Remarks on Hyperbolic Awning Design (MATH 6702 Final Exam Problem 3)

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Abstract

This is a report for Kendra and James Riddle of the Sonshine Awning Company, Pheonix Arizona, concerning their proposed design of an awning satisfying a certain Dirichlet boundary value problem for the hyperbolic PDE $u_{yy} - u_{xx} = 0$ on a rectangle. We introduce the problem, give some preliminary discussion of techniques which can be applied to understand the possibilities and properties of solutions, consider the consequences for this particular design, and offer some conclusions and design alternatives.

1 Introduction

The proposed design suggests finding an awning with shape determined by the boundary value problem

$$\begin{cases} u_{xx} = u_{yy} \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x,0) = a^2/4 - x^2, \ u(\pm a/2, y) = 0, \ u(x,b) = a^2/16 - x^2/4. \end{cases} \tag{1}$$

The partial differential equation appearing in (1) is hyperbolic, that is, it is a version of the wave equation, and it is customary to think of one of the variables as time. We will generally think of the second variable y as time. In this framework, the condition

$$u(x,0) = u_0(x) = \frac{a^2}{4} - x^2$$

may be considered as an **initial condition** (with respect to the time variable $t = y$). The homogeneous condition

$$u(\pm a/2, y) = 0 \tag{2}$$

is a relatively natural condition for this PDE which may be interpreted as a requirement that the endpoints of a "vibrating string" are kept fixed. The last condition

$$u(x,b) = u_1(x) = \frac{a^2}{16} - \frac{x^2}{4}$$

is somewhat unnatural with respect to this interpretation we have imposed on the problem, however, it is clear that time could be reversed and one could take this as an initial condition. One could also view x as the "time" variable and take either of the homogeneous conditions (2) as an initial condition, but let us proceed under the time assumption compatible with the given coordinates and the assumption that y represents "time" so that time $y = 0$ corresponds to the side of the awning closest to the building.

In this framework, it is natural to take the initial "velocity"

$$v_0(x) = u_y(x,0)$$

as given. We will assume this value represents a function which may be chosen within certain design parameters. For example it should be noted that a choice of $v_0(x)$ with $v_0(x) > 0$ will direct water back toward the building. This may require additional considerations concerning drainage and gutters. We note that such designs are sometimes used; see for example

`https://www.cityofsydney.nsw.gov.au/__data/assets/pdf_file/0014/120371/Co.`

Based on the other boundary values, we may assume the initial "velocity" v_0 is an even function, though this also may be relaxed if desired.

2 Preliminaries

Perhaps the simplest situation for the wave equation (as we have imposed it on this design) involves prescribing $u_0 = u_0(x)$ and $v_0 = v_0(x)$ along the entire real line. Then the equation may be solved uniquely as follows: We write

$$w = u_y - u_x \qquad \text{so that} \qquad w_y + w_x = 0.$$

In addition, subject to differentiability of u_0 which we have for $|x| \le a/2$ and we will (for now) assume for any extension $u_0 : \mathbb{R} \to \mathbb{R}$, we can write

$$w(x,0) = w_0(x) = u_y(x,0) - u_x(x,0) = v_0(x) - u_0'(x).$$

Restricting to a parameterized path $\gamma(t) = (\xi + t, t)$ starting at $(\xi, 0)$, we have

$$\frac{d}{dt} w(\xi + t, t) = w_x(\xi + t, t) + w_y(\xi + t, t) = 0$$

according to the first order PDE for w. Thus, solving $x = \xi + t$ and $y = t$ for ξ and t so that $\xi = x - y$, we can say

$$w(x, y) = w(\xi, 0) = w_0(\xi) = w_0(x - y) = v_0(x - y) - u_0'(x - y).$$

Similarly,

$$\frac{d}{dt}u(\xi - t, t) = -u_x(\xi - t, t) + u_y(\xi - t, t) = w(\xi - t, t),$$

so that

$$u(x, y) = u(\xi, 0) + \int_0^t w(\xi - \tau, \tau)\, d\tau$$

where this time $\xi = x + y$ and $t = y$. That is,

$$\begin{aligned} u(x, y) &= u_0(x + y) + \int_0^y [v_0(x + y - 2\tau) - u_0'(x + y - 2\tau)]\, d\tau \\ &= \frac{1}{2}[u_0(x + y) + u_0(x - y)] + \frac{1}{2}\int_{x-y}^{x+y} v_0(\xi)\, d\xi. \end{aligned}$$

This is d'Alembert's solution.

Now, we note that the values of d'Alembert's solution $u(x, y)$ at a particular point (x, y) depend on the values of u_0 at $x - y$ and $x + y$ as well as the values of $v_0(\xi)$ on the entire interval $x - y < \xi < x + y$. Of course, we do not know v_0 at all. Nevertheless, we make the following observation: If (x, y) satisfies $y < a/2 - |x|$ as indicated in Figure 1, then at least the values of u_0 are given by the desired boundary/initial value $u_0(x) = a^2/4 - x^2$.

Before attempting to understand conditions imposed on v_0 and, in particular if v_0 may be chosen so that d'Alembert's formula produces the appropriate boundary values $u(x, b) = u_1(x)$ for some $b < a/2$, we proceed to consider the possibility of extending this solution to the regions $R_2^{\pm}$ indicated in Figure 1 using only the values of u_0 and v_0 on the interval $[-a/2, a/2]$ and the lateral homogeneous values. In order to accomplish this, another approach to the derivation of d'Alembert's solution on R_1 is instructive. Consider the region U determined by the point $(x, y) \in R_1$ as indicated in Figure 1, namely

$$U = \{(\xi, \eta) : 0 < \eta < y - |x|,\ x - y < \xi < x + y\}.$$

Letting $\mathbf{v}$ be the field on U defined by

$$\mathbf{v}(\xi, \eta) = (u_x, -u_y) = (u_x(\xi, \eta), -u_y(\xi, \eta))$$

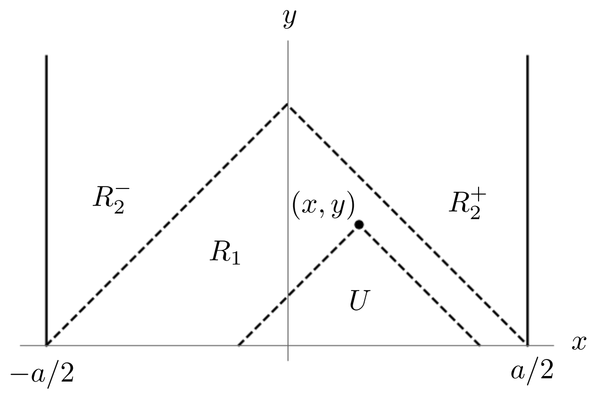

Figure 1: A region where the value of u is determined by the values of u_0 and v_0 on the interval $[-a/2, a/2]$ according to d'Alembert's formula.

we note that $\operatorname{div} \mathbf{v} = -\Box u = u_{xx} - u_{yy} = 0$. Thus, we may apply the divergence theorem to see

$$\int_{\partial U} \mathbf{v} \cdot n = 0.$$

In this case, the region U is bounded by the three segments of a triangle which we consider one by one. Letting Γ^- denote the left side of the triangle,

$$\begin{aligned}
\int_{\Gamma^-} \mathbf{v} \cdot n &= \int_0^y (u_x(x-y+t,t), -u_y(x-y+t,t)) \cdot \frac{(-1,1)}{\sqrt{2}} \sqrt{2}\, dt \\
&= -\int_0^y \frac{d}{dt} u(x-y+t,t)\, dt \\
&= -[u(x,y) - u(x-y,0)].
\end{aligned}$$

Similarly, the flux integral along the right side Γ^+ is

$$\begin{aligned}
\int_{\Gamma^+} \mathbf{v} \cdot n &= \int_0^y (u_x(x+t,y-t), -u_y(x+t,y-t)) \cdot (1,1)\, dt \\
&= \int_0^y \frac{d}{dt} u(x+t,y-t)\, dt \\
&= u(x+y,0) - u(x,y).
\end{aligned}$$

Finally, the integral along the interval $[x-y, x+y]$ is

$$\int_{x-y}^{x+y} (u_x(t,0), -u_y(t,0)) \cdot (0,-1)\, dt = \int_{x-y}^{x+y} v_0(t)\, dt.$$

Summing these three flux integrals, we have

$$-2u(x,y) + u_0(x-y) + u_0(x+y) + \int_{x-y}^{x+y} v_0(t)\,dt = 0$$

which is d'Alembert's formula.

Let us apply the same approach with regard to a point

$$(x,y) \in R_2^- = \{(\xi,\eta) : a/2 - |\xi| < \eta < a/2 - \xi,\ -a/2 < \xi < 0\}.$$

This time, we take U to be a rectangular region as indicated in Figure 2 with

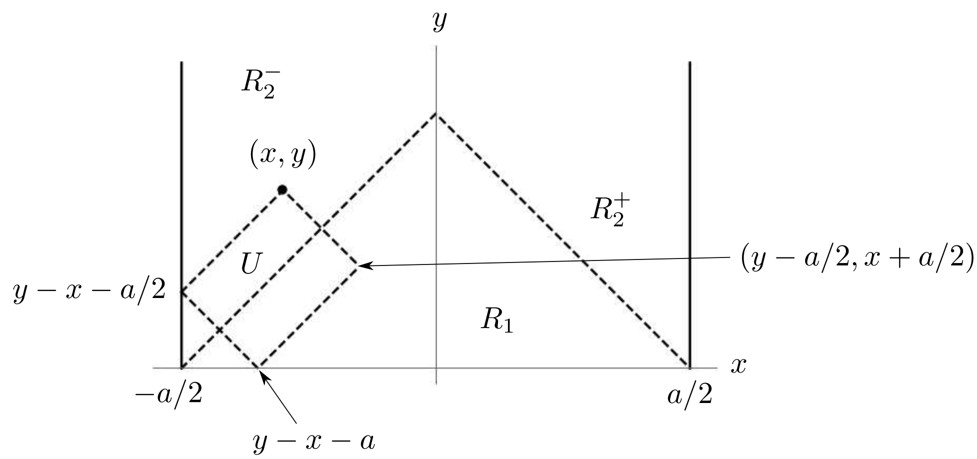

Figure 2: Determining the value of $u(x,y)$ be applying the divergence theorem on the rectangular region U.

corners at (x,y), $(-a/2, y-x-a/2)$, $(y-x-a, 0)$, and $(y-a/2, x+a/2)$. We have four path integrals to compute. Let us start at (x,y) and proceed counterclockwise to each corner as listed above:

$$\begin{aligned}
&\int_0^{x+a/2} (u_x(-a/2+t, y-x-a/2+t), -u_y(-a/2+t, y-x-a/2+t) \cdot (-1,1)\,dt \\
&\qquad = -\int_0^{x+a/2} \frac{d}{dt} u(-a/2+t, y-x-a/2+t)\,dt \\
&\qquad = u(-a/2, y-x-a/2) - u(x,y) \\
&\qquad = -u(x,y).
\end{aligned}$$

$$\int_0^{y-x-a/2} (u_x(-a/2+t, y-x-a/2-t), -u_y(-a/2+t, y-x-a/2-t) \cdot (-1,-1)\, dt$$

$$\begin{aligned} &= -\int_0^{y-x-a/2} \frac{d}{dt} u(-a/2+t, y-x-a/2-t)\, dt \\ &= u(-a/2, y-x-a/2) - u(y-x-a, 0) \\ &= -u_0(y-x-a). \end{aligned}$$

$$\int_0^{x+a/2} (u_x(y-x-a+t, t), -u_y(y-x-a+t, t) \cdot (1,-1)\, dt$$

$$\begin{aligned} &= \int_0^{x+a/2} \frac{d}{dt} u(y-x-a+t, t)\, dt \\ &= u(y-a/2, x+a/2) - u(y-x-a, 0) \\ &= u(y-a/2, x+a/2) - u_0(y-x-a). \end{aligned}$$

$$\int_0^{y-x-a/2} (u_x(x+t, y-t), -u_y(x+t, y-t) \cdot (1,1)\, dt$$

$$\begin{aligned} &= \int_0^{y-x-a/2} \frac{d}{dt} u(x+t, y-t)\, dt \\ &= u(y-a/2, x+a/2) - u(x,y). \end{aligned}$$

Summing these four integrals, we find

$$-2u(x,y) - 2u_0(y-x-a) + 2u(y-a/2, x+a/2) = 0$$

or

$$u(x,y) = u(y-a/2, x+a/2) - u_0(y-x-a).$$

The point $(y-a/2, x+a/2)$ lies in region R_1 and the value $u(y-a/2, x+a/2)$ is given by d'Alembert's formula:

$$\begin{aligned} u(x,y) &= \frac{1}{2}[u_0(x+y) + u_0(y-x-a)] + \frac{1}{2}\int_{y-x-a}^{x+y} v_0(\xi)\, d\xi - u_0(y-x-a) \\ &= \frac{1}{2}[u_0(x+y) - u_0(y-x-a)] + \frac{1}{2}\int_{y-x-a}^{x+y} v_0(\xi)\, d\xi. \end{aligned}$$

Naturally a symmetric formula may be obtained by assuming v_0 is even and that the solution $u = u(x, y)$ is even in x:

$$u(x,y) = \frac{1}{2}[u_0(-x+y) - u_0(y+x-a)] + \frac{1}{2}\int_{x+y-a}^{-x+y} v_0(\xi)\,d\xi$$
$$= \frac{1}{2}[u_0(x-y) - u_0(a-x-y)] + \frac{1}{2}\int_{x-y}^{a-x-y} v_0(\xi)\,d\xi.$$

This formula may also be obtained directly without the assumption that v_0 is even and is correct without that assumption. This suffices to consider some special cases and the compatibility of the "initial" condition $u(x, 0) = u_0(x)$ with the "final" condition $u(x, b) = u_1(x)$.

A simple explicit solution

We have obtained above formulas for awning shapes satisfying the PDE $u_{yy} - u_{xx} = 0$ and three of the four boundary conditions for relatively short awnings whose length satisfies $b < a/2$. These shapes depend on the initial "velocity" $u(x, 0) = v_9(x)$ on the interval $|x| \le a/2$ and, in principle, on a choice of $b < a/2$. If we take $v_0 \equiv 0$, then we obtain a specific shape given by

$$u(x,y) = \begin{cases} a^2/2 + a(x-y) - 2xy, & a/2 + x \le y \le b,\ -a/2 \le x \le 0 \\ a^2/4 - x^2 - y^2, & 0 \le y \le \min\{b, a/2 - |x|\} \\ a^2/2 - a(x+y) + 2xy, & a/2 - x \le y \le b,\ 0 \le x \le a/2. \end{cases}$$

Note that by a homogeneous scaling in x and y, we may assume the width a measures one unit. This shape is shown by $b = a/2$ in Figure 3. We observe that for this solution

$$u_x(x,y) = \begin{cases} a - 2y, & a/2 + x \le y \le b,\ -a/2 \le x \le 0 \\ -2x, & 0 \le y \le \min\{b, a/2 - |x|\} \\ -a + 2y, & a/2 - x \le y \le b,\ 0 \le x \le a/2, \end{cases}$$

so that the solution satisfies $u \in C^1(R)$, but

$$u_{xx}(x,y) = \begin{cases} 0, & a/2 + x < y \le b,\ -a/2 \le x \le 0 \\ -2, & 0 \le y < \min\{b, a/2 - |x|\} \\ 0, & a/2 - x < y \le b,\ 0 \le x \le a/2. \end{cases}$$

This means, first of all, that even though we assumed the solution satisfied $u \in C^2(R)$ in order to derive the formula, the formula we obtained does not provide a classical solution for the problem. Presumably, we have here a weak

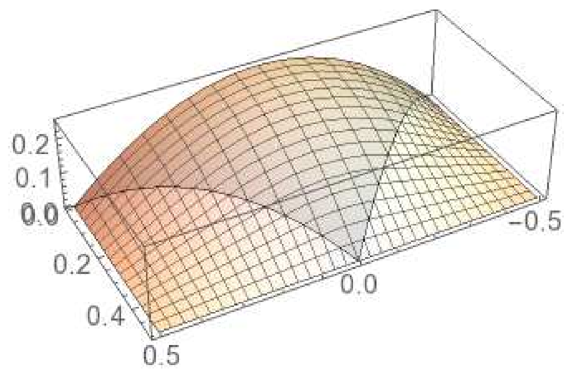

Figure 3: A simple awning obtained by taking $v_0 \equiv 0$

C^1 solution of the PDE. Nevertheless, the awning shape does appear to be practically and aesthetically viable.

a second observation is that it is not possible for one of these shapes to match the desired "final" boundary value $u(x,b) = u_1(x) = a^2/16 - x^2/4$. It is enough to observe, as the calculations above indicate, that $u(x,y)$ is affine in x for fixed y on the regions $R_2^{\pm}$. We may still ask if it is possible to achieve the final value for $|x| \le a/2 - b$ for some b with $0 < b < a/2$. The value of the solution in this case is given by

$$u(x,b) = \frac{a^2}{4} - b^2 - x^2 \neq \frac{a^2}{16} - \frac{x^2}{4}.$$

So this is not possible. Choosing b to minimize $\|u(x,b) - u_1(x)\|_{L^2(-b/2,b/2)}$ however, we obtain a unique value $b^* \approx (0.43172)a$ (fairly close to the maximum value $b = a/2$ under consideration). Figure 4 shows how well (or how poorly) this simple solution matches the desired end profile.

At this point, there are several natural conjectures:

1. The target end profile given by $u_1(x) = a^2/16 - x^2/4$ can be more closely approximated by a solution associated with an appropriate choice of (nonzero but still even) v_0.
2. Even with the assumption $v_0(x) \equiv 0$, it should be possible to minimize the overall deviance from the desired "initial" value and the desired "final" value to obtain a (possibly) more satisfactory shape.
3. The form of d'Alembert's solution giving

$$u(0,a/2) = \int_{-a/2}^{a/2} v_0(\xi)\, d\xi$$

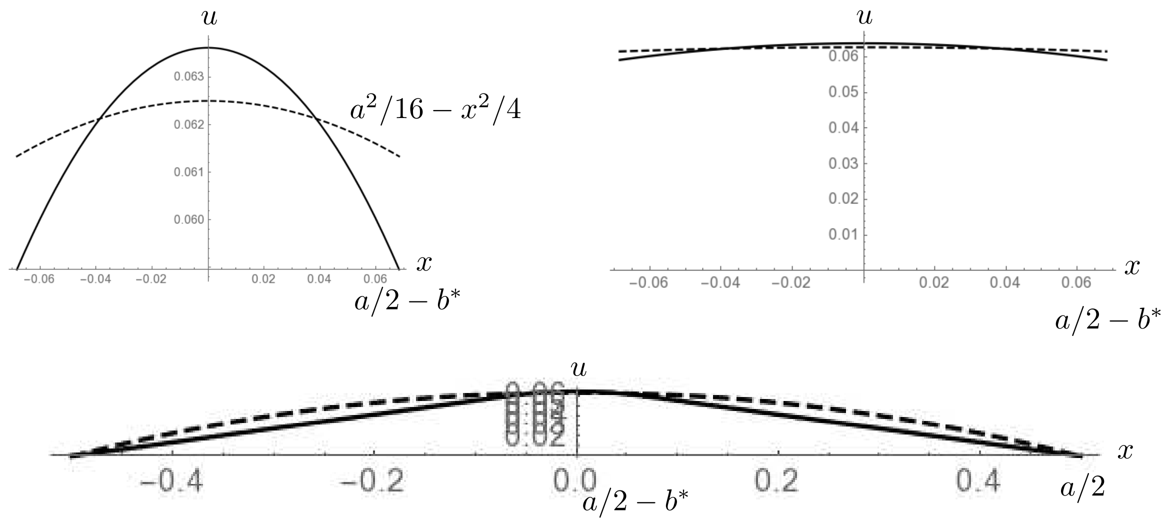

Figure 4: Plots of $u(x, b^*)$ (solid line) in comparison to $u_1(x) = a^2/16 - x^2/4$ (dashed).

suggests that apart from extreme positive initial velocity, it will be difficult to obtain a very long awning with adequate height at the outer edge.

3 Fourier series approximation

We can also attempt a separated variables solution $u(x, y) = A(x)B(y)$ according to which we obtain separation equations

$$\frac{A''}{A} = \frac{B''}{B} = -\lambda.$$

Taking the implied boundary conditions $A(\pm a/2) = 0$ associated with a nonzero solution $A(x)B(y)$, we arrive at

$$A_j = \cos\frac{(2j+1)\pi x}{a} \qquad \text{and} \qquad B_j = a_j \cos\frac{(2j+1)\pi y}{a} + b_j \sin\frac{(2j+1)\pi y}{a}$$

for $j = 0, 1, 2, \ldots$. Thus, our superposition takes the form

$$u(x, y) = \sum_{j=0}^{\infty} \left[a_j \cos\frac{(2j+1)\pi y}{a} + b_j \sin\frac{(2j+1)\pi y}{a} \right] \cos\frac{(2j+1)\pi x}{a}.$$

Differentiating this series formally, we find

$$u_y(x,y) = \frac{\pi}{a}\sum_{j=0}^{\infty}(2j{+}1)\left[-a_j \sin\frac{(2j+1)\pi y}{a} + b_j \cos\frac{(2j+1)\pi y}{a}\right]\cos\frac{(2j+1)\pi x}{a}$$

so that

$$v_0(x) = u_y(x,0) = \frac{\pi}{a}\sum_{j=0}^{\infty}(2j+1)b_j \cos\frac{(2j+1)\pi x}{a}.$$

The coefficients a_j are determined by the "initial condition" and integration using the L^2 orthonormality of the cosine basis since

$$u(x,0) = \sum_{j=0}^{\infty} a_j \cos\frac{(2j+1)\pi x}{a}.$$

In particular,

$$a_j = \frac{2}{a}\int_{-a/2}^{a/2} u_0(x)\cos\frac{(2j+1)\pi x}{a}\,dx = \frac{2}{a}\int_{-a/2}^{a/2}\left[\frac{a^2}{4} - x^2\right]\cos\frac{(2j+1)\pi x}{a}\,dx.$$

Computing we find

$$\int_0^{a/2}\cos\frac{(2j+1)\pi x}{a}\,dx = \frac{a}{(2j+1)\pi}\sin\frac{(2j+1)\pi x}{a}\Big|_{x=0}^{a/2} = \frac{(-1)^j a}{(2j+1)\pi}.$$

Also,

$$\begin{aligned}\int_0^{a/2} x^2\cos\frac{(2j+1)\pi x}{a}\,dx &= \frac{a}{(2j+1)\pi}x^2\sin\frac{(2j+1)\pi x}{a}\Big|_{x=0}^{a/2} \\ &\qquad - \frac{2a}{(2j+1)\pi}\int_0^{a/2} x\sin\frac{(2j+1)\pi x}{a}\,dx \\ &= \frac{(-1)^j a^3}{4(2j+1)\pi} - \frac{2a^2}{(2j+1)^2\pi^2}\int_0^{a/2}\cos\frac{(2j+1)\pi x}{a}\,dx \\ &= \frac{(-1)^j a^3}{4(2j+1)\pi} - \frac{2(-1)^j a^3}{(2j+1)^3\pi^3}.\end{aligned}$$

We conclude that

$$a_j = \frac{4}{a}\left[\frac{a^2}{4}\frac{(-1)^j a}{(2j+1)\pi} - \frac{(-1)^j a^3}{4(2j+1)\pi} + \frac{2(-1)^j a^3}{(2j+1)^3\pi^3}\right] = \frac{8(-1)^j a^2}{(2j+1)^3\pi^3}.$$

Assuming the coefficients a_j are known, we can attempt to determine the coefficients b_j from the "final condition" $u(x,b) = u_1(x)$. We have

$$u(x,b) = \sum_{j=0}^{\infty} \left[a_j \cos \frac{(2j+1)\pi b}{a} + b_j \sin \frac{(2j+1)\pi b}{a} \right] \cos \frac{(2j+1)\pi x}{a}.$$

Noting that the desired ending values satisfy $u_1(x) = u_0(x)/4$, we obtain the desired relation

$$a_j \cos \frac{(2j+1)\pi b}{a} + b_j \sin \frac{(2j+1)\pi b}{a} = \frac{a_j}{4}$$

or

$$b_j \sin \frac{(2j+1)\pi b}{a} = \left[\frac{1}{4} - \cos \frac{(2j+1)\pi b}{a} \right] a_j.$$

Clearly, we are going to have a problem if b/a is an integer or even if b/a is a rational with odd integer denominator. On the other hand, for many choices of the ratio b/a, the factor $\sin(2j+1)/\pi b/a$ will be non-vanishing and we can write

$$\begin{aligned} b_j &= \frac{1}{\sin \frac{(2j+1)\pi b}{a}} \left[\frac{1}{4} - \cos \frac{(2j+1)\pi b}{a} \right] a_j \\ &= \frac{1}{\sin \frac{(2j+1)\pi b}{a}} \left[\frac{1}{4} - \cos \frac{(2j+1)\pi b}{a} \right] \frac{8(-1)^j a^2}{(2j+1)^3 \pi^3}. \end{aligned}$$

In particular, if $b/a = (2k+1)/2$ for some natural number k, then b_j takes a particularly simple (nonzero) value, namely,

$$b_j = \frac{2(-1)^k a^2}{(2j+1)^3 \pi^3},$$

so that

$$u(x,y) = \frac{2a^2}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^3} \left[4(-1)^j \cos \frac{(2j+1)\pi y}{a} + (-1)^k \sin \frac{(2j+1)\pi y}{a} \right] \cos \frac{(2j+1)\pi x}{a}.$$

The fully singular coefficients b_j may also be characterized as those for which

$$b = \frac{k}{2j+1} a \qquad \text{for some } j \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}.$$

Let us isolate these two collections of lengths:

$$\text{The fully nonsingular lengths:} \qquad N = \left\{ \frac{2k+1}{2} a : k \in \mathbb{N}_0 \right\}.$$

$$\text{The fully singular lengths:} \qquad S = \left\{ \frac{k}{2j+1}\, a : j \in \mathbb{N}_0,\ k \in \mathbb{N} \right\}.$$

There are two immediate questions about the singular lengths and the lengths $b \in (0,\infty)\backslash(N \cup S)$:

1. Is the singular behavior a direct result of something inherent in the PDE or is this an extraneous artifact of the technique of Fourier expansion?
2. If we can understand the fully singular behavior, can we understand the situation with the indeterminate lengths $b \in (0,\infty)\backslash(N \cup S)$?

4 Heuristics and properties

We appear to have constructed a great many awnings meeting the nominal design requirements of the boundary value problem (1). Unlike with awnings arising from the heat equation, we do not expect these solutions to decay to zero. Taking the "energy" of a solution to be

$$e(y) = \|u_t\|^2_{L^2(-a/2,a/2)} + \|u_x\|^2_{L^2(-a/2,a/2)} = \int_{-a/2}^{a/2} \left([u_t(x,y)]^2 + [u_x(x,y)]^2 \right)\, dx$$

as a measure of the total "elastic and kinetic energy" of the cross-section of the awning, we can calculate as before

$$\begin{aligned} e'(y) &= 2\int_{-a/2}^{a/2} \left(u_y(x,y)\, u_{yy}(x,y) + u_x(x,y)\, u_{xy}(x,y)\right)\, dx \\ &= 2\int_{-a/2}^{a/2} u_y(x,y)\left(u_{yy}(x,y) - u_{xx}(x,y)\right)\, dx + u_x(x,y)u_y(x,y)\Big|_{x=-a/2}^{a/2} \\ &= 0 \end{aligned}$$

since $u(\pm a/2, y) \equiv 0$ implies $u_y(\pm a/2, y) = 0$. This means the "energy" remains constant over "time."

We recall from the previous problem that the initial profile will be approximated according to Figure 5. In particular, using one term in the Fourier expansion gives an overall rough qualitative approximation of the desired profile while three terms in the Fourier series gives an initial profile almost indistinguishable from the desired profile. In addition, the form of our approximation, when the first three terms or nonsingular, will give precisely the same approximation (up to a factor of $1/4$) at $y = b$. In particular, if we take the first fully nonsingular value $b = a/2$, we obtain an awning shape as indicated in Figure 6

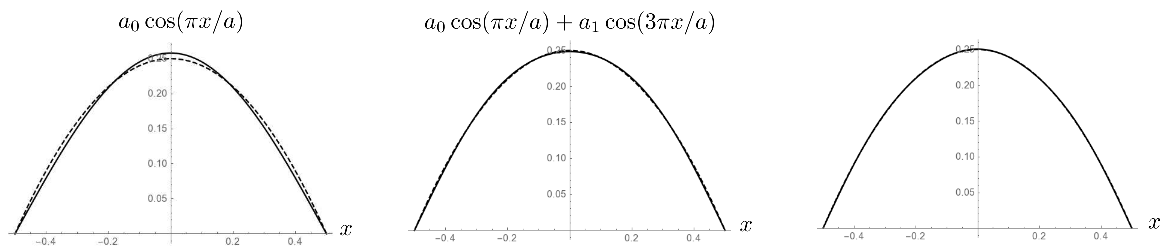

Figure 5: Initial cross-sections of Fourier approximations with one, two, and three terms.

(using the first three terms of the approximation). This solution gives supporting evidence for conjectures 1 and 3 above: The end profile data is matched precisely at the expense of including a nonzero positive initial inclination. It will be noted that this can be expected to direct runoff toward the building. The magnitude of the initial positive inclination does not appear to be severe, but the length is also fixed at the specific nonsingular length $b = a/2$.

As noted above, taking the initial velocity zero in the Fourier approximation is equivalent to taking (all) the coefficients $b_j = 0$. As expected, we may take, for example, the first term of the Fourier approximation under this assumption ($b_0 = 0$) and obtain a length $b = b_* \approx 0.41957\, a$ minimizing $\|u(x,b) - u_1(x)\|_{L^2(-b/2,b/2)}$ where $u(x,y)$ is taken to be the approximation

$$u(x,y) = a_0 \cos\frac{\pi y}{a} \cos\frac{\pi x}{a}.$$

This length is somewhat shorter than the previous length used to motivate our second conjecture, but the approximation of the ending values is much better. In fact, it can be expected that this approximation is, again, precisely the approximation of the leftmost illustration in Figure 4 (scaled by 1/4). These observations are indicated in Figure 7.

The single term approximation, and the observation that restriction to a specific "time" $y = b$ corresponds to a simple scaling of the initial value in this case, can also be useful in addressing the third conjecture and the first question. What we may do is temporarily abandon the prescriptions $u(x,0) = u_0(x)$ and $u(x,b) = u_1(x)$ and replace them with $u(x,0) = a_0 \cos(\pi x/a)$ and $u(x,b) = (a_0/4)\cos(\pi x/a)$. Then we have certainly a simple explicit solution of the equation

$$u(x,y) = \left[a_0 \cos\frac{\pi y}{a} + b_0 \sin\frac{\pi y}{a}\right] \cos\frac{\pi x}{a}. \tag{3}$$

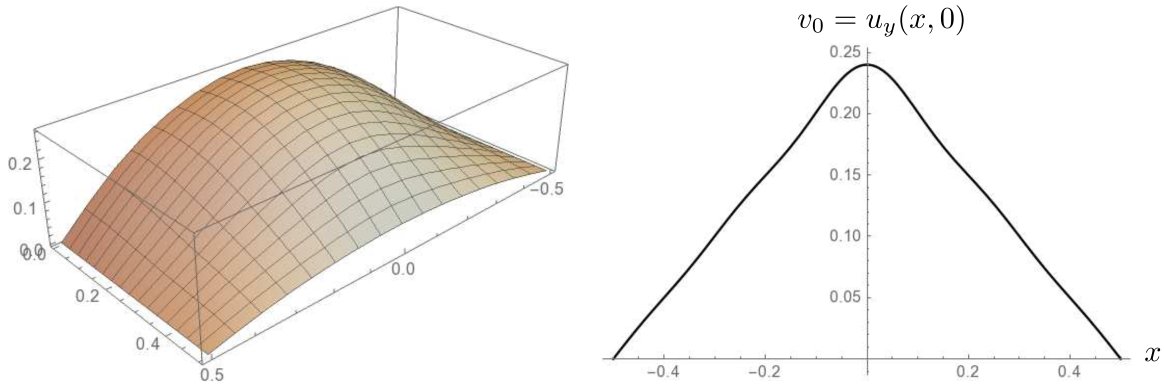

Figure 6: An awning determined by the wave equation with length $b = a/2$ (left); the initial inclination/velocity associated with this awning (right). We have used 5 terms in the approximation for the initial velocity $v_0(x) = u_y(x, 0)$ for additional accuracy in the illustration.

Furthermore, the end profile requirement gives

$$b_0 \sin \frac{\pi b}{a} = \left[\frac{1}{4} - \cos \frac{\pi b}{a}\right] a_0 = \left[\frac{1}{4} - \cos \frac{\pi b}{a}\right] \frac{8a^2}{\pi^3}.$$

Thus, it is seen that the coefficient b_0 is singular when $b/a \in \mathbb{N}$. This is (apparently) a **feature of the PDE** rather than a peculiarity of Fourier expansion. Moreover, by taking $b = \alpha a$ for $0 < \alpha < 1$, we can see the nature of the singular behavior. In Figure 8 we have plotted these solutions for a sequence of lengths approaching the singular value $b = a$ from below and then for corresponding values larger than $b = a$. It will be observed that indeed, greater and greater initial velocities as well as greater and greater overall amplitude for the awning shape is required as the singular value is approached. In particular, for length values just below the singular value $b = a$, the awnings have extreme initial inclinations and large arching shapes, though the (trigonometric) initial and final profiles are attained exactly. **There is no solution of the problem for the critical length $b = a$.** In addition to large oscillations for $b > a$, we also see the formation of an undesirable "bowl" which will persist for all $b > a$. Thus, in this modified problem, and presumably in the general problem, these "bowls" will persist for large length awnings making this particular PDE unsuitable for such long awnings.

Figure 9 shows the three term awning approximations corresponding to the fully nonsingular lengths $b = 3a/2$ and $b = 5a/2$.

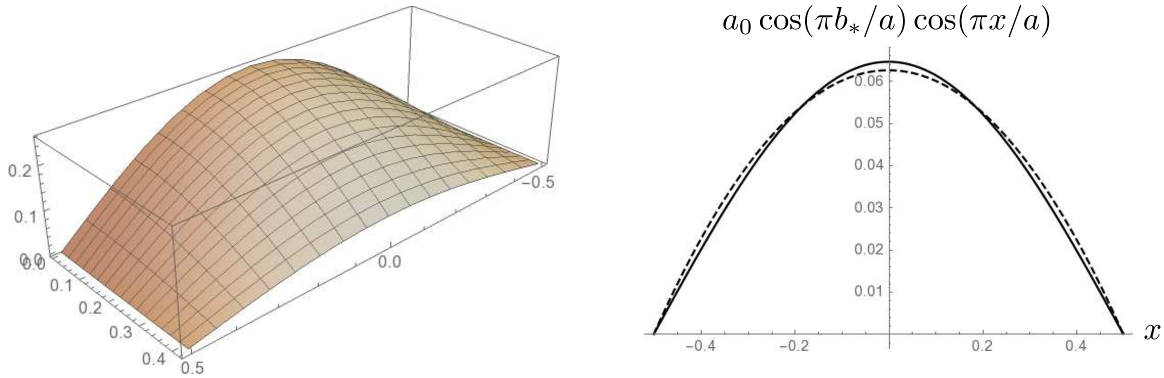

Figure 7: An awning determined by the first term of the Fourier approximation (left) with the error in the ending profile at $b \approx 0.41957\,a$.

The general situation is roughly as follows: The higher modes include more and more features of the quadratic profiles $u_0(x)$ and $u_1(x)$. These are attainable exactly for the fully nonsingular lengths with only the smallest fully nonsingular length $b = a/2$ having no undesirable "bowls." The first three terms give a reasonably good approximation for most lengths of practical interest. Using these three terms the fully nonsingular lengths $b = a/5$, $b = a/3$, $b = 2a/5$, $b = 3a/5$, $b = 2a/3$, etc. will become evidently excluded. For a precise solution, there will be some singular expressions in the full expansion (in the sense of large values of the coefficient b_j for some j). If the length b under consideration is close to a fully nonsingular length of interest, i.e., $a/2$, then the large coefficient b_j will also occur for a large index j, and the coefficient a_j will be extremely small eliminating the singular behavior. Precise estimates may be given, but I will not give them here at the moment.

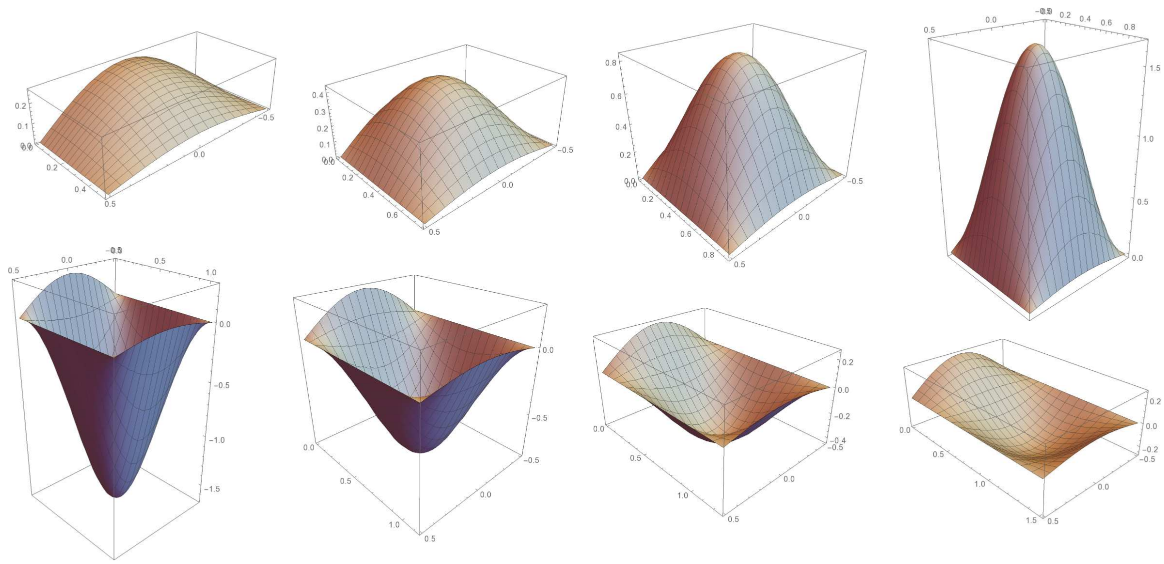

Figure 8: From left to right the exact solution given by (3) for $b = a/2, 3a/4, 7a/8, 15a/16, (1+1/16)a, (1+1/8)a, (1+1/4)a$, and $3a/2$.

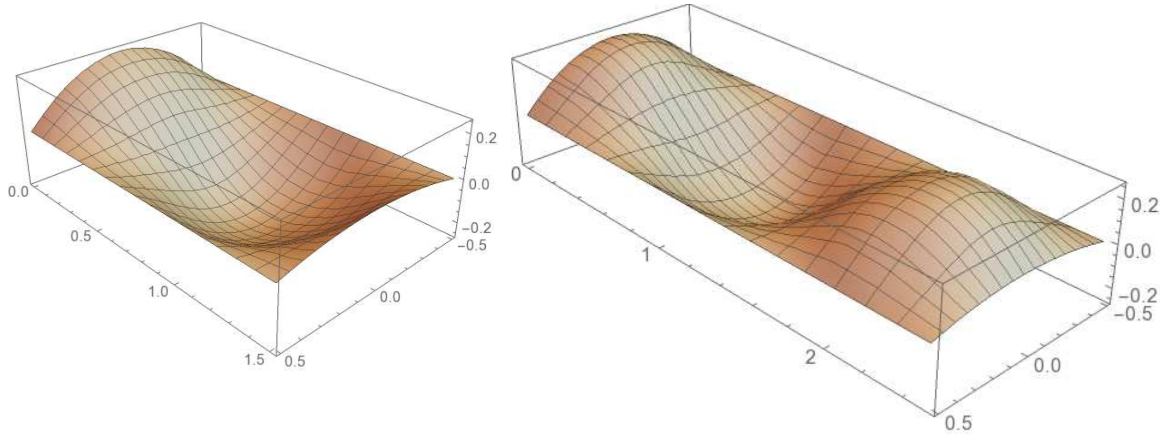

Figure 9: Long awnings with undesirable runoff catching bowls.