Assignment 9: Integration Due Wednesday, March 29, 2023

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Problem 1 (Hooke's constant) This is the third in a series of problems on Hooke's constant.

Recall Problem 2 of Assignment 8. Say you are given a spring of (equilibrium) length a modeled using the horizontal model of Problem 2 (assuming no gravity and no spring mass). Recall the model position of the fixed end was $x = -a$ and the equilibrium for the extended/compressed end was at $x = 0$. Assume this spring has Hooke's constant k, so that the force corresponding to the position x is modeled by

$$F = -kx.$$

(a) If you cut this spring in half to obtain two springs of length $a/2$ made of precisely the same "material," i.e., the same coils and elastic properties, can you model the uniform extensions and compressions of one of these half springs with the same spring constant?

(b) What does your answer to part **(a)** tell you about Hooke's constant. Hint: Hooke's constant may be used to model the force exerted by a compressed or extended spring, but Hooke's constant is **not**. . . (what?).

(c) (important) Recall my solution of Problem 1 of Assignment 1. Model the uniform compression and extension of the spring of length a using the extension $u : [0, a] \to [0, L]$. Notice the force exerted is now modeled by

$$F = -k(L - a). \tag{1}$$

Write down a/the formula for $u(x)$ and reexpress the model force (1) in terms of u. Hint: Use the derivative of u.

(d) Based on your force formula from part **(c)**, postulate the existence of a different constant (not the Hooke's constant) associated with a certain kind of **spring material** (rather than a particular spring of a given length) called the **elastic modulus**. Hint(s): Think about units/physical dimensions. What are the physical dimensions $[k]$ of the Hooke's constant? The elastic modulus for spring material should have units/physical dimenions of force.

Remark(s): Parts **(c)** and **(d)** are crucial. It is important (I think) to understand that Hooke's law and its attendant constant are restricted to the context of uniform deformations of a spring. If you want to model non-uniform deformations, like that displayed in the hanging slinky, you need something else. If you think carefully about the suggestion in part **(c)**, it will tell you how to do part **(d)**. This is a relatively hard task, but it is one that is absolutely crucial to good mathematical modeling of physical systems. You need to understand the driving mechanisms behind your system and the applicable "laws" to use to model it. In this case your answer to part **(d)** is what is called a **constitutive relation**. Once you have formulated something for part **(d)**, you need to go back and ask yourself: Is this reasonable? Does it behave in a manner consistent with the modeling of the physical system/slinky? Another example of a constitutive relation is Fourier's law of heat conduction. They are generally considered kind of "magic," so if you were able to find the constitutive relation in part **(d)** above, congratulate yourself.

Problem 2 (slinky mass considerations) Problem 1 above should take you a long way toward understanding the elastic aspect of the slinky. Another important aspect is the modeling of the mass. Let us assume an initial linear density ρ_0 (constant) for the equilibrium slinky so that the total mass of the slinky is $M = \rho_0 L_0$.

(a) Consider a/the vertical model for a slinky based on an extension $u : [0, L_0] \to [0, L]$, so that $y : [0, L_0] \to \mathbb{R}$ is given by $y(x) = -u(x)$. Approximate the (non-constant linear) mass density $\rho : [0, L_0] \to \mathbb{R}$ of the extension between two points $u(x_1)$ and $u(x_2)$ in the extension.

(b) Find a mass density $\rho : [0, L] \to \mathbb{R}$ on the extension so that

$$M = \int_0^L \rho(\xi)\, d\xi.$$

(c) Express the gravitational potential energy associated with a hanging slinky modeled in the vertical model by $y = -u$. Hint(s): Take a reference level $y = y_0$ and find the work (against gravity) to move the mass element $\rho_0/u'(-y_j^*)(y_{j+1} - y_j)$ from height $y = y_0$ to height $y = y_j^*$. Remember work is force times distance. Form a Riemann sum.

Problem 3 (slinky) Determine and plot a set of height measurment data for the hanging slinky.

Problem 4 (slinky: first model) If you understood and were able to solve Problem 1 above, especially parts **(c)** and **(d)**, then you should be in a position to give an elementary model for the hanging slinky. We should find a better model later.

(a) Pose an intial value problem for a first order ODE to model the hanging slinky with modeling function the extension $u : [0, L_0] \to [0, L]$. Hint: The tension force in the slinky at each position $y = -u(x)$ should balance the force mg associated with the mass of the slinky hanging hanging below that point.

(b) Determine any constants from the model using the measured data.

(c) Compare your model prediction with the measured data by plotting each on the same axes.

Problem 5 (continuous weak solutions of the wave equation) There are various kinds of weak solutions for various kinds of PDE. Most decent notions of weak solution have two properties:

(i) Classical solutions should also be weak solutions.

(ii) Any weak solution is unique.

In the context of differentiability classes it is often nice to make sense of and have a third property:

(iii) If a weak solution happens to be (appropriately) classically differentiable, then it is a classical solution.

Sometimes, for example with linear elliptic PDE like Laplace's equation, weak solutions turn out to (always) be classical solutions. This might seem to imply the notion of weak solutions in this case is irrelevant, but it does not. The reason is that it is rather difficult to show classical solutions of linear elliptic PDE exist. On the other hand, it is relatively easy to show weak solutions of linear elliptic PDE exist. It is not easy to show weak solutions of linear elliptic PDE are classical solutions, but it turns out to be possible, and this is (basically) the only way known to do it, or at least it is the standard way. But this problem is about hyperbolic equations, and the wave equation in particular. Here is a definition:

Definition 1 (weak solutions of the wave equation, cf. Angenant) A function $w \in C^0(\mathbb{R} \times [0, \infty))$ is a **continuous weak solution** of the initial value problem

$$\begin{cases} u_{tt} = u_{xx}, & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_x(x,0) = g(x), & x \in \mathbb{R} \end{cases} \tag{2}$$

for the 1-D wave equation if

$$\int_{\mathbb{R}\times(0,\infty)} w \,\Box\phi = \int_{-\infty}^{\infty} g(x)\phi(x,0)\,dx - \int_{-\infty}^{\infty} f(x)\phi_t(x,0)\,dx$$

for every $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$.

Notice the set $C_c^\infty(\mathbb{R} \times [0, \infty))$ is a little different from the set(s) $C_c^\infty(U)$ considered before where U is an open subset of $\mathbb{R}^n$. In this case, the support of ϕ is required to be a compact set and to be a subset of $\mathbb{R} \times [0, \infty)$. This allows points in the support of ϕ to be (in a compact subset) on the $t = 0$ boundary.

(a) Show that if $u \in C^2(\mathbb{R} \times [0, \infty))$ is a classical solution of (2), i.e., the conditions of (2) simply hold as stated, then u is a weak solution. Hint(s): Multiply the PDE by by a test function, integrate over all of $\mathbb{R} \times (0, \infty)$, express the resulting area integral as appropriate iterated integrals, and integrate by parts.

(b) Show that if $w \in C^2(\mathbb{R} \times [0, \infty))$ is a weak solution of (2), then w is a classical solution.

Problem 6 (Problem 5.2.1 in Boas) Calculate $\int_A f$ where $A = (0, 1) \times (2, 4)$ is a rectangle and $f : A \to \mathbb{R}$ is given by $f(x, y) = 3x$.

Problem 7 (Problem 5.2.3 in Boas) Calculate $\int_A f$ where $A = \{(x, y) : 0 < y < 2,\ 2y < x < 4$ is a triangle and $f : A \to \mathbb{R}$ given by $f(x, y) \equiv 1$ is constant.

Problem 8 (Problem 5.2.6 in Boas) Calculate $\int_A f$ where $A = \{(x, y) : 1 < y < 2,\ \sqrt{y} < x < y^2$ and $f : A \to \mathbb{R}$ given by $f(x, y) = x$.

Problem 9 (Problem 5.2.14 in Boas) Calculate $\int_A f$ where A is the region bounded by the curves

$$\{(x, 1/x) : x \in \mathbb{R}\backslash\{0\}\}, \qquad \{(x, 1/x^2) : x \in \mathbb{R}\backslash\{0\}\}, \qquad \text{and} \qquad \{(\ln 4, y) : y \in \mathbb{R}\},$$

and $f : A \to \mathbb{R}$ given by $f(x, y) = x^2 e^{x^2 y}$.

Problem 10 (Problem 5.2.30 in Boas) Calculate the iterated integral

$$\int_{x\in(0,2)} \left(\int_{y\in(x,2)} e^{-y^2/2} \right)$$

by expressing the value first as an integral over a region in the plane and re-expressing the value as an iterated integral with the dependence of integration reversed.