Math 6702, Assignment 8

Laplace's Equation on a Rectangle

1. (Exam 1 Problem 7) Consider again the boundary value problem for Laplace's equation on the rectangle $U = [0, L] \times [0, M]$ where L and M are positive numbers.

$$\begin{cases} \Delta u = 0, \\ u(x, 0) = 0,\ u(L, y) = 0,\ u(x, M) = x(x - L),\ u(0, y) = 0 \end{cases} \tag{1}$$

   (a) Find a function $g \in C^\infty([0, L] \times [0, M])$ such that

$$g\Big|_{\partial U \setminus \{y=M\}} \equiv 0 \qquad \text{and} \qquad g(x, M) = g_3(x) = x(x - L).$$

   Hint: Take a convex combination of $g_1 \equiv 0$ and g_3.

   (b) Let $w = u - g$ and write down the boundary value problem for Poisson's equation satisfied by w.

   (c) Consider the Fourier basis

$$\{\phi_{jk}\}_{j,k=1}^\infty \qquad \text{with} \qquad \phi_{jk}(x, y) = \sin \frac{j\pi x}{L} \sin \frac{k\pi y}{M}.$$

   Expand $-\Delta g$ in a Fourier series

$$-\Delta g = \sum_{j,k=1}^\infty a_{jk}\phi_{jk}.$$

   (d) Let w_{jk} solve

$$\begin{cases} \Delta w = \phi_{jk}, \\ w\Big|_{\partial U} \equiv 0. \end{cases} \tag{2}$$

   Hint: Compute $\Delta \phi_{jk}$.

   (e) Take the specific values $L = 1$ and $M = 0.5$ and plot enough terms of

$$u(x, y) = w(x, y) + g(x, y) \qquad \text{where} \qquad w = \sum_{j,k=1}^\infty a_{jk} w_{jk}$$

   to convince yourself (and me) that you have obtained a series solution for the problem. (Postscript/Note: The plots might look a little better with $L = 1$ and $M = 3$. Or you could do $L = 2$ and $M = 5$ for example, but these kinds of aspect ratios may be easier for the visualization.)

2. Let $u \in W^1(U)$ have weak (partial) derivatives g_j for $j = 1, 2, \ldots, n$. Assume U_0 is an open subset of U on which u has a weak (or classical) derivative $D_j u$. Show $D_j u(x) = g_j(x)$ for $x \in U_0$. Hint: Use the fundamental lemma of the calculus of variations.

Solution: Let $\phi \in C_c^\infty(U_0)$. Then we may extend ϕ to $\bar{\phi} : U \to \mathbb{R}$ by setting

$$\bar{\phi}(x) = \begin{cases} \phi(x), & x \in U_0 \\ 0, & x \in U \backslash U_0. \end{cases}$$

Note that $\bar{\phi} \in C_c^\infty(U)$. By the definition of weak derivative

$$\int_U g_j \bar{\phi} = -\int_U u \bar{\phi}'.$$

Since $\phi \in C_c^\infty(U_0)$, this implies

$$\int_{U_0} g_j \phi = -\int_{U_0} u \phi'. \tag{3}$$

Also, since $D_j u$ is a derivative on U_0, we know

$$\int_{U_0} D_j u \phi = -\int_{U_0} u \phi'. \tag{4}$$

We either get (4) by definition if $D_j u$ is a weak derivative or by integration by parts if $D_j u$ is a classical derivative on U_0. Alternatively, we can quote the fact that a classical derivative is a weak derivative. In any case, subtracting (3) from (4) we have

$$\int_{U_0} [D_j u - g_j] \phi = 0 \qquad \text{for all } \phi \in C_c^\infty(U_0).$$

By the fundamental theorem of the calculus of variations, we conclude $D_j u = g_j \in L^1_{loc}(U_0)$.

Actually, the statement should be corrected slightly to read: $D_j u(x) = g_j(x)$ for almost every $x \in U_0$ or $D_j u = g_j \in L^1_{loc}(U_0)$ which amounts to the same thing. If g_j may be assumed continuous and $D_j u$ may be assumed continuous, of course, then you get pointwise equality at all points $x \in U_0$ by the simple version of the fundamental lemma.

3. Consider the modified tent function $T \in \text{Lip}[0, L]$ given by

$$T(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(L-x)/(L-a) + \epsilon, & a \le x \le L \end{cases}$$

where $\epsilon > 0$. Show $T \notin W^1(0, L)$. Hints: Assume T has a weak derivative g and get a contradiction. Use the previous problem.

§4.11 Change of Variables

4. (4.11.1) Consdider the second order PDE

$$\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} = 0$$

for $u = u(x, y)$ defined on a domain U in the plane $\mathbb{R}^2$.

   (a) Use the change of variables

   $$\begin{cases} s = y + 2x \\ t = y + 3x \end{cases}$$

   to define a function $w(s, t) = u(x(s, t), y(s, t))$.

   (b) Assume $u \in C^2(U)$ satisfies the PDE above, and find a PDE satisfied by w.

   (c) Solve the PDE satisfied by w.

   (d) Solve the original second order PDE for u.

§5.4 Change of Variables in Integrals

5. (5.4.1) Let $B_a(0) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$ model a physical disk of constant density δ. Use polar coordinates to find the following:

   (a) The centroid of the first quadrant of the disk.

   (b) The moment of inertia of the disk about the diameter.

6. (5.4.2) Consider the disk $\{(x, y) \in \mathbb{R}^2 : (x - a)^2 + y^2 < a^2\}$.

   (a) Find the equation of the boundary of this disk in polar coordinates.

   (b) Use polar coordinates to compute the model mass of this disk if the density is modeled by $\delta(x, y) = \sqrt{x^2 + y^2}$.

7. (5.4.20) Use the change of variables

$$\begin{cases} x = (r - s)/2 \\ y = (r + s)/2 \end{cases}$$

to evaluate the iterated integral

$$\int_0^{1/2} \int_x^{1-x} \left(\frac{x-y}{x+y}\right)^2 dydx.$$

Hints: Sketch the region of integration and the new region of integration in the r, s-plane.