Assignment 8: Partial Differential Equations (Laplace's PDE) Due Wednesday, March 15, 2023

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Problem 1 (A problem in geometric ODEs) This is the fifth and final problem in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Remember Problems 2 and 3 of Assignment 7. The basic point of this problem is for you to realize that there is no reason the curve we have been considering as a graph—the curve with signed curvature equal to the signed arclength from the center—should be restricted to being a graph.

(a) Assume a curve $\gamma : (-\ell, \ell) \to \mathbb{R}^2$ is parameterized by arclength s, so that $\|\dot{\gamma}\| = 1$. Show the signed curvature defined in my notes as

$$\dot{\gamma}^{\perp} \cdot \ddot{\gamma}$$

agrees with the signed curvature of the graph defined previously as

$$\frac{f''}{(1 + f'^2)^{3/2}}.$$

Hint: Start with a parameterization of the graph $\alpha(x) = (x, f(x))$ and then reparameterize by arclength.

(b) Assume there is a function $\theta \in C^1(-\ell, \ell)$ defined along the (domain of the) parameterized curve of part **(a)** such that

$$\begin{cases} \cos\theta = \dot{\gamma}_1 \\ \sin\theta = \dot{\gamma}_2 \\ \theta(0) = 0. \end{cases}$$

Show the signed curvature of the parameterized curve γ is given by $\dot{\theta}$.

(c) Find a first order system of ODEs determining a curve $\gamma : (-\ell, \ell) \to \mathbb{R}^2$ parameterized by arclength for which the signed curvature is equal to the signed arclength s and $\gamma(0) = (0, 0)$ and $\dot{\gamma}(0) = (1, 0)$.

(d) Plot a numerical approximation of your solution and find, for example, a numerical approximation of $a = \max_{s \in \mathbb{R}} \gamma_1(s)$.

Problem 2 (Hooke's constant—warming up to modeling the slinky) Let me suggest to you that the modeling of a spring force using Hooke's constant only applies to homogeneous extensions of a spring and is of limited use with regard to the slinky problem. Some of your answers to problems 1 and 10 of Assignment 1 suggest you are inclined to think otherwise, and I'm certainly willing to try to argue with you about that.

In any case, this is the first in a series of problems concerning how I think about Hooke's constant (along with some aspects that may come to you as "discoveries").

I'm going to model the homogeneous deformations (compressions and expansions) of a physical spring having one fixed end and one moveable end. I introduce a model variable x corresponding to the measured position of the **moveable end** which may be considered a function of time or may be considered independent. In particular, I assume the moveable end(point) of the spring corresponds to a position $x \in \mathbb{R}$, so that at equilibrium (when there is no tension or compression force in the spring nor at the end) the end is located at $x = 0$. See Figure 1 (left). More generally, I assume moving the end of the spring to a measured location corresponding to a position x results in a force (tension or compression) both at the end and uniformly along the length of the spring modeled by the **force function**

$$f = -kx \tag{1}$$

where $k > 0$ is a constant (a Hooke's constant).

(a) Assume the fixed end of the spring modeled above corresponds to a position $x = -a$ for some $a > 0$ and the intermediate points of the spring correspond to the points on the interval $(-a, x)$. Taking numerical values for the positive constants m, a, and k, Use Newton's second law to model the motion of a mass corresponding to the measured value m released from rest at a position $x = a/2$.

(b) What is your expectation for comparison of the model considered in part **(a)** to an actual physical spring if the initial extension were instead $x = 2a$?

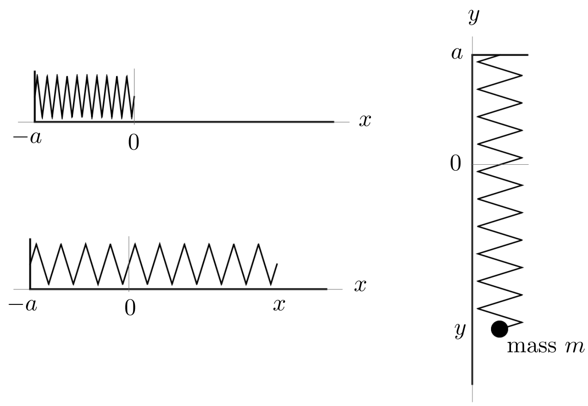

Figure 1: Modeling massless springs: Horizontal model (left). Vertical model with a mass (right).

(c) Let us now take the same spring and hang the mass corresponding to measurement m from it with the fixed end corresponding to a new coordinate $y = a$. Assuming the mass of the spring itself is negligible, find the model location corresponding to the mass in equilibrium, and find the force rule corresponding to (1) in the new y-coordinate. (Hint: I'm assuming $g = -9.8$ m/sec^2 or some equivalent will play a role here.)

(d) Compute the (potential) energy modeled by the system assumed to be stationary, i.e., having no kinetic energy, with

(i) position x in the horizontal model.

(ii) position y in the vertical model.

Problem 3 (Hooke's constant—warming up to modeling the slinky (part 2)) Find an interesting relationship between the equilibrium you found in the horizontal case of part **(c)** of Problem 2 above and the energy you found in part **(ii)** of part **(d)** of the same problem.

Problem 4 (wave equation; Boas 13.1.2(b)) Let u satisfy the wave equation in $U \times (0, \infty)$ where $U = \mathbb{R}^3 \backslash \{(0,0,0)\}$.

(a) Consider $v : (0, \infty) \times (0, \pi) \times \mathbb{R} \times (0, \infty)$ by

$$v(r, \phi, \theta, t) = u(r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi, t).$$

Find a PDE satisfied by v. Hint: Check out (7.10) in Boas. This PDE may be called the wave equation in spherical coordinates.

(b) Show any function of the form $v(r, \phi, \theta, t) = f(r - ct)/r$ where $f \in C^2(\mathbb{R})$ and $c \in \mathbb{R}$ satisfies the PDE you found in part **(a)** above.

Problem 5 (heat flow) If U is an open subset of $\mathbb{R}^n$, a **vector field** on U is a function $\mathbf{v} : U \to \mathbb{R}^n$. We may also consider a time dependent vector field $\mathbf{v} : U \times [0, T) \to \mathbb{R}^n$ for some $T > 0$.

(a) (velocity field; physical dimensions; mass flow) A vector field $\mathbf{v} : U \to \mathbb{R}^n$ may be used to model a velocity field, e.g., when $n = 3$ and one models the time independent motion of a liquid moving within a region modeled by U. We denote the **physical dimensions** of a model quantity by square brackets so that

$$[\mathbf{v}] = \frac{L}{T} = \frac{\text{length}}{\text{time}}$$

in terms of **fundamental dimensions** which are often taken to include mass, length, and time. Note that physical dimensions as we are using them here are distinct from **units**. For example, a unit of the physical dimension time is the second.

If we introduce a spatially dependent mass density function $\rho : U \to \mathbb{R}$ to model a moving liquid with velocity $\mathbf{v}$, what are the natural physical dimensions of ρ?

$$[\rho] =$$

(b) Given $\mathbf{v} : U \times [0, T) \to \mathbb{R}^3$ with $U \subset \mathbb{R}^3$ modeling a time dependent liquid velocity and $\rho : U \times [0, T) \to \mathbb{R}$ modeling the time dependent mass density of the same liquid, give a physical interpretation of

$$\epsilon^2 \rho(\mathbf{x}_0, t_0) \mathbf{v}(\mathbf{x}_0, t_0) \cdot \mathbf{e}_1 \tag{2}$$

where ϵ is a (small) positive number with $[\epsilon] = L$, $\mathbf{x}_0 \in U$, $t_0 \in (0, T)$ and $\mathbf{e}_1 = (1, 0, 0)$ is the first standard unit basis vector as usual. Hint: What are the physical dimensions of $\rho \mathbf{v}$?

(c) The field $\rho\mathbf{v}$ in the previous part of this problem is called a **mass flux field**. Repeat part **(b)** above when $U \subset \mathbb{R}^n$. Hint: You'll need to modify the form of the mass flux field in (2).

(d) When modeling heat flow one usually postulates a model **thermal energy flux field** $\vec{\phi} : U \times [0,T) \to \mathbb{R}^n$ directly with

$$[\vec{\phi}] = \frac{[\text{energy}]}{L^{n-1} \cdot T}.$$

In addition, one can introduce a model quantity $u : U \times [0,T) \to \mathbb{R}$ taken to have fundamental units

$$[u] = [\text{temperature}] = Temp.$$

Fourier's law of heat conduction states the following:

> *The thermal energy flux is (instantaneously and pointwise) proportional to, and in the opposite direction of, the spatial gradient of the temperature.*

(i) Express [energy] in terms of fundamental units. Hint: $[\text{energy}] = [\text{force}] \cdot L$ and $[\text{force}] = M \cdot [\text{acceleration}]$.

(ii) Express Fourier's law of heat conduction as a model relation among (the) functions ($\vec{\phi}$ and u) with a constant of proportionality $K > 0$ called the **thermal conductivity**.

(iii) Find the physical units of K.

Problem 6 (a solution of Laplace's equation) Show $u(x,y) = x^2 - y^2$ satisfies

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Problem 7 (solutions of Laplace's equation) You may recall that a function $f : \Omega \to \mathbb{C}$ where Ω is an open subset of the complex plane $\mathbb{C}$ is **differentiable** if for each $z \in \Omega$ the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists as a complex number $f'(z) \in \mathbb{C}$. Whether you remember this or not, it is true that many of the functions (of one variable) you know from calculus can be considered also as functions of one complex variable, and they are differentiable. For example, you (perhaps) encountered the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ in calculus, and there is also a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2xyi$$

and the complex square function is also (complex) differentiable. Whenever this happens and we write the real and imaginary parts $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ of a complex differentiable function

$$f = u + iv$$

as real valued functions of the two real variables x and y in $z = x + iy$, then the functions u and v are **harmonic**, i.e., they satisfy Laplace's equation.

(a) The harmonic function u from Problem 6 above is the real part of the complex square function. Verify that the imaginary part of $f(z) = z^2$ is also harmonic.

(b) What harmonic functions are the real and imaginary parts of $f(z) = z^3$?

(c) What harmoinci functions are the real and imaginary parts of $f(z) = \sin z$? Hint:

$$\sin(z) = \frac{e^{iz} + e^{-iz}}{2i}.$$

Separation of variables

Problems 8-10 below constitute a sequence of steps leading you to a solution of the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{on } U \\ u(x,0) = 0 = u(x,M), & 0 \le x \le L \\ u(0,y) = \sin(\pi y/M), & 0 \le y \le M \\ u(L,y) = 0, & 0 \le y \le M \end{cases} \tag{3}$$

for a function $u \in C^2(U) \cap C^0(\overline{U})$ where $U = (0,L) \times (0,M) \subset \mathbb{R}^2$ for some $L, M > 0$ denotes an open rectangle.

Problem 8 Assume the problem (3) has a solution of the form $u(x, y) = A(x)B(y)$ for some $A \in C^2(0, L) \cap C^0[0, L]$ and $B \in C^2(0, M) \cap C^0[0, M]$.

(a) Write down the PDE in terms of A and B.

(b) Algebraically rearrange the your answer from part **(a)** above in the form

$$\Phi(x) = \Psi(y). \tag{4}$$

(c) Show $\Phi(x) = \lambda$ for some constant λ. Hint: Differentiate (4) with respect to x. The constant λ here is called a **separation constant**.

Problem 9 (Sturm-Liouville Problem) Let A and B and λ satisfy the conditions described in the previous problem.

(a) Find a boundary value problem (depending on λ) involving an ODE for B. This kind of problem is called a **Sturm-Liouville problem**.

(b) By considering various cases, find all (possible) solutions of the Sturm-Liouville problem for B:

- **(i)** Assume $\lambda < 0$.
- **(ii)** Assume $\lambda = 0$.
- **(iii)** Assume $\lambda > 0$.

Hint: You should get a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of separation constants giving corresponding solutions B_j for $j = 1, 2, 3, \ldots$.

(c) For each solution B_j (corresponding to λ_j you found in part **(b)** above, find the general solution of an ODE for A. Call this general solution A_j. Each of the functions $u_j(x, y) = A_j(x)B_j(y)$ is called a **separated variables solution** of the PDE (though they do not all solve the boundary value problem).

Problem 10 (separated variables solution of Laplace's equation) Solve (3).