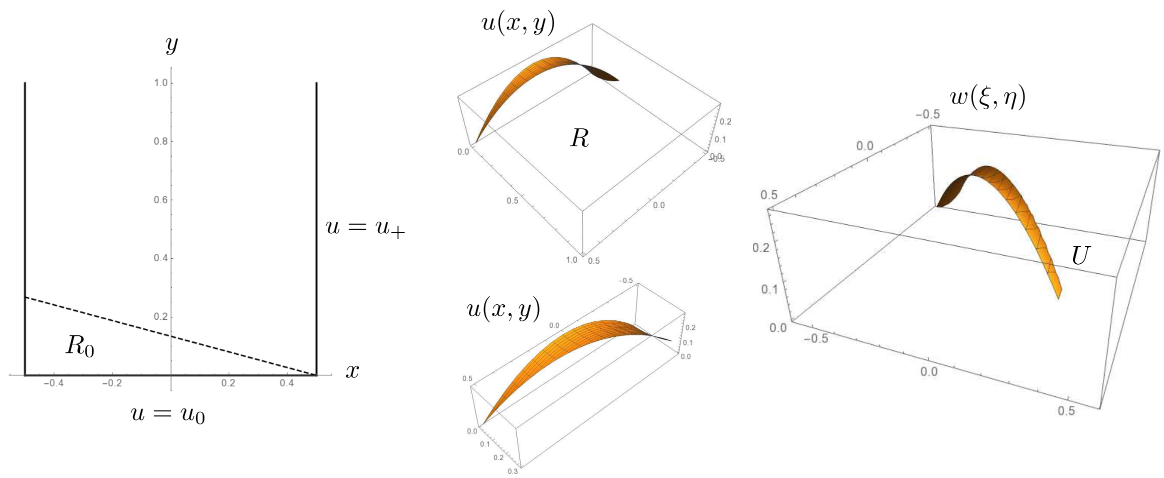

Figure 3: A d'Alembert type solution u over a first portion R_0 of R. Shown on the right is the auxiliary function w satisfying the wave equation over the domain U_0. Here we have taken boundary values $u_0(x) = 1/4 - x^2$ and $v_0(x) \equiv 0$ for $|x| \le 1/2$. It will be noted that the characteristics impinge on the left boundary $x = -1/2$ but the values are determined entirely by u_0. Thus, any prescription $u_-(y)$ along the portion of $x = -1/2$ in ∂R_0 beyond that determined by the formula is not possible.

We can use the formula above for $W_0(\xi_1, \eta_1)$, and now the left boundary starts to play a role since the term

$$\begin{aligned} w(-1/2, \eta_0) &= u(A(-1/2, \eta_0)) \\ &= u(-1/2, 1 + \sqrt{3}\eta_0) \\ &= u_-(1 + \sqrt{3}\eta_0) \\ &= u_-(1 - \sqrt{3}(\xi - \eta) - \sqrt{3}/2) \end{aligned}$$

appears in the expression for w. Here we have used the convention that $A(-1/2, \eta_0) = [A(-1/2, \eta_0)^T]^T$. In particular, the formula gives $w(-1/2, \eta_0) = w_-(\eta_0) = u_-(1 - \sqrt{3}(\xi - \eta) - \sqrt{3}/2)$ along the left boundary of U_1. Likewise, the boundary value $u_- = u_-(y)$ along the left edge of the corresponding region R_1 may be prescribed.

It will be noted from Figure 4 that this portion of the awning over R_1 does not meet continuously the portion obtained above over R_0. Nevertheless, the two pieces together look like the start of a possibly stylish two-piece awning (assuming there is a practical way for the second piece to be supported). The

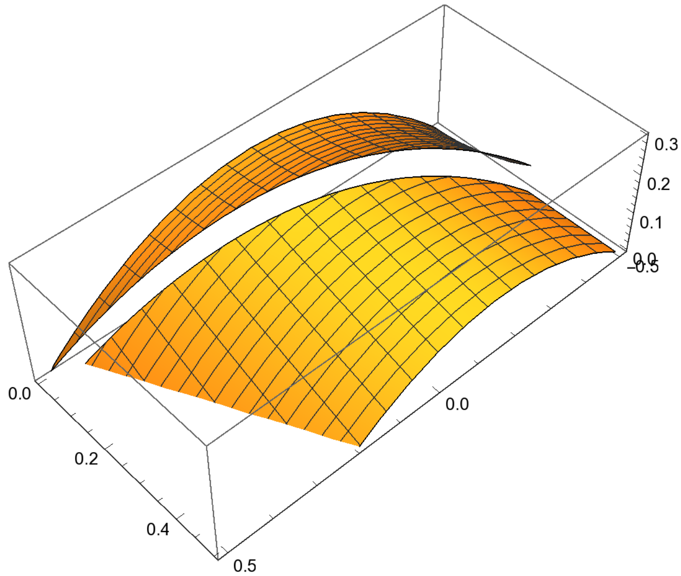

Figure 4: Two pieces of a potential multi-piece awning given by solutions of the equation $u_{xx} - 4u_{xy} + u_{yy} = 0$.

appearance of a discontinuity here should be no surprise because the rectangular domain of integration used to obtain the second piece does not agree with the triangular domain T_0 for points (ξ, η) along the common edge $\partial U_0 \cap \partial U_1$. There is no obvious way to modify one of the approaches we have used for points in the regions U_0 and U_1 to match the other. Some reflection, however, suggests there are at least two ways to redo the solution over both regions to give a continuous result/union.

The other observation that is worth making at this point is that the width a of the awning should not be ignored in this problem. In the previous problem, the width a played an essentially homogeneous role, but here as we shall see, due to the uneven impingement of the characteristics on the lateral boundaries, the actual width a compared to the coefficients $1w_{\xi\xi} - 1w_{\eta\eta} = 0$ in the PDE is relevant.

Before, we investigate a unified approach to obtaining a solution w (which yields continuous solutions u and w), let's compute a formula for the next natural region.

2.3 Third Portion

Given a point $(\xi, \eta) \in U$ with $\eta > 1/\sqrt{3} - (x + 1/2)$ we can consider a rectangle with vertices

$$\begin{aligned}
&(\xi, \eta) \in U_2, \\
&(-1/2, \eta_0) = (-1/2, -(\xi - \eta) - 1/2), \\
&(1/2, \eta_1) = (1/2, \xi + \eta - 1/2), \quad \text{and} \\
&(\xi_2, \eta_2) = 2(2 - \sqrt{3})(\xi + \eta)(-1, 1) + (\eta, -\xi) - (1/2, 1/2) \in U_1.
\end{aligned}$$

We include here the full derivation for the values of η_0, η_1 and (ξ_2, η_2). First for η_0, we have

$$\left\{ \begin{array}{rcl} -1/2 + t_0 & = & \xi \\ \eta_1 + t_0 & = & \eta \end{array} \right. \implies \eta_0 + \frac{1}{2} = -(\xi - \eta) \implies \eta_0 = -(\xi - \eta) - \frac{1}{2}.$$

Associated with this side we also compute

$$\begin{aligned}
&\int_0^{t_0} Qw(-1/2 + t, \eta_0 + t) \cdot (-1, 1)\, dt \\
&\qquad = -w(\xi, \eta) + w(-1/2, \eta_0) \\
&\qquad = -w(\xi, \eta) + u(A((-1/2, \eta_0)) \\
&\qquad = -w(\xi, \eta) + u_-(\sqrt{3}\eta_0 + 1) \\
&\qquad = -w(\xi, \eta) + u_-(-\sqrt{3}(\xi - \eta) - \sqrt{3}/2 + 1).
\end{aligned}$$

The computation(s) for η_1 are similar:

$$\left\{ \begin{array}{rcl} 1/2 - t_1 & = & \xi \\ \eta_1 + t_1 & = & \eta \end{array} \right. \implies \eta_1 + \frac{1}{2} = \xi + \eta \implies \eta_1 = \xi + \eta - \frac{1}{2}.$$

Associated with this side we also compute

$$\int_0^{t_1} Qw(1/2 - t, \eta_1 + t) \cdot (1, 1)\, dt$$

$$-w(\xi, \eta) + w(1/2, \eta_1)$$

$$\begin{aligned}
&= -w(\xi, \eta) + u(A((1/2, \eta_1)) \\
&= -w(\xi, \eta) + u_+(\sqrt{3}\eta_1 - 1) \\
&= -w(\xi, \eta) + u_-(\sqrt{3}(\xi + \eta) - \sqrt{3}/2 - 1).
\end{aligned}$$

We consider the two remaining sides together to solve for (ξ_2, η_3). We have

$$\left\{\begin{array}{rcl} \xi_2 - t_2 &=& -1/2 \\ \eta_2 + t_2 &=& \eta_0 = -(\xi-\eta) - 1/2 \end{array}\right. \qquad \left\{\begin{array}{rcl} \xi_2 + t_3 &=& 1/2 \\ \eta_2 + t_3 &=& \eta_1 = \xi+\eta - 1/2 \end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{rcl} \xi_2 + \eta_2 &=& -(\xi-\eta) - 1 \\ \xi_2 - \eta_2 &=& -(\xi+\eta) + 1 \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} \xi_2 &=& -\xi \\ \eta_2 &=& \eta - 1. \end{array}\right.$$

The associated integrals are

$$\begin{aligned} &\int_0^{t_2} Qw(\xi_2 - t, \eta_2 + t)\cdot(-1,-1)\,dt \\ &\qquad = w(-1/2,\eta_0) - w(\xi_2,\eta_2) \\ &\qquad = u_-(-\sqrt{3}(\xi-\eta) - \sqrt{3}/2 + 1) - W_1(-\xi, \eta - 1), \end{aligned}$$

and

$$\begin{aligned} &\int_0^{t_3} Qw(\xi_3 + t, \eta_2 + t)\cdot(1,-1)\,dt \\ &\qquad = w(1/2,\eta_1) - w(\xi_2,\eta_2) \\ &\qquad = u_+(\sqrt{3}(\xi+\eta) - \sqrt{3}/2 - 1) - W_1(-\xi, \eta - 1), \end{aligned}$$

Summing these results and rearranging, we obtain a solution defined for $(\xi, \eta) \in U_2$ given by

$$W_2(\xi,\eta) = u_-(-\sqrt{3}(\xi-\eta)-\sqrt{3}/2+1)+u_+(\sqrt{3}(\xi+\eta)-\sqrt{3}/2-1)-W_1(-\xi,\eta-1).$$

With the usual transformation formula $u(x, y) = W_2(B(x, y))$ we obtain a third portion of the awning. We have plotted all three portions of the graph of w and the graph of u extending to length $y = b = 1$ in Figure 5. It will be noted that while the third portion is obtained using regions of integration that continuously transform to the regions used for the second region, the third portion does not meet the second portion continuously. This is because, in the continuous deformation of a rectangle with top vertex at $(\xi, \eta) \in U_2$ to one with top vertex $(\xi, \eta) \in U_1$ across the boundary between U_1 and U_2, the corresponding bottom vertex $(\xi_2, \eta_2) \in U_1$ crosses the boundary between U_1 and U_0 to become a bottom vertex $(\xi_1, \eta_1) \in U_0$. Thus, values $W_1(\xi_2, \eta_2)$ are used before the transition and the (discontinuously different) values $W_0(\xi_1, \eta_1)$ are used after the transition, leaving a kind of residual discontinuity. That

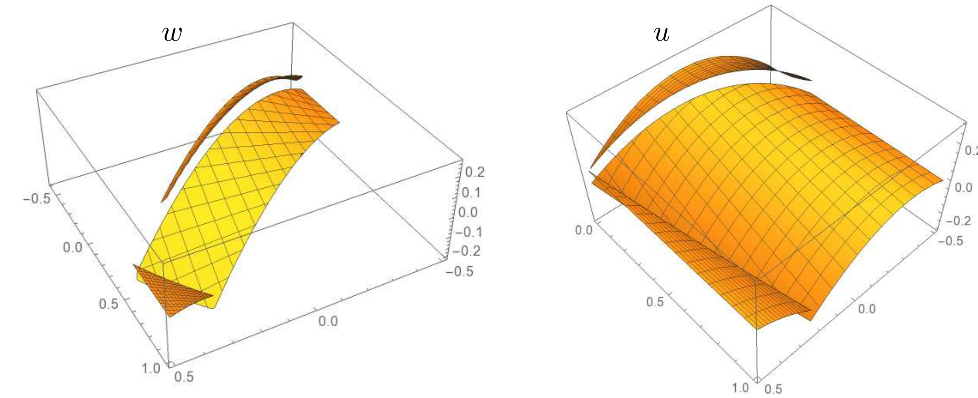

Figure 5: A three–piece awning given by solutions of the equation $u_{xx} - 4u_{xy} + u_{yy} = 0$.

is, even though the regions of integration for $(\xi, \eta) \in U_2$ and $(\xi, \eta) \in U_1$ are chosen in a consistent and unified manner, the fact that the regions chosen for $(\xi, \eta) \in U_2$ in a fundamentally different way still influences the result and gives a discontinuity here.

There does not appear to be a way to choose a region of integration for $(\xi, \eta) \in U_0$ in a manner consistent with the choice we have taken for $(\xi, \eta) \in U_1 \cup U_2$. Indeed, we can consider the appropriate rectangle with $(\xi, \eta) \in U_0$, but the bottom vertex (ξ_0, η_0) in this case will still lie in U_0, so we will not have a value to assign to such a vertex, and integration does not determine a solution value $u(\xi, \eta)$.

There are several viable alternatives leading to different awnings with differing properties. The approach used for $(\xi, \eta) \in U_0$, where a segments are extended from (ξ, η) along the characteristic directions $(1, -1)$ and $(1, 1)$ to meet ∂U, could be extended to cases in which $(\xi, \eta) \in U_1 \cup U_2$. It has been noted that this approach does not allow any control of the boundary behavior along the left boundary. In order to take account boundary values along the left boundary, it is natural to consider characteristics extending to the left from (ξ, η) along the direction $(-1, -1)$ to ∂U. All such characteristics meet ∂U along the left edge $\xi = -1/2$. There are two obvious alternatives starting with these initial characteristic edges. In order to illustrate these approaches it is convenient to change the aspect ratio of the parallelogram shaped domain U under consideration. We take a new domain U with bottom edge along $\eta = 2\xi$. In this case, the physical awning region $R = (-1/2, 1/2) \times (0, 2)$ transforms to

U according to the matrices

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The corresponding PDE on R is then $u_{xx} - 4u_{xy} + 3u_{yy} = 0$, but our primary interest at this point is in the equation $\Box w = w_{\xi\xi} - w_{\eta\eta} = 0$ on the domain U indicated in Figure 6. Another aspect which we are going to emphasize now (and have not emphasized above) is the ending length of the awning represented by a specific top edge of our parallelogram. In this case, we've taken $b = 2$ resulting in the top edge $\{(x, 2x+2) : |x| \leq 1/2\}$ for U.

Figure 6 illustrates a first approach. In contrast to the previous approaches

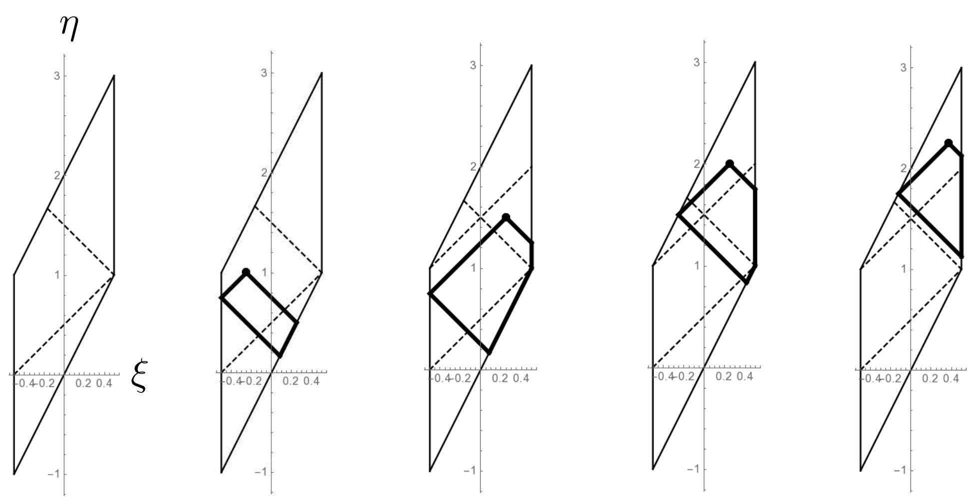

Figure 6: A unified approach to integration on U leading to a continuous solution w of $w_{\xi\xi} - w_{\eta\eta} = 0$. The point (ξ, eta) at which the value $w(\xi,\eta)$ is to be calculated is indicated by the large point vertex of the polygonal boundary.

no use of the value of w previously calculated at points interior to U is used. On the other hand, pentagonal as well as quadrilateral (and no triangular nor rectangular) domains are used. There is a very strong asymmetry in this approach which groups the left and top edges of U (and correspondingly R) and the bottom and right edges. As before, it is natural to assume here given values of $w(x,0) = w_0(x)$ and $w_\eta(x,0) = \phi_0(x)$. (Here we use $\phi = \phi(x)$ to denote initial "velocity" to avoid conflict with the previous use $v_0(x) = u_y(x,0)$. The natural relations between these quantities using the chain rule and the matrices A and B still persist.)

When we integrate on the right side, say from $\eta = \eta_0$ to $\eta = \eta_1$, we get an expression

$$\int_{\eta_0}^{\eta_1} Qw(1/2,\eta)\cdot \mathbf{e}_1\, d\eta = \int_{\eta_0}^{\eta_1} w_\xi(1/2,\eta)\, d\eta.$$

Thus, we see that, rather than having the nominally prescribed value $w_+ = w_+(\eta)$ representing $w(1/2,\eta)$, we see only the influence of the "velocity" w_ξ. Denoting this quantity by $\phi_+ = \phi_+(\eta)$, we have $w(\xi,\eta) = u(A(\xi,\eta)) = u(\xi,-2\xi+\eta)$ and

$$\phi_+(\eta) = w_\xi(1/2,\eta) = u_x(1/2,\eta-1) - 2u_y(1/2,\eta-1) = v_+(\eta-1) - 2u_+'(\eta-1),$$

where $u_+(y) = u(1/2,y)$ and $v_+(y) = u_y(1/2,y)$ are nominally prescribed for $0 \le y \le 2$. Of course, these formulas would change along with the matrices A and B for the original aspect ratio and PDE. We conclude under the current assumptions, however, that

$$\begin{aligned}\int_{\eta_0}^{\eta_1} Qw(1/2,\eta)\cdot \mathbf{e}_1\, d\eta &= \int_{\eta_0}^{\eta_1} v_+(\eta-1)\, d\eta - 2\int_{\eta_0}^{\eta_1} u_+'(\eta-1)\, d\eta \\ &= 2u_+(\eta_0-1) - 2u_+(\eta_1-1) + \int_{\eta_0}^{\eta_1} v_+(\eta-1)\, d\eta.\end{aligned}$$

This illustrates that both some initial value u_+ and some initial velocity v_+ along the right edge of R have an influence in this approach. In addition, an edge parameterized by $(1/2,\eta_1)+t(-1,1)$ with $(1/2-t_1,\eta_1+t_1) = (\xi,\eta)$ can be anticipated in this case with integral giving

$$\begin{aligned}\int_0^{t_1} Qw(1/2-t,\eta_1+t)\cdot(1,1)\, dt &= -w(\xi,\eta) + w(1/2,\eta_1) \\ &= -w(\xi,\eta) + w_+(\eta_1) \\ &= -w(\xi,\eta) + u_+(\eta_1-1).\end{aligned}$$

In particular, the value $w_+(\eta_1)$ does have an influence on the value $w(\xi,\eta)$.

In contrast, the left and top edge have no velocity influence in this approach with only the values $w_- = w_-(\eta)$ and $w_1 = w_1(\xi)$, nominally prescribing $w(-1/2,\eta)$ and $w(\xi,2\xi+2)$ respectively, appearing in the ultimate formula for $w(\xi,\eta)$. Since, however, the regions of integration change continuously throughout the entire process, we should arrive at a continuous (weak) solution.

A second approach which endeavors to treat the various nominally prescribed boundary values in a more symmetric manner is indicated in Figure 7. Here initial values and initial velocity values w_-, $\phi_- = w_\xi$, w_+, $\phi_+ = w_\xi$, w_0, $\phi_0 = w_\eta$, w_1, and $\phi_1 = w_\eta$ will be required along the left, right, bottom,

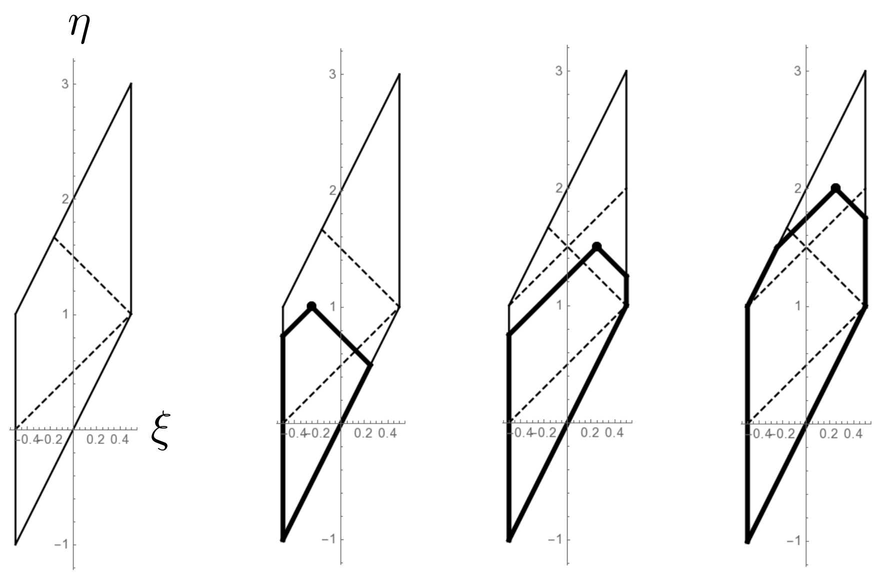

Figure 7: A second unified approach to integration on U leading to a continuous solution w of $w_{\xi\xi} - w_{\eta\eta} = 0$. The point (ξ, η) at which the value $w(\xi, \eta)$ is to be calculated is indicated by the large point vertex of the polygonal boundary.

and top boundaries of U. The integration takes place over quadrilaterals, pentagons, and hexagons, though only two transitions (three different integration processes) are required. Again, no internal (previously calculated) values of w are required.

3 Alternative approaches

Taking into account the various comments above, we begin with a rectangular domain of arbrtrary width a and fixed length b (representing the footprint of the awning):

$$R = \{(x, y) : |x| < a/2,\ 0 < y < b\} = (-a/2, a/2) \times (0, b).$$

We change variables using a matrix B of the form

$$B = \begin{pmatrix} 1 & 0 \\ \lambda & \mu \end{pmatrix} \qquad \text{with} \qquad A = B^{-1} = \begin{pmatrix} 1 & 0 \\ -\lambda/\mu & 1/\mu \end{pmatrix}$$

where $\lambda > 1$ and $\mu > 0$. The resulting domain $U = \{B(x,y) : (x,y) \in R\}$ is a parallelogram

$$U = \{(\xi,\eta) : |\xi| < a/2,\ \lambda\xi < \eta < \mu b + \lambda\xi\}$$

with "bottom" edge along $\eta = \lambda\xi$ having slope greater than the characteristic slope represented by $\eta = \xi$ for the PDE $\Box w = w_{\xi\xi} - w_{\eta\eta} = 0$. We have again used the convention $B(x,y) = [B(x,y)^T]^T$. Changing variables with $w(\xi,\eta) = u(A(\xi,\eta))$, we see that we are treating all PDEs of the form

$$u_{xx} - \frac{2\lambda}{\mu}\, u_{xy} + \frac{1}{\mu^2}(\lambda^2 - 1)\, u_{yy} = 0$$

for $\lambda > 1$ and $\mu > 0$. This includes the original PDE $u_{xx} - 4u_{xy} + u_{yy} = 0$ when $\lambda = 2/\sqrt{3}$ and $\mu = 1/\sqrt{3}$ and the second PDE $u_{xx} - 4u_{xy} + 3u_{yy} = 0$ for $\lambda = 2$ and $\mu = 1$.

We will assume here a full set of boundary values with velocities:

$$\left\{\begin{array}{l} w(-a/2,\sigma) = w_-(\sigma) \\ \qquad = u_-\left(\frac{\lambda a/2+\sigma}{\mu}\right) = u\left(-\frac{a}{2}, \frac{\lambda a/2+\sigma}{\mu}\right) \\ w_\xi(-a/2,\sigma) = \phi_-(\sigma) \\ \qquad = v_-\left(\frac{\lambda a/2+\sigma}{\mu}\right) - \frac{\lambda}{\mu} u_-'\left(\frac{\lambda a/2+\sigma}{\mu}\right) \\ \qquad = u_x\left(-\frac{a}{2}, \frac{\lambda a/2+\sigma}{\mu}\right) - \frac{\lambda}{\mu} u_y\left(-\frac{a}{2}, \frac{\lambda a/2+\sigma}{\mu}\right) \end{array}\right. \quad \text{on the left edge,}$$

$$\left\{\begin{array}{l} w(\tau,\lambda\tau) = w_0(\tau) = u_0(\tau) = u(\tau,0) \\ w_\eta(\tau,\lambda\tau) = \phi_0(\tau) = \frac{1}{\mu}\, v_0(\tau) = \frac{1}{\mu}\, u_y(\tau,0) \end{array}\right. \quad \text{on the bottom}$$

$$\left\{\begin{array}{l} w(a/2,\sigma) = w_+(\sigma) \\ \qquad = u_+\left(\frac{-\lambda a/2+\sigma}{\mu}\right) = u\left(\frac{a}{2}, \frac{-\lambda a/2+\sigma}{\mu}\right) \\ w_\xi(a/2,\sigma) = \phi_+(\sigma) \\ \qquad = v_+\left(\frac{-\lambda a/2+\sigma}{\mu}\right) - \frac{\lambda}{\mu} u_+'\left(\frac{-\lambda a/2+\sigma}{\mu}\right) \\ \qquad = u_x\left(\frac{a}{2}, \frac{-\lambda a/2+\sigma}{\mu}\right) - \frac{\lambda}{\mu} u_y\left(\frac{a}{2}, \frac{-\lambda a/2+\sigma}{\mu}\right) \end{array}\right. \quad \text{on the right edge, and}$$

$$\left\{\begin{array}{l} w(\tau,\lambda\tau+\mu b) = w_1(\tau) = u_1(\tau) = u(\tau,b) \\ w_\eta(\tau,\lambda\tau+\mu b) = \phi_1(\tau) = \frac{1}{\mu}\, v_1(\tau) = \frac{1}{\mu}\, u_y(\tau,b) \end{array}\right. \quad \text{on the top.}$$

There are three subregions U_0, U_1, and U_2 on which the integrations giving the value $w(\xi,\eta)$ for $(\xi,\eta)\in U_j$, $j=0,1,2$ are distinct. These regions are

$$U_0=\left\{(\xi,\eta)\in U:\eta\le\min\left\{\xi-\frac{\lambda-1}{2}\,a+\mu\,b,\frac{\lambda+1}{2}\,a-\xi\right\}\right\},$$
$$U_1=\left\{(\xi,\eta)\in U:\frac{\lambda+1}{2}\,a-\xi\le\eta\le\xi-\frac{\lambda-1}{2}\,a+\mu\,b\right\},$$
$$U_2=\left\{(\xi,\eta)\in U:\eta\ge\xi-\frac{\lambda-1}{2}\,a+\mu\,b\right\}.$$

For points (ξ,η) in the first region U_0 we determine boundary points $(-a/2,\eta_0)$ and $(\xi_0,\lambda\xi_0)$ in the left and bottom boundary segments respectively according to the equations

$$\left\{\begin{array}{rcl}-a/2+t_0&=&\xi\\ \eta_0+t_0&=&\eta\end{array}\right.\qquad\text{and}\qquad\left\{\begin{array}{rcl}\xi_0-t_1&=&\xi\\ \lambda\xi_0+t_1&=&\eta.\end{array}\right.$$

It follows that

$$\eta_0=-\frac{a}{2}-(\xi-\eta)\qquad\text{and}\qquad\xi_0=\frac{\xi+\eta}{\lambda+1}.$$

We have then four path integrals to evaluate. Integrating on the segment from $(-a/2,\eta_0)$ to (ξ,η),

$$\begin{aligned}\int_0^{t_0}Qw(-a/2+t,\eta_0+t)\cdot(-1,1)\,dt&=-w(\xi,\eta)+w(-a/2,\eta_0)\\&=-w(\xi,\eta)+w_-(\eta_0)\\&=-w(\xi,\eta)+u(-a/2,(\lambda a/2+\eta_0)/\mu)\\&=-w(\xi,\eta)+u_-\left(\frac{1}{\mu}\left[\lambda a/2+\eta_0\right]\right)\\[2ex]&=-w(\xi,\eta)+u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]\right).\end{aligned}$$

Integrating from the lower left corner $(-a/2,-\lambda a/2)$ to $(-a/2,\eta_0)$, we use the change of variables formuala

$$w(\xi,\eta) = u(A(\xi,\eta)) = u\left(\xi, -\frac{\lambda}{\mu}\,\xi + \frac{1}{\mu}\,\eta\right)$$

according to which

$$w_\xi = u_x - \frac{\lambda}{\mu} u_y,$$

and find

$$\begin{aligned}
&\int_{-\lambda a/2}^{\eta_0} Qw(-a/2,\sigma)\cdot(-1,0)\,d\sigma \\
&\qquad = -\int_{-\lambda a/2}^{\eta_0} w_\xi(-a/2,\sigma)\,d\sigma \\
&\qquad = -\int_{-\lambda a/2}^{\eta_0} u_x\left(-\frac{a}{2}, \frac{\lambda}{2\mu}\,a + \frac{1}{\mu}\,\sigma\right)\,d\sigma + \frac{\lambda}{\mu}\int_{-\lambda a/2}^{\eta_0} u_y\left(-\frac{a}{2}, \frac{\lambda}{2\mu}\,a + \frac{1}{\mu}\,\sigma\right)\,d\sigma \\
&\qquad = -\int_{-\lambda a/2}^{\eta_0} v_-\left(\frac{\lambda}{2\mu}\,a + \frac{1}{\mu}\,\sigma\right)\,d\sigma + \frac{\lambda}{\mu}\int_{-\lambda a/2}^{\eta_0} u_-'\left(\frac{\lambda}{2\mu}\,a + \frac{1}{\mu}\,\sigma\right)\,d\sigma \\
&\qquad = -\mu\int_0^{(\lambda a/2+\eta_0)/\mu} v_-(s)\,ds + \lambda\left[u_-\left(\frac{1}{\mu}(\lambda a/2+\eta_0)\right) - u_-(0)\right] \\
&\qquad = -\mu\int_0^{\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]} v_-(s)\,ds + \lambda\,u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]\right) - \lambda\,u_-(0).
\end{aligned}$$

Integrating from $(-a/2,-\lambda a/2)$ to $(\xi_0,\lambda\xi_0)$ along the bottom edge, we have

$$\begin{aligned}
&\int_{-a/2}^{\xi_0} Qw(\tau,\lambda\tau)\cdot(\lambda,-1)\,d\tau \\
&\qquad = \lambda\int_{-a/2}^{\xi_0} w_\xi(\tau,\lambda\tau)\,d\tau + \int_{-a/2}^{\xi_0} w_\eta(\tau,\lambda\tau)\,d\tau \\
&\qquad = \lambda\int_{-a/2}^{\xi_0} u_x(t,0)\,dt - \frac{\lambda^2}{\mu}\int_{-a/2}^{\xi_0} u_y(t,0)\,dt + \frac{1}{\mu}\int_{-a/2}^{\xi_0} u_y(t,0)\,dt \\
&\qquad = \lambda\int_{-a/2}^{\xi_0} u_0'(t)\,dt - \frac{\lambda^2-1}{\mu}\int_{-a/2}^{\xi_0} v_0(t)\,dt \\
&\qquad = \lambda\,u_0\left(\frac{\xi+\eta}{\lambda+1}\right) - \lambda\,u_0(-a/2) - \frac{\lambda^2-1}{\mu}\int_{-a/2}^{\frac{\xi+\eta}{\lambda+1}} v_0(t)\,dt.
\end{aligned}$$

Finally, integrating from $(\xi_0, \lambda\xi_0)$ to (ξ, η),

$$\begin{aligned}\int_0^{t_1} Qw(\xi_0 - t, \lambda\xi_0 + t) \cdot (1,1)\, dt &= -w(\xi,\eta) - w(\xi_0, \lambda\xi_0) \\ &= -w(\xi,\eta) - u_0\left(\frac{\xi+\eta}{\lambda+1}\right).\end{aligned}$$

Since the sum of these four integrals should vanish, we obtain

$$\begin{aligned}w(\xi,\eta) &= \frac{1}{2}\, u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\, a - (\xi-\eta)\right]\right) \\ &\quad - \frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\frac{\lambda-1}{2}\, a-(\xi-\eta)\right]} v_-(s)\, ds + \frac{\lambda}{2}\, u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\, a - (\xi-\eta)\right]\right) - \frac{\lambda}{2}\, u_-(0) \\ &\quad + \frac{\lambda}{2}\, u_0\left(\frac{\xi+\eta}{\lambda+1}\right) - \frac{\lambda}{2}\, u_0(-a/2) - \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{\frac{\xi+\eta}{\lambda+1}} v_0(t)\, dt \\ &\quad - \frac{1}{2}\, u_0\left(\frac{\xi+\eta}{\lambda+1}\right) \\ &= \frac{\lambda+1}{2}\, u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\, a - (\xi-\eta)\right]\right) - \frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\frac{\lambda-1}{2}\, a-(\xi-\eta)\right]} v_-(s)\, ds \\ &\quad - \frac{\lambda}{2}[u_-(0) + u_0(-a/2)] + \frac{\lambda-1}{2}\, u_0\left(\frac{\xi+\eta}{\lambda+1}\right) - \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{\frac{\xi+\eta}{\lambda+1}} v_0(t)\, dt.\end{aligned}$$

For $(\xi,\eta) \in U_1$, the point $(-a/2, \eta_0)$ used above and the associated integrals remain the same. A new point $(a/2, \eta_1)$ along the right edge is required satisfying

$$\begin{cases} a/2 - t_2 = \xi \\ \eta_1 + t_2 = \eta \end{cases}$$

so that

$$\eta_1 = \xi + \eta - \frac{a}{2}.$$

The integral computed above from the lower left corner $(-a/2, -\lambda/2)$ to $(\xi_0, \lambda\xi_0)$

may be extended to the point $(a/2, \lambda a/2)$ to obtain

$$\begin{aligned}
&\int_{-a/2}^{a/2} Qw(\tau, \lambda\tau)\cdot(\lambda, -1)\, d\tau \\
&\qquad = \lambda \int_{-a/2}^{a/2} w_\xi(\tau, \lambda\tau)\, d\tau + \int_{-a/2}^{a/2} w_\eta(\tau, \lambda\tau)\, d\tau \\
&\qquad = \lambda \int_{-a/2}^{a/2} u_x(t, 0)\, dt - \frac{\lambda^2}{\mu}\int_{-a/2}^{a/2} u_y(t, 0)\, dt + \frac{1}{\mu}\int_{-a/2}^{a/2} u_y(t, 0)\, dt \\
&\qquad = \lambda \int_{-a/2}^{a/2} u_0'(t)\, dt - \frac{\lambda^2 - 1}{\mu}\int_{-a/2}^{a/2} v_0(t)\, dt \\
&\qquad = \lambda\, u_0\,(a/2) - \lambda\, u_0(-a/2) - \frac{\lambda^2-1}{\mu}\int_{-a/2}^{a/2} v_0(t)\, dt.
\end{aligned}$$

Two additional integrals arise for a total of 5 integrals along boundary segments. One of these is from $(a/2, \lambda a/2)$ to $(a/2, \eta_1)$ and is given by

$$\begin{aligned}
&\int_{\lambda a/2}^{\eta_1} w_\xi(a/2, \sigma)\, d\sigma \\
&\quad = \int_{\lambda a/2}^{\eta_1} u_x\left(\frac{a}{2}, \frac{1}{\mu}(-\lambda a/2 + \sigma)\right) d\sigma - \frac{\lambda}{\mu}\int_{\lambda a/2}^{\eta_1} u_y\left(\frac{a}{2}, \frac{1}{\mu}(-\lambda a/2 + \sigma)\right) d\sigma \\
&\quad = \int_{\lambda a/2}^{\eta_1} v_+\left(\frac{1}{\mu}(-\lambda a/2 + \sigma)\right) d\sigma - \frac{\lambda}{\mu}\int_{\lambda a/2}^{\eta_1} u_+'\left(\frac{1}{\mu}(-\lambda a/2 + \sigma)\right) d\sigma \\
&\quad = \mu \int_0^{\frac{1}{\mu}(-\lambda a/2+\eta_1)} v_+\,(s)\ ds - \lambda \int_0^{\frac{1}{\mu}(-\lambda a/2+\eta_1)} u_+'(s)\, ds \\
&\quad = \mu \int_0^{\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]} v_+\,(s)\ ds - \lambda\, u_+\left(\frac{1}{\mu}\left[\xi + \eta - \frac{\lambda+1}{2}\,a\right]\right) + \lambda\, u_+(0).
\end{aligned}$$

The remaining integral is from $(a/2, \eta_1)$ to (ξ, η) resulting in

$$\begin{aligned}
\int_0^{t_2} Qw(a/2 - t, \eta_1 + t)\cdot(1,1)\, dt &= -w(\xi, \eta) + w(a/2, \eta_1) \\
&= -w(\xi, \eta) + u_+\left(\frac{1}{\mu}(-\lambda a/2 + \eta_1)\right) \\
&= -w(\xi, \eta) + u_+\left(\frac{1}{\mu}\left[\xi + \eta - \frac{\lambda+1}{2}\,a\right]\right).
\end{aligned}$$

Setting the sum of these five integrals equal to zero, we obtain a formula valid on the second region U_1:

$$\begin{aligned}
w(\xi,\eta) = {} & \frac{1}{2}\,u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]\right) \\
& -\frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]} v_-(s)\,ds+\frac{\lambda}{2}\,u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]\right)-\frac{\lambda}{2}\,u_-(0) \\
& +\frac{\lambda}{2}\left[u_0\left(a/2\right)-u_0(-a/2)\right]-\frac{\lambda^2-1}{2\mu}\int_{-a/2}^{a/2} v_0(t)\,dt \\
& +\frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]} v_+\left(s\right)\,ds-\frac{\lambda}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right)+\frac{\lambda}{2}\,u_+(0) \\
& +\frac{1}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right) \\
= {} & \frac{\lambda+1}{2}\,u_-\left(\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]\right)-\frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\frac{\lambda-1}{2}\,a-(\xi-\eta)\right]} v_-(s)\,ds \\
& -\frac{\lambda}{2}[u_-(0)+u_0(-a/2)]+\frac{\lambda}{2}[u_0(a/2)+u_+(0)]-\frac{\lambda^2-1}{2\mu}\int_{-a/2}^{a/2} v_0(t)\,dt \\
& -\frac{\lambda-1}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right)+\frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]} v_+\left(s\right)\,ds.
\end{aligned}$$

For $(\xi,\eta)\in U_2$ the point $(a/2,\eta_1)$ and its associated integrals remain unchanged, but a new point $(\xi_2,\lambda\xi_2+\mu b)$ is required satisfying

$$\left\{\begin{array}{rcl} \xi_2+t_3 & = & \xi \\ \lambda\xi_2+\mu b+t_3 & = & \eta \end{array}\right.$$

so that

$$\xi_2=\frac{-(\xi-\eta)-\mu b}{\lambda-1}.$$

The integral along the left edge from $(-a/2,-\lambda a/2)$ to $(-a/2,\eta_0)$ calcuated in the previous two cases is extended in this case to the corner $(-a/2,-\lambda a/2+$

μb) to yield

$$
\begin{aligned}
&\int_{-\lambda a/2}^{-\lambda a/2+\mu b} Qw(-a/2,\sigma)\cdot(-1,0)\,d\sigma \\
&\quad = -\int_{-\lambda a/2}^{-\lambda a/2+\mu b} w_\xi(-a/2,\sigma)\,d\sigma \\
&\quad = -\int_{-\lambda a/2}^{-\lambda a/2+\mu b} u_x\left(-\frac{a}{2},\frac{\lambda}{2\mu}\,a+\frac{1}{\mu}\,\sigma\right)\,d\sigma+\frac{\lambda}{\mu}\int_{-\lambda a/2}^{-\lambda a/2+\mu b} u_y\left(-\frac{a}{2},\frac{\lambda}{2\mu}\,a+\frac{1}{\mu}\,\sigma\right)\,d\sigma \\
&\quad = -\int_{-\lambda a/2}^{-\lambda a/2+\mu b} v_-\left(\frac{\lambda}{2\mu}\,a+\frac{1}{\mu}\,\sigma\right)\,d\sigma+\frac{\lambda}{\mu}\int_{-\lambda a/2}^{-\lambda a/2+\mu b} u_-'\left(\frac{\lambda}{2\mu}\,a+\frac{1}{\mu}\,\sigma\right)\,d\sigma \\
&\quad = -\mu\int_0^b v_-(s)\,ds+\lambda\,[u_-\,(b)-u_-(0)] \\
&\quad = -\mu\int_0^b v_-(s)\,ds+\lambda\,u_-\,(b)-\lambda\,u_-(0).
\end{aligned}
$$

The three boundary integrals from $(-a/2,-\lambda a/2)$ to $(a/2,\lambda a/2)$, from $(a/2,\lambda a/2)$ to $(a/2,\eta_1)$, and from $(a/2,\eta_1)$ to (ξ,η) all have the same form.

The new integral from $(\xi_2,\lambda\xi_2+\mu b)$ to (ξ,η) is given by

$$
\begin{aligned}
\int_0^{t_3} Qw(\xi_2+t,\lambda\xi_2+\mu b+t)\cdot(-1,1)\,dt &= -w(\xi,\eta)+w(\xi_2,\lambda\xi_2+\mu b) \\
&= -w(\xi,\eta)+u(\xi_2,b) \\
&= -w(\xi,\eta)+u_1(\xi_2) \\
&= -w(\xi,\eta)+u_1\left(\frac{-(\xi-\eta)-\mu b}{\lambda-1}\right).
\end{aligned}
$$

The new integral from $(-a/2, -\lambda a/2 + \mu b)$ to $(\xi_2, \lambda\xi_2 + \mu b)$ is

$$\begin{aligned}
&\int_{-a/2}^{\xi_2} Qw(\tau, \lambda\tau + \mu b) \cdot (-\lambda, 1)\, d\tau \\
&\qquad = -\lambda \int_{-a/2}^{\xi_2} w_\xi(\tau, \lambda\tau + \mu b)\, d\tau - \int_{-a/2}^{\xi_2} w_\eta(\tau, \lambda\tau + \mu b)\, d\tau \\
&\qquad = -\lambda \int_{-a/2}^{\xi_2} u_x(\tau, b)\, d\tau + \frac{\lambda^2}{\mu} \int_{-a/2}^{\xi_2} u_y(\tau, b)\, d\tau - \frac{1}{\mu} \int_{-a/2}^{\xi_2} u_y(\tau, b)\, d\tau \\
&\qquad = -\lambda \int_{-a/2}^{\xi_2} u_1'(t)\, dt + \frac{\lambda^2 - 1}{\mu} \int_{-a/2}^{\xi_2} v_1(t)\, dt \\
&\qquad = \lambda\, u_1(-a/2) - \lambda\, u_1\left(\frac{-(\xi - \eta) - \mu b}{\lambda - 1}\right) + \frac{\lambda^2 - 1}{\mu} \int_{-a/2}^{\frac{-(\xi-\eta)-\mu b}{\lambda - 1}} v_1(t)\, dt.
\end{aligned}$$

The expression for w on the final region U_2 is determined by setting the sum of these six boundary segment integrals to zero:

$$\begin{aligned}
w(\xi,\eta) = &-\frac{\mu}{2}\int_0^b v_-(s)\,ds + \frac{\lambda}{2}\,u_-\,(b) - \frac{\lambda}{2}\,u_-(0) \\
&+ \frac{\lambda}{2}\,[u_0\,(a/2) - u_0(-a/2)] - \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{a/2} v_0(t)\,dt \\
&+ \frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]} v_+\,(s)\;ds - \frac{\lambda}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right) + \frac{\lambda}{2}\,u_+(0) \\
&+ \frac{1}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right) \\
&+ \frac{1}{2}\,u_1\left(\frac{-(\xi-\eta)-\mu b}{\lambda-1}\right) \\
&+ \frac{\lambda}{2}\,u_1(-a/2) - \frac{\lambda}{2}\,u_1\left(\frac{-(\xi-\eta)-\mu b}{\lambda-1}\right) + \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{\frac{-(\xi-\eta)-\mu b}{\lambda-1}} v_1(t)\,dt \\
= &\,\frac{\lambda}{2}\,[u_-\,(b) + u_1(-a/2)] - \frac{\lambda}{2}\,[u_-(0) + u_0(-a/2)] - \frac{\mu}{2}\int_0^b v_-(s)\,ds \\
&+ \frac{\lambda}{2}[u_0(a/2) + u_+(0)] - \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{a/2} v_0(t)\,dt \\
&- \frac{\lambda-1}{2}\,u_+\left(\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]\right) + \frac{\mu}{2}\int_0^{\frac{1}{\mu}\left[\xi+\eta-\frac{\lambda+1}{2}\,a\right]} v_+\,(s)\;ds \\
&- \frac{\lambda}{2}\,u_1\left(\frac{-(\xi-\eta)-\mu b}{\lambda-1}\right) + \frac{\lambda^2-1}{2\mu}\int_{-a/2}^{\frac{-(\xi-\eta)-\mu b}{\lambda-1}} v_1(t)\,dt.
\end{aligned}$$

4 Preliminary Testing

First of all, it should be noted that the approach we have used is based on the (incorrect) assumption that there exists a solution with given boundary values. As a consequence, we should not expect to be able to achieve any particular prescribed boundary values, but remember the problem said we could "choose" the boundary values. I'm going to start with a choice of domain $R = (-1/2, 1/2) \times (0, 1)$ and nominal boundary values determined by the function $g(x, y) = (1 - y)(a^2/4 - x^2) + y(a^2/16 - x^2/4)$. This function can be used to give continuous values for all the boundary functions $u_\pm$, $v_\pm$, u_0, v_0, u_1, etc. used above. While (as I said) we cannot expect the formulas above to provide a solution achieving the utilized boundary conditions, we should get a solution, and that solution should be continuous on the entire region.

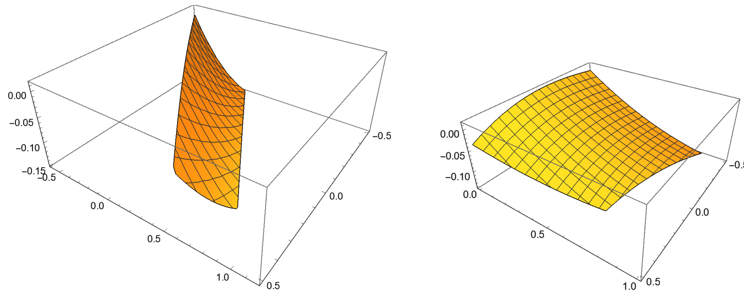

Figure 8: The preliminary plots of w over the region U_0 and u over the corresponding subregion of R. Notice that these plots look promising.

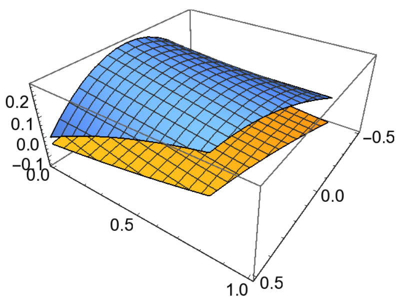

Figure 9: The graph of u over the region U_0 compared to the graph of g.

I have also taken the original PDE, which means $\lambda = 2/\sqrt{3}$ and $\mu = 1/\sqrt{3}$ (in addition to $a = b = 1$).

Here are (attempted) plots of the resulting awning corresponding to the regions U_0 and U_1. As you will see there are some problems; this is why this section is "preliminary." There is some error here.

Here is a comparison of the first portion compared to the natural extension of the boundary values. The boundary values are not met (as expected), but the result is not disasterous.

Finally, here is the plot I'm getting of w over regions U_0 (as above) and U_1.

As you can see, these two portions do not fit together to form a continous surface, and (if I'm understanding things correctly) they should. Presumably this means I've got an error in the derivation somewhere (perhaps hopefully in the derivation of the expression for w on U_1, as at least the result on U_0 lead to a reasonable looking portion of awning; I'm not sure what this second portion

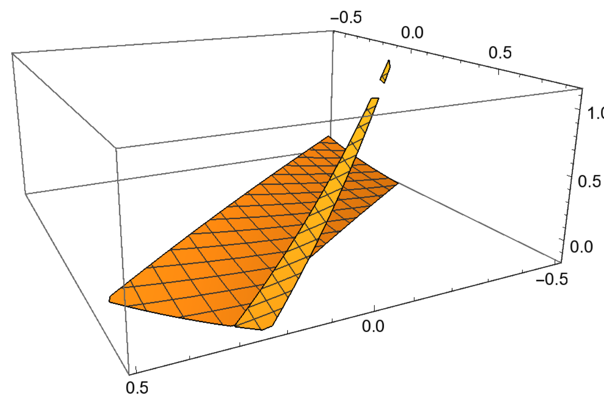

Figure 10: Plots of w over regions U_0 and U_1; evidence of an error.

leads to for u, but clearly the magnitude is large and it slopes only from one side to the other) or else there is a problem in my mathematica coding.

There is also the possibility that I'm missing something more fundamental in the setup, but I'm not seeing that at the moment.