Assignment 10: Final Exam (draft)
Due Friday, April 25, 2025, 2:10 PM

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Problem 1 (convexity and continuity) For this problem, let a and b be real numbers with $a < b$.

(a) Formulate a definition of **convexity** for a function of one real variable

$$f : [a, b] \to \mathbb{R},$$

and recall the definition of continuity and the vector space $C^0[a, b]$.

(b) Denote by $\mathcal{C}[a, b]$ the collection of convex functions $f : [a, b] \to \mathbb{R}$. Is $\mathcal{C}[a, b]$ a vector space? Explain your answer clearly and in detail.

(c) Find an example of a function $f \in \mathcal{C}[a, b]\backslash C^0[a, b]$.

Problem 2 (differentiability) Recall From Problem 8 of Assignment 2 the defintion of differentiability for $f : \Omega \to \mathbb{R}$ where Ω is an open subset of $\mathbb{R}^n$. Recall also the definition of **directional differentiability**:

Definition 1 *Given* $\mathbf{x} \in \Omega$ *and* $\mathbf{u} \in \mathbb{S}^{n-1}$*, the function* f *is said to have a* **directional derivative** *at* $\mathbf{x}$ *in the direction* $\mathbf{u}$ *if the limit*

$$\lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{u}) - f(\mathbf{x})}{v}$$

exists. In this case we write

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{u}) - f(\mathbf{x})}{v}.$$

(a) Show that if a function $f : \Omega \to \mathbb{R}$ has a directional derivative in every direction $\mathbf{u} \in \mathbb{S}^{n-1}$ at a point $\mathbf{x} \in \Omega$, then f has partial derivatives

$$\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})$$

at $\mathbf{x}$.

(b) Assume a function $f : \Omega \to \mathbb{R}$ has a directional derivative in every direction $\mathbf{u} \in \mathbb{S}^{n-1}$ at every point $\mathbf{x} \in \Omega$. Determine if this implies the function f is differentiable (on Ω). If f is diffrentiable, prove it. If there is an example of a function f with this property which is not differentiable at some point $\mathbf{x}$, then give that example.

(c) Assume $f : \Omega \to \mathbb{R}$ has a directional derivative in every direction $\mathbf{u} \in \mathbb{S}^{n-1}$ at a point $\mathbf{x} \in \Omega$ and denote by $Df(\mathbf{x})$ the vector of partial derivatives at $\mathbf{x}$, i.e., the gradient of f at $\mathbf{x}$, as usual. Assume further that the forumula

$$Df_{\mathbf{u}}(\mathbf{x}) = Du(\mathbf{x}) \cdot \mathbf{u} = \sum_{j=1}^{n} u_j \frac{\partial f}{\partial x_j}(\mathbf{x})$$

for a directional derivative in terms of the gradient holds for every $\mathbf{u} \in \mathbb{S}^{n-1}$ and every $\mathbf{x} \in \Omega$. Show f is differentiable at every point in Ω.