Math 6702, Assignment 9

Weak Derivatives

1. Show that if $u \in W^1(-1,1)$, then $u \in C^0(-1,1)$.

Warning: To complete this problem requires, perhaps, a bit more detailed understanding of the nature and properties of measurable functions than I had intended. I included this problem because someone (of you students) asked about it—and rightly so. It's a very interesting question, namely:

Is it always true that a weakly differentiable function $u \in W^1(U)$ is continuous?

Basically for $U \subset \mathbb{R}^1$, the answer is "yes," but for $U \subset \mathbb{R}^n$ with $n \geq 2$, the answer is "no." I have attempted to take up the first situation in this homework assignment and the other situation in the next.

I think I can basically reduce this problem to two, rather believable, facts for you. So, for now, I suggest you just use these facts, and I will write up some more detailed notes at least outlining why these facts are true under some heading like "the fundamental theorem of calculus for measurable functions." See my notes on integration.

Fact 1: If $g \in L^1(a,b)$, then $v : (a,b) \to \mathbb{R}$ by

$$v(x) = \int_{(a,x)} g$$

has $v \in C^0(a,b)$.

Fact 2: If $w \in L^1(a,b)$ and

$$\int_{(a,b)} w\, \phi' = 0 \qquad \text{for every } \phi \in C_c^\infty(a,b),$$

then there exists a constant c such that $w(x) = c$ for almost every $x \in (a,b)$. That is, $\mu\{x \in (a,b) : w(x) \neq c\} = 0$ where μ is Lebesgue measure, or put another way, w is **essentially constant**.

I gave a proof of Fact 2 in my lecture. At least I reduced the assertion to an application of the fundamental lemma of the calculus of variations for L^1_{loc} functions. Incidentally, in studying up to (re)compose this problem, I've discovered that Fact 2 is also called *the second fundamental lemma of the calculus of variations*. You learn something new every day! All of these results rest, more or less on another, more basic, fact I mentioned briefly in my lecture, namely that almost every point in the domain of an L^1_{loc} function is a **Lebesgue point**. More precisely, if $u \in L^1_{loc}(U)$ with U an open subset of $\mathbb{R}^n$, then

$$\lim_{r \searrow 0} \frac{1}{\mu B_r(x_0)} \int_{B_r(x_0)} |u(x) - u(x_0)| = 0 \qquad \text{for almost every } x_0 \in U. \tag{1}$$

As always, when I say almost every $x_0 \in U$, I mean **except on a set of Lebesgue measure zero**. In this case,

$$\mu\{x_0 \in U : \text{ (1) does not hold}\} = 0.$$

This result is called the **Lebesgue differentiation theorem**.

Beyond Fact 1 and Fact 2 you need to know (or understand) the paradoxical fact that while the pointwise values of a "function" $u \in L^1(U)$ are **defined nowhere** (i.e., at no single point!), the pointwise values of such a function may still be, **and are**, uniquely defined almost everywhere. The trick to understanding this paradox, is to realize one is actually talking about not a single function but a certain equivalence class of functions. That is to say: The standard meaning of "$u \in L^1(U)$" is the following:

> There is a measurable function $u : U \to \mathbb{R}$ with
>
> $$\int_U |u| < \infty,$$
>
> **and** you might (actually) get **any function** $\tilde{u} : U \to \mathbb{R}$ which is measurable and satisfies
>
> $$\int_U |\tilde{u} - u| = 0.$$

Using the standard meaning, the **set** of all measurable functions $\tilde{u} : U \to \mathbb{R}$ with $\int_U |\tilde{u} - u| = 0$ is an **equivalence class of functions**. From the measure theoretic point of view, the functions u and $\tilde{u}$ are indistinguishable; they are "the same function." This means, that one should be careful when one says something about $u \in L^1(U)$ because anything you say, generally speaking, should be true for every function in the equivalence class determined by u.

In particular, if one takes any one point $x_0 \in U$ and any particular value $c \in \mathbb{R}$, then the function $\tilde{u} : U \to \mathbb{R}$ by

$$\tilde{u}(x) = \begin{cases} u(x), & x \neq x_0 \\ c, & x = x_0, \end{cases}$$

satisfies $\tilde{u} = u \in L^1(U)$, so the pointwise value of u at x_0 clearly has no meaning. Now the statement of the problem given here, concluding $u \in C^0(-1, 1)$, obviously departs from the standard meaning. When this kind of thing is done, first of all, one is said to be considering the "fine properties of measurable functions," and things can be expected to get a little delicate. In particular, what we are asserting here is the following:

> There exists **some specific** $u_1 : (-1, 1) \to \mathbb{R}$ for which the following hold
>
> **(i)** $u_1 \in C^0(-1, 1)$ and
>
> **(ii)** $u_1 = u \in L^1(-1, 1)$.

Technically, condition (ii) should be $u_1 = u \in W^1(-1, 1)$ which (technically) means $u_1 = u \in L^1(K)$ for every compact set K with $K \subset (-1, 1)$. Remember that $u \in W^1(-1, 1)$ technically means $u \in L^1_{loc}(-1, 1)$ with a weak derivative also in $L^1_{loc}(-1, 1)$. If you want, you can just assume u and the weak derivative of u are in $L^1(-1, 1)$, then condition (ii) holds as it is. More could be said, but let me leave it for the notes.

Hint(s): Remember that if $u \in W^1(-1,1)$, then there is some $g \in L^1_{loc}(-1,1)$ with

$$-\int u\phi' = \int g\phi \qquad \text{for every } \phi \in C_c^\infty(-1,1). \tag{2}$$

Let $a, b \in \mathbb{R}$ with $-1 < a < b < 1$. Then

$$u = u\Big|_{(a,b)} \in L^1(a,b) \qquad \text{and} \qquad g = g\Big|_{(a,b)} \in L^1(a,b)$$

with g a weak derivative of u in $W^{1,1}(a,b)$.

Consider $v : (a,b) \to \mathbb{R}$ by

$$v(x) = \int_{(a,x)} g.$$

Use Fubini's theorem to show

$$\int_{(a,b)} (v-u)\,\phi' = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

Hint for the hint:

$$\int_{x\in(a,b)} \int_{t\in(a,x)} g(t)\,\phi'(x) = \int_T g(t)\,\phi'(x)$$

where $T = \{(x,t) : a < x < b,\ a < x < x\}$ is a triangular domain in the plane. Be careful: You can write

$$\int_{x\in(a,b)} \int_{t\in(a,x)} g(t)\,\phi'(x) = \int_{x\in(a,b)} \phi'(x) \int_{t\in(a,x)} g(t)$$

without Fubini's theorem, but you can't evaluate the integral due to the x dependence in the inner t integral.

Solution: By Fubini's theorem

$$\begin{aligned}\int_{(a,b)} v\,\phi' &= \int_{x\in(a,b)} \int_{t\in(a,x)} g(t)\,\phi'(x) \\ &= \int_{t\in(a,b)} \int_{x\in(t,b)} g(t)\,\phi'(x) \\ &= \int_{t\in(a,b)} g(t) \int_{x\in(t,b)} \phi'(x) \\ &= -\int_{t\in(a,b)} g(t)\,\phi(t).\end{aligned}$$

Therefore,

$$\int_{(a,b)} (v-u)\,\phi' = \int_{(a,b)} v\,\phi' - \int_{(a,b)} u\,\phi' = -\int_{(a,b)} g\,\phi + \int_{(a,b)} g\,\phi = 0.$$

By the second fundamental lemma of the calculus of variations, there is a constant c such that $u = v - c \in L^1(a,b)$. Now, if we have (or assume) $a = -1$ and $b = 1$, then we are essentially done. If not, we have a little more work to do.

Consider the compactly contained subintervals $I_j = [a_j, b_j] = [-1 + 1/j, 1 - 1/j]$ for $j = 2, 3, 4, \ldots$. On each such subinterval we obtain a continuous function $w_j = v_j - c_j \in C^0(I_j)$ and a constant c_j with $u = w_j \in L^1(a_j, b_j)$. In view of the strict nesting

$$(a_{j-1}, b_{j-1}) \subset (a_j, b_j) \subset (a_{j+1}, b_{j+1})$$

we have $w_j, w_{j+1} \in C^0[a_{j-1}, b_{j-1}]$ with $w_j = w_{j+1} = u \in L^1(a_{j-1}, b_{j-1})$. In particular, for the continuous functions w_j and w_{j+1}, we have

$$\int_{(a_{j-1}, b_{j-1})} |w_{j+1} - w_j| = 0.$$

It follows that

$$w_{j+1}(x) \equiv w_j(x) \qquad \text{for every } x \in (a_{j-1}, b_{j-1}),$$

and

$$w(x) = \lim_{j \to \infty} w_j(x)$$

is well-defined for every $x \in (-1,1)$. Furthermore, the function $w \in C^0(-1,1)$ with

$$\int_{I_j} |w - u| = 0 \qquad \text{for every } j = 2, 3, 4, \ldots.$$

Thus, $\int_K |w - u| = 0$ for every compact set $K \subset (-1,1)$, and this is what it means to have $w = u \in L^1_{loc}(-1,1)$. Finally, it is also clear that g is a weak derivative for w since

$$\left| \int_{(-1,1)} w\phi' - \int_{(-1,1)} u\phi' \right| \le \|\phi'\|_{C^0} \int_{\operatorname{supp} \phi'} |w - u| = 0.$$

This is what it means to have $w = u \in W^1(-1,1)$. $\square$

Fundamental Solution

2. Find all axially symmetric solutions $u(x, y) = u_0(x^2 + y^2)$ of $\Delta u = 0$ on $\mathbb{R}^2 \backslash \{(0,0)\}$. Note that u is a function of two variables and u_0 is a function of one variable; the form I have given you is what it means to be axially symmetric.

3. Let $\Phi = \Phi(x, y)$ satisfy $\Phi(x, y) = \Phi_0(x^2+y^2)$ and "$-\Delta\Phi = \delta_0$" on $\mathbb{R}^2$ in the distributional sense where δ_0 represents a point impulse at the origin.
   (a) Write down what this means as a condition involving test functions.
   (b) Let $U \subset \mathbb{R}^2$ be a bounded open domain, and consider $\xi \in U$ fixed. Setting $\mathbf{x} = (x, y)$, consider $v(\mathbf{x}, \xi) = \Phi_0(|\mathbf{x} - \xi|^2)$. Show that

$$v(\mathbf{x}, \xi) = v(\xi, \mathbf{x}) \qquad \text{and} \qquad -\int_{\mathbf{x}\in U} v(\mathbf{x}, \xi)\Delta\phi(\mathbf{x}) = \phi(\xi) \quad \text{for every } \phi \in C_c^\infty(U).$$

Solution:

(a)

$$-\int_{\mathbb{R}^2} \Phi\Delta\phi = \phi(0) \qquad \text{for every } \phi \in C_c^\infty(\mathbb{R}^2). \tag{3}$$

(b) For the first assertion we have $v(\mathbf{x}, \xi) = \Phi_0(|\mathbf{x} - \xi|^2) = \Phi_0(|\xi - \mathbf{x}|^2) = v(\xi, \mathbf{x})$.

For the second assertion, let $\phi \in C_c^\infty(U)$. Then note that $\psi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\psi(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}), & \mathbf{x} \in U \\ 0, & \mathbf{x} \notin U \end{cases}$$

satisfies $\psi \in C_c^\infty(\mathbb{R}^2)$. Therefore, changing variables with $\eta = \mathbf{x} - \xi$,

$$\begin{aligned} -\int_{\mathbf{x}\in U} v(\mathbf{x}, \xi)\Delta\phi(\mathbf{x}) &= -\int_{\mathbf{x}\in\mathbb{R}^2} \Phi_0(|\mathbf{x} - \xi|^2)\Delta\psi(\mathbf{x}) \\ &= -\int_{\eta\in\mathbb{R}^2} \Phi_0(|\eta|^2)\Delta\psi(\eta + \xi). \end{aligned}$$

Note then that $\mu : \mathbb{R}^2 \to \mathbb{R}$ by $\mu(\mathbf{x}) = \psi(\mathbf{x} + \xi)$ has $\mu \in C_c^\infty(\mathbb{R}^2)$, and we can write

$$\begin{aligned} -\int_{\mathbf{x}\in U} v(\mathbf{x}, \xi)\Delta\phi(\mathbf{x}) &= -\int_{\eta\in\mathbb{R}^2} \Phi_0(|\eta|^2)\Delta\psi(\eta + \xi) \\ &= -\int_{\mathbf{x}\in\mathbb{R}^2} \Phi(\mathbf{x})\Delta\mu(\mathbf{x}) \\ &= \mu(0) \\ &= \psi(\xi) \\ &= \phi(\xi). \end{aligned}$$

Note that we have used (3) with ϕ replaced by μ.

§4.12 Differentiating Integrals

4. (4.12.9,13) Compute the derivatives using the chain rule and without evaluating the integral first.

$$\frac{d}{dx}\int_0^x \sin(xt)\,dt.$$

$$\frac{d}{dx}\int_{1/x}^{2/x} \frac{\sin(xt)}{t}\,dt.$$

(You can check your answer to the first one by evaluating the integral first and then differentiating.)

§5.5 Surface Integrals

5. (5.4.3) Find the area of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 \leq 9$.

6. Find the area of the helicoid parameterized by

$$X(u, v) = (u\cos v, u\sin v, v) \qquad \text{on } [0, 1] \times [0, n\pi].$$