Remarks on Parabolic Awning Design (MATH 6702 Final Exam Problem 2)

John McCuan

April 19, 2020

Abstract

This is a report for Kendra and James Riddle of the Sonshine Awning Company, Pheonix Arizona, concerning their proposed design of an awning satisfying a certain Dirichlet boundary value problem for the parabolic PDE $u_y = u_{xx}$ on a rectangle. We introduce the problem, give some preliminary discussion of techniques which can be applied to understand the possibilities and properties of solutions, consider the consequences for this particular design, and offer some conclusions and design alternatives.

1 Introduction

The proposed design suggests finding an awning with shape determined by the boundary value problem

$$\begin{cases} u_{xx} = u_y \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x, 0) = a^2/4 - x^2, \ u(\pm a/2, y) = 0, \ u(x, b) = a^2/16 - x^2/4. \end{cases} \tag{1}$$

The partial differential equation appearing in (1) is parabolic, that is, it is a version of the heat equation, and it is customary to think of the variable y as "time." In this framework, the condition

$$u(x, 0) = u_0(x) = \frac{a^2}{4} - x^2$$

may be considered as an **initial condition** (with respect to the time variable $t = y$). The homogeneous condition

$$u(\pm a/2, y) = 0 \tag{2}$$

is a relatively natural condition for this PDE which may be interpreted as a requirement that temperature at the endpoints of a "thin heated rod" is maintained to be constant and zero. The last condition

$$u(x, b) = u_1(x) = \frac{a^2}{16} - \frac{x^2}{4}$$

is somewhat unnatural with respect to this interpretation we have imposed on the problem. In fact, we expect the initial condition should determine the value $u(x, b)$ at $y = b$ for every b independent of the prescription $u(x, b) = u_1(x)$. On the other hand, We could also presumably "start" with the condition $u(x, b) = u_1(x)$ as an initial condition and attempt to solve a "backwards heat equation."

We begin with parts (b) and (c) justifying our intuition/claim above that the value $u(x, b)$ is already determined by the initial condition and lateral boundary conditions.

2 Uniqueness

As suggested in the hint, we consider the non-negative function $g : [0, \infty) \to \mathbb{R}$ given by

$$g(y) = \int_{-a/2}^{a/2} [w(x, y)]^2 \, dx$$

which we think of as a function of "time" $t = y$. Differentiating under the integral sign and assuming w satisfies the PDE $w_y = w_{xx}$, we find

$$g'(y) = 2 \int_{-a/2}^{a/2} w(x, y) \, w_y(x, y) \, dx = 2 \int_{-a/2}^{a/2} w(x, y) \, w_{xx}(x, y) \, dx.$$

We can integrate by parts to obtain

$$g'(y) = 2w(x, y) w_x(x, y)\Big|_{x=-a/2}^{a/2} - 2 \int_{-a/2}^{a/2} [w_x(x, y)]^2 \, dx.$$

Therefore, if we know in addition that

$$w(\pm a/2, y) = 0,$$

then we have $g'(y) \le 0$. In particular, if we take $w(x, y) = u(x, y) - \tilde{u}(x, y)$ where u and $\tilde{u}$ are two solutions of (1), then $w(x, 0) \equiv 0$ and it follows that $g(y) \equiv 0$. This implies $w(x, y) \equiv 0$ and solutions of (1) are unique. It will be noted have not used the ending/boundary condition $u(x, b) = u_1(x)$, and the uniqueness we have obtained applies to solutions of the problem (1) which omit this condition, which we might well presume must be omitted.

3 Approximations

We may proceed to approximate solutions of the abbreviated problem

$$\begin{cases} u_{xx} = u_y \text{ on } R = (-a/2, a/2) \times (0, b) \\ u(x,0) = a^2/4 - x^2, \;\; u(\pm a/2, y) = 0, \end{cases} \tag{3}$$

using Fourier series. This approach arises as a superposition of separated variables solutions as follows: We consider the possibility of finding a solution $u(x,y) = A(x)B(y)$ of (3). The PDE gives the separation equations

$$\frac{A''}{A} = \frac{B'}{B} = -\lambda.$$

Taking the implied boundary conditions $A(\pm a/2) = 0$ associated with a nonzero solution $A(x)B(y)$, we arrive at

$$A_j = \cos\frac{(2j+1)\pi x}{a} \qquad \text{and} \qquad B_j = e^{-(2j+1)^2\pi^2 y/a^2}$$

for $j = 0, 1, 2, \ldots$. Thus, our superposition takes the form

$$u(x,y) = \sum_{j=0}^{\infty} a_j e^{-(2j+1)^2\pi^2 y/a^2} \cos\frac{(2j+1)\pi x}{a}.$$

The coefficients a_j are determined by integration using the L^2 orthonormality of the cosine basis. In particular,

$$a_j = \frac{2}{a}\int_{-a/2}^{a/2} u_0(x)\cos\frac{(2j+1)\pi x}{a}\,dx = \frac{2}{a}\int_{-a/2}^{a/2}\left[\frac{a^2}{4} - x^2\right]\cos\frac{(2j+1)\pi x}{a}\,dx.$$

Computing we find

$$\int_0^{a/2}\cos\frac{(2j+1)\pi x}{a}\,dx = \frac{a}{(2j+1)\pi}\sin\frac{(2j+1)\pi x}{a}\Big|_{x=0}^{a/2} = \frac{(-1)^j a}{(2j+1)\pi}.$$

Also,

$$\begin{aligned}\int_0^{a/2} x^2\cos\frac{(2j+1)\pi x}{a}\,dx &= \frac{a}{(2j+1)\pi}x^2\sin\frac{(2j+1)\pi x}{a}\Big|_{x=0}^{a/2} \\ &\quad - \frac{2a}{(2j+1)\pi}\int_0^{a/2} x\sin\frac{(2j+1)\pi x}{a}\,dx \\ &= \frac{(-1)^j a^3}{4(2j+1)\pi} - \frac{2a^2}{(2j+1)^2\pi^2}\int_0^{a/2}\cos\frac{(2j+1)\pi x}{a}\,dx \\ &= \frac{(-1)^j a^3}{4(2j+1)\pi} - \frac{2(-1)^j a^3}{(2j+1)^3\pi^3}.\end{aligned}$$

We conclude that

$$a_j = \frac{4}{a}\left[\frac{a^2}{4}\frac{(-1)^j a}{(2j+1)\pi} - \frac{(-1)^j a^3}{4(2j+1)\pi} + \frac{2(-1)^j a^3}{(2j+1)^3\pi^3}\right] = \frac{8(-1)^j a^2}{(2j+1)^3\pi^3}.$$

We have obtained a (Fourier cosine) series for the solution of (3):

$$u(x,y) = \frac{8a^2}{\pi^3}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^3}\,e^{-(2j+1)^2\pi^2 y/a^2}\cos\frac{(2j+1)\pi x}{a}.$$

The general appearance of the resulting awning is indicated in Figure 1. For

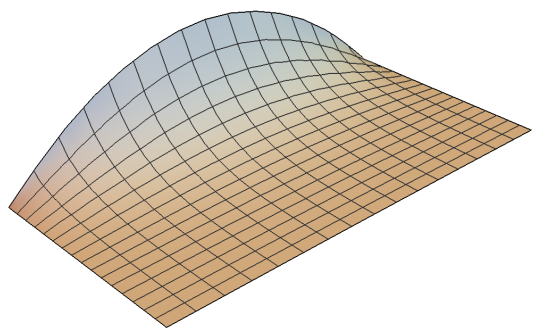

Figure 1: An awning determined by the heat equation.

this graphic, we have taken only three terms $j = 0, 1, 2$ in the series expansion. The approximations of $u(x, 0)$ for these first three approximations are indicated in Figure 2.

4 Heuristics and properties

Due to the fixed "temperatures" $u(\pm a/2, y) = 0$ at the endpoints and the absence of a forcing term in the PDE, we expect "energy dissipation" meaning that the awning should become flatter and flatter with increasing length b. Indeed, taking the "energy" used above

$$h(y) = \|u\|^2_{L^2(-a/2,a/2)} = \int_{-a/2}^{a/2}[u(x,y)]^2\,dx$$

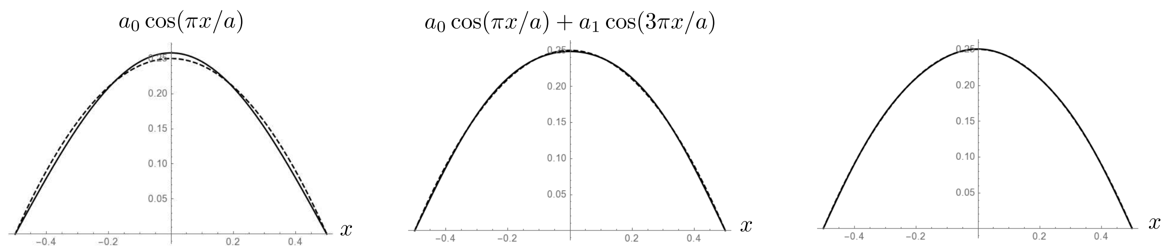

Figure 2: Initial cross-sections of Fourier approximations with one, two, and three terms.

as a measure of the total "heat energy" of the cross-section of the awning, we can calculate as before

$$h'(y) = 2\int_{-a/2}^{a/2} u(x,y)\,u_y(x,y)\,dx = 2\int_{-a/2}^{a/2} u(x,y)\,u_{xx}(x,y)\,dx = -2\int_{-a/2}^{a/2} [u_x(x,y)]^2\,dx.$$

This means $h(y)$ is a decreasing non-negative quantity, and certainly, on average, the magnitude/"energy" of the cross-section of the awning diminishes with length. Indeed, this behavior is clearly visible in Figure 1.

It is not entirely clear that the awning produced by this procedure slopes down away from the building, i.e., that $u_y(x,y) = u_{xx}(x,y) < 0$, at all points. This does seem to be what the Fourier series approximation indicates.

Similarly, there do not appear to be any "bowls" in these solutions.

It strikes me that it is a very natural question to ask for conditions under which a solution $u = u(x,t)$ of the heat equation is spatially concave. In particular, if one knows $u(x,0)$ satisfies $u_{xx}(x,0) < 0$, as we know here, then we would like to know $u_{xx}(x,t) < 0$ for $t > 0$. In our investigation of the behavior of the Green's function for this problem as a solution of the heat equation satisfying "$G(x,0) = \delta_\xi$," we observed that it is possible to have cooler areas become warmer as time moves forward, i.e., $u_t = u_{xx} > 0$, while $u > 0$ at all points. Our observations, however, did not contradict the following conjecture:

Conjecture 1 *If u satisfies*

$$\begin{cases} u_t = u_{xx} \ \text{on}\ U = (-a/2,a/2)\times(0,T) \\ u(x,0) \ \text{is even and concave:}\ u_{xx}(x,0) < 0, \\ u(\pm a/2,t) = 0, \end{cases} \tag{4}$$

then $u_{xx}(x,t) < 0$ for $0 \le t < T$.

Since it is easy to see that $u(-x,t)$ is also a solution, our uniqueness result gives immediately that such a solution (and our solution for the awning equation (3)) is spatially even. An even concave function clearly cannot have "bowls" in its graph. Thus, the problem here gives a nice practical example in which the conjecture above would be useful. I don't know, nor do I know how to prove, this result. Initially, one would not think it should be difficult, and maybe it is not. A cursory internet search for "spatial concavity of solutions of the heat equation" yields a paper by Andreucci and Ishige (Annali di Matematica Pura ed Applicata, (2013) **3**) which apparently treats a somewhat different (and perhaps basically easier) problem and gives a weaker result. Nevertheless, they do seem to be interested in this basic issue, and it strikes me as a very natural question to consider, so perhaps someone has done it. The references in the paper of Andreucci and Ishige look promising, but I didn't have time to follow up with further reading.

5 Final boundary condition

Finally we address briefly the desired/proposed condition $u(x,b) = u_1(x) = a^2/16 - x^2/4$.

5.1 Approximation determined by length

Taking the first three terms of the Fourier series expansion as an approximation of the desired awning shape, we may plot $u_1(x)$ as a function of x and y along with $u(x,y)$ to see there is a fairly well-defined length b for which $u(x,b)$ is close to $u_1(x)$. See Figure 3.

In fact, taking $f(b) = \|u(x,b) - u_1(x)\|_{L^2(-a/2,a/2)}$ as a measure of the disparity of u and u_1 at $y = b$, we find a unique value $b_* \approx 0.14046a$ for which this measurement of a error is a minimum. See Figure 4. It will be observed that this gives a relatively short awning.

5.2 Backwards heat evolution

An alternative approach for obtaining a precise fit $u(x,b) = u_1(x)$ at a given length $y = b$, is to consider again, the superposition

$$u(x,y) = \sum_{j=0}^{\infty} b_j e^{-(2j+1)^2\pi^2 y/a^2} \cos\frac{(2j+1)\pi x}{a}$$

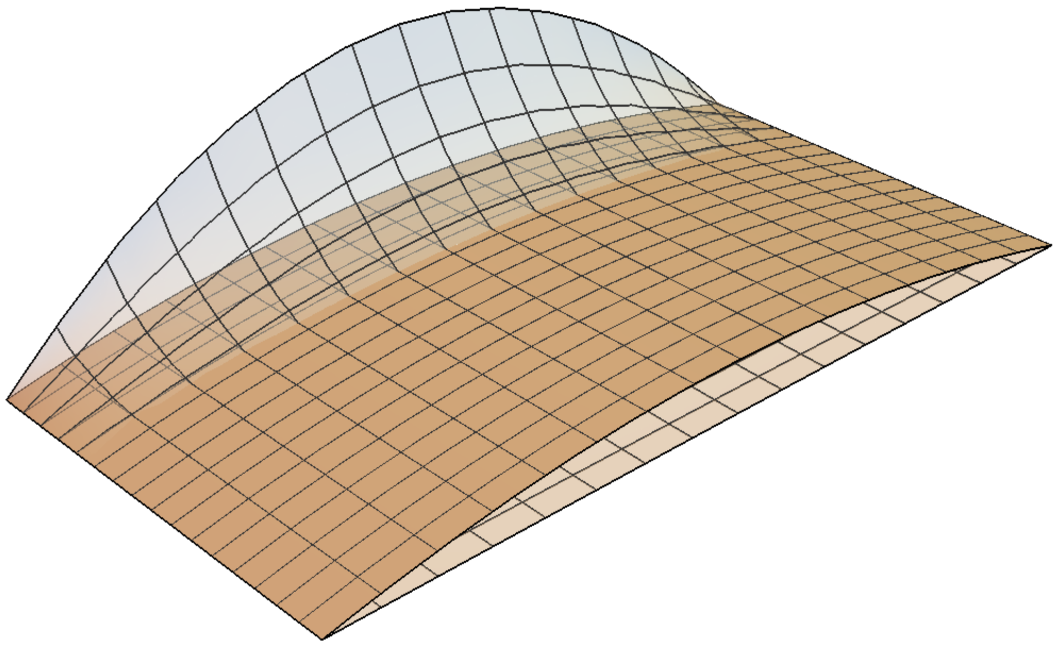

Figure 3: Comparison of $u(x, y)$ with $u_1(x)$

but determine the coefficients b_j so that $u(x, b) = u_1(x)$. Noting that $u(x, b) = u_1(x) = u_0(x)/4$. That is, we could take

$$b_j = \frac{a_j e^{(2j+1)^2\pi^2 b/a^2}}{4} = \frac{2(-1)^j a^2 e^{(2j+1)^2\pi^2 b/a^2}}{(2j+1)^3\pi^3}$$

and

$$u(x, y) = \frac{2}{a^2\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^3} e^{(2j+1)^2\pi^2(b-y)/a^2} \cos\frac{(2j+1)\pi x}{a}.$$

Assuming backwards uniqueness for solutions of the heat equation (which is also true) we can expect that a choice of b near b_* will likely lead to a starting profile close to the desired quadratic one. Unfortunately, it will be observed that the exponentials in this series grow very quickly with j for $b - y > 0$. Consequently, the series is apparently unstable in backwards time and does not provide a viable means of approximation.

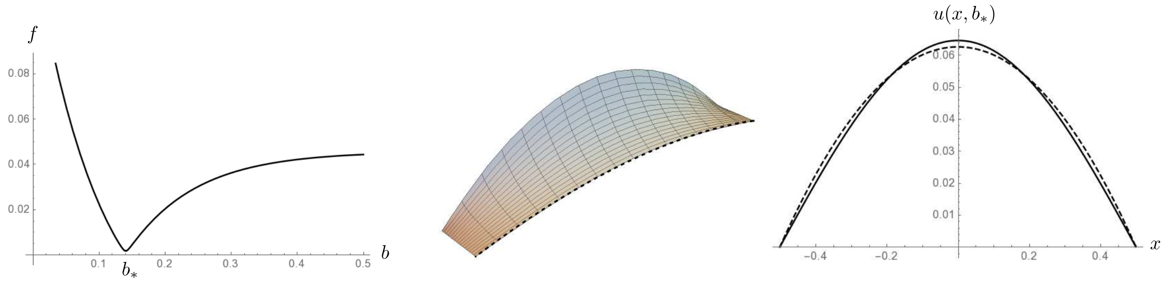

Figure 4: The awning $u(x, y)$ with $R = (-a/2, a/2) \times (0, b_*)$. The desired end values $u_1(x)$ are shown dashed.

5.3 Trigonometric boundary values

Finally, we observe that the desired starting and ending profiles u_0 and u_1 above satisfy the relation $u_1(x) = u_0(x)/4$ which is a homogeneity of sclaing relation between u_0 and u_1. The Fourier series expansion suggests this homogeneity is not possible with a superposition of solutions in general achieving $u(x, 0) = u_0(x)$. This is also in accord with what we expect from uniqueness. This precise homogeneity, $u(x, b) = u(x, 0)/4$ is possible and easy to achieve, however, if only one term in the Fourier series approximation is used (and the corresponding boundary/initial/final conditions are adopted. Precisely, we can take

$$u(x, y) = v_0(x, y) = a_0 \, e^{-\pi^2 y/a^2} \cos \frac{\pi x}{a} = \frac{8a^2}{\pi^3} \, e^{-\pi^2 y/a^2} \cos \frac{\pi x}{a}.$$

We see that $v_0(x, 0) \neq u_0(x)$, but the fit is reasonably close as indicated in the leftmost comparison of Figure 2. Furthermore, we can solve

$$v_0(x, b) = v_0(x, 0)/4, \qquad \text{that is} \qquad e^{-\pi^2 b/a^2} = \frac{1}{4}$$

for a specific value $b = b_* = a \ln 4/\pi^2 \approx 0.14046\pi a$ which is essentially the same length as the best approximation given above. The error/approximation at $y = b_*$ will now be essentially identical to that shown on the left in Figure 2 with all values simply scaled by $1/4$. The appearance of the trigonometric awning is reasonably similar to that with three terms as indicated in Figure 5. Furthermore, all desired properties such as $u_y(x, y) = u_{xx}(x, y) < 0$ either clearly hold or are easy to check. We note that while the homogeneity relation between starting and ending profiles could be achieved using any one term of the Fourier approximation, only the first term satisfies what may be assumed

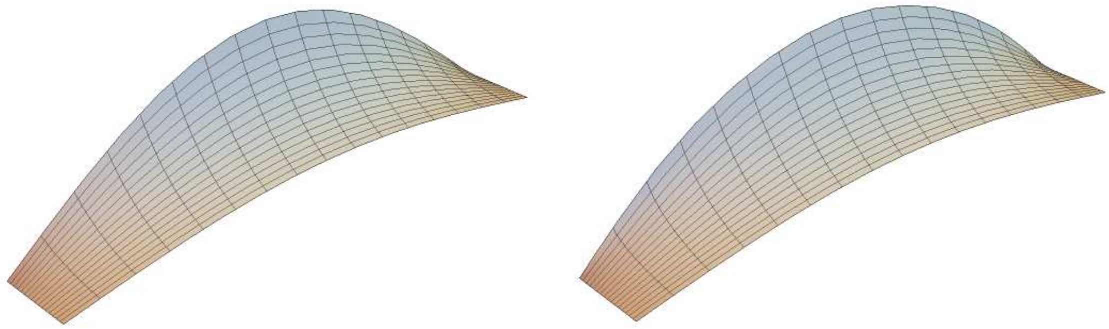

Figure 5: The awning $u(x,y) = v_0(x,y) = a_0\, e^{-\pi^2 y/a^2} \cos \pi x/a$ with $R = (-a/2, a/2) \times (0, b_*)$ (left) with the three term Fourier approximation for a solution matching the initial boundary values exactly (right).

desirable properties of an awning. For example, a scaled plot of the second term in the Fourier approximation indicates a trough hanging down in the middle of the awning against the wall as indicated in Figure 6.

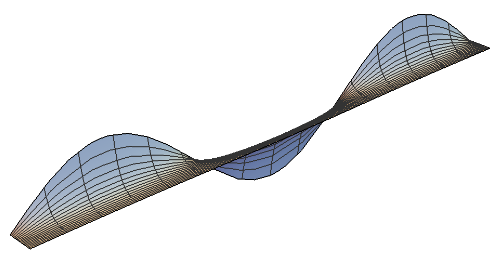

Figure 6: The second approximation term $v_1(x,y) = a_1\, e^{-9\pi^2 y/a^2} \cos 3\pi x/a$ (vertically scaled by a factor of 10 for visibility).