Assignment 7: solutions Due Friday, March 28, 2025

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For this assignment unless stated otherwise the "heat equation" will refer to the equation

$$u_t = \Delta u. \tag{1}$$

Problem 1 (the heat equation) In Problem 5 of Assignment 6 you derived a heat equation different from (1).

(a) Can you realize equation (1) as a special case of the equation you derived in Problem 5 of Assignment 6?

(b) What is the most general physical conduction problem one can model with equation (1)? What is required for the modeling process outlined in Problem 5 of Assignment 6 to lead to equation (1)?

(c) Note that $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ by $u(x, t) = e^{-t} \sin x$ satisfies (1) in one space dimension. Find the total thermal energy (modeled) in the one dimensional continuum $[0, \pi]$ as a function of time and the thermal flux at $x = \pi$.

Problem 2 (weak maximum principle) Recall that by the extreme value theorem if $u \in C^0(\overline{\Omega})$ for some bounded open set $\Omega \subset \mathbb{R}^n$, then the maximum values

$$M = \max_{\mathbf{x} \in \overline{\Omega}} u(\mathbf{x}) \qquad \text{and} \qquad m = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x})$$

are well-defined finite real numbers. The **weak maximum principle** for Laplace's equation asserts that if $\Delta u \geq 0$, then $M \leq m$. Assume by way of contradiction that

$$M = \max_{\mathbf{x} \in \overline{\Omega}} u(\mathbf{x}) > m = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x}) \tag{2}$$

and complete the following steps to prove the weak maximum principle for Laplace's equation, cf. Problems 7 and 8 of Assignment 4.

(a) Observe that if (2) holds, then there is some $\mathbf{p} \in \Omega$ with $u(\mathbf{p}) > m$. Show there exists some $\epsilon > 0$ for which the function $v \in C^\infty(\mathbb{R}^n)$ with values

$$v(\mathbf{x}) = u(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2$$

satisfies

$$\min_{\mathbf{x} \in \partial\Omega} v(\mathbf{x}) > m = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x}). \tag{3}$$

(b) Let $\epsilon > 0$ be a fixed positive number for which (3) holds. Show there exists some **nonnegative** number $\delta \geq 0$ and some point $\mathbf{q} \in \Omega$ for which the function $w \in C^\infty(\mathbb{R}^n)$ with values

$$w(\mathbf{x}) = v(\mathbf{x}) + \delta$$

satisfies

$$w(\mathbf{x}) \geq u(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Omega \text{ and} \qquad w(\mathbf{q}) = u(\mathbf{q}). \tag{4}$$

(c) Show $\Delta u(\mathbf{q}) \leq \Delta w(\mathbf{q}) < 0$ to obtain a contradiction.

Solution:

(a) Let $\mathbf{p} \in \overline{\Omega}$ with

$$u(\mathbf{p}) = M = \max_{\mathbf{x} \in \overline{\Omega}} u(\mathbf{x}).$$

Let $\mathbf{y} \in \partial\Omega$ with

$$u(\mathbf{y}) = m = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x}).$$

If $M > m = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x})$, then $\mathbf{p} \notin \partial\Omega$, so $\mathbf{p} \in \Omega$.

Set

$$\mu = \operatorname{dist}(\mathbf{p}, \partial\Omega) = \inf\{|\mathbf{x} - \mathbf{p}| : \mathbf{x} \in \partial\Omega\} > 0.$$

Then $\mu > 0$. Let

$$\epsilon_0 = \frac{M - m}{2\operatorname{dist}(\mathbf{p}, \partial\Omega)^2} = \frac{u(\mathbf{p}) - u(\mathbf{y})}{2\mu^2} > 0.$$

Then for each $\mathbf{x} \in \partial\Omega$ and each $\epsilon \leq \epsilon_0$ there holds

$$\begin{aligned} v(\mathbf{x}) &= u(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2 \\ &\geq u(\mathbf{p}) - \frac{u(\mathbf{p}) - u(\mathbf{y})}{2\mu^2}|\mathbf{x} - \mathbf{p}|^2 \\ &\geq u(\mathbf{p}) - \frac{u(\mathbf{p}) - u(\mathbf{y})}{2} \\ &= u(\mathbf{y}) + u(\mathbf{p}) - u(\mathbf{y}) - \frac{u(\mathbf{p}) - u(\mathbf{y})}{2} \\ &= m + \frac{M - m}{2}. \end{aligned}$$

Thus, for

$$0 < \epsilon \leq \frac{u(\mathbf{p}) - m}{2\mu^2}$$

there holds

$$\min_{\mathbf{x} \in \partial\Omega} v(\mathbf{x}) \geq m + \frac{M - m}{2} > m.$$

(b) Let

$$\delta = \max_{\mathbf{x} \in \overline{\Omega}}[u(\mathbf{x}) - v(\mathbf{x})]$$

and let $\mathbf{q} \in \overline{\Omega}$ with $u(\mathbf{q}) - v(\mathbf{q}) = \delta$.

Since $v(\mathbf{p}) = u(\mathbf{p})$ it follows that $\delta \geq 0$. Also, since $u(\mathbf{x}) < v(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$, it follows that $\mathbf{q} \notin \partial\Omega$, so $\mathbf{q} \in \Omega$.

For $\mathbf{x} \in \overline{\Omega}$ we have

$$w(\mathbf{x}) = v(\mathbf{x}) + \delta \geq v(\mathbf{x}) + u(\mathbf{x}) - v(\mathbf{x}) = u(\mathbf{x}).$$

Also, $w(\mathbf{q}) = v(\mathbf{q}) + \delta = v(\mathbf{q}) + u(\mathbf{q}) - v(\mathbf{q}) = u(\mathbf{q})$.

(c) Let $\nu = \text{dist}(\mathbf{q}, \partial\Omega) > 0$. For $j = 1, 2, \ldots, n$ consider the function $f \in C^2(-\nu, \nu)$ with values

$$f(h) = u(\mathbf{q} + h\mathbf{e}_j) - w(\mathbf{q} + h\mathbf{e}_j).$$

Note that $f(0) = 0$ and $f(h) \leq 0$ for all $h \in (-\nu, \nu)$. It follows that $f'(0) = 0$ and $f''(0) \leq 0$. To see that $f'(0) = 0$, note that on the one hand

$$f'(0) = \lim_{h \nearrow 0} \frac{f(h)}{h} \geq 0 \qquad \text{since } f(h) \leq 0 \text{ and here } h < 0,$$

and on the other hand,

$$f'(0) = \lim_{h \searrow 0} \frac{f(h)}{h} \leq 0 \qquad \text{since } f(h) \leq 0 \text{ and here } h > 0.$$

To see $f''(0) \leq 0$, assume $f''(0) > 0$ and note that

$$f''(0) = \lim_{h \to 0} \frac{f'(h)}{h}.$$

In particular, there exist values $h > 0$ with $f'(h) > 0$. By the mean value theorem, there are also values h_* with $0 < h_* < h$ and $f(h_*) > 0$ contradicting the fact that $f(h) \leq 0$ for all h.

We conclude

$$0 \geq f''(0) = \frac{\partial^2 u}{\partial x_j^2}(\mathbf{q}) - \frac{\partial^2 w}{\partial x_j^2}(\mathbf{q})$$

or

$$\frac{\partial^2 u}{\partial x_j^2}(\mathbf{q}) \leq \frac{\partial^2 w}{\partial x_j^2}(\mathbf{q}) = -2\epsilon.$$

This means $\Delta u(\mathbf{q}) \leq \Delta w(\mathbf{q}) = -2n\epsilon < 0$, which is a contradiction. Thus,

$$M = u(\mathbf{p}) \leq m = \min_{x \in \partial\Omega} u(\mathbf{x}).$$

Problem 3 (uniqueness for solutions of Poisson's equation) Use the weak maximum principle of Problem 2 above to show the boundary value problem

$$\begin{cases} \Delta u(\mathbf{x}) = -f(\mathbf{x}), & \mathbf{x} \in \Omega \\ u\Big|_{\mathbf{x}\in\partial\Omega} = g, \end{cases}$$

where $g \in C^0(\partial\Omega)$ can have at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ where Ω is a bounded open subset of $\mathbb{R}^n$.

Solution: Let u and $\tilde{u}$ be two solutions. Then $w = \pm v = \pm(u - \tilde{u})$ satisfies

$$\begin{cases} \Delta w \equiv 0, & \text{on } \Omega \\ w\Big|_{\partial\Omega} \equiv 0. \end{cases}$$

By the weak maximum principle,

$$w(\mathbf{p}) \leq \min_{\mathbf{x}\in\partial\Omega} w(\mathbf{x}) = 0.$$

That is $u - \tilde{u} \leq 0$ and $-(u - \tilde{u}) \leq 0$ or

$$u \leq \tilde{u} \qquad \text{and} \qquad \tilde{u} \leq u.$$

We conclude $u \equiv \tilde{u}$ on Ω.

Problem 4 weak maximum principle for the heat equation) Let $T > 0$, and let Ω be a bounded open subset of $\mathbb{R}^n$. Consider a function $u \in C^2(\overline{\Omega} \times [0, T])$ satisfying the initial/boundary value problem

$$\begin{cases} u_t(\mathbf{x}, t) \leq \Delta u(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, T) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega \\ u(\mathbf{x}, t) = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \partial\Omega \times [0, T]. \end{cases} \tag{5}$$

Complete the following steps to prove a weak maximum principle for the heat equation.

(a) Let $v \in C^2(\overline{\Omega} \times [0, T])$ have values given by

$$v(\mathbf{x}, t) = u(\mathbf{x}, t) - \epsilon t$$

where $\epsilon > 0$ is some positive constant. Find an initial/boundary value problem satisfied by the function v.

(b) Let

$$M = \max_{(\mathbf{x},t) \in \overline{\Omega} \times [0,T]} v(\mathbf{x}, t) \qquad \text{and} \qquad m = \max \left\{ \max_{\mathbf{x} \in \overline{\Omega}} v(\mathbf{x}, 0), \max_{(\mathbf{x},t) \in \partial\Omega \times [0,T]} v(\mathbf{x}, t) \right\}.$$

Express m as a single maximum

$$m = \max_{(\mathbf{x},t) \in A} v(\mathbf{x}, t)$$

over a single compact set $A \subset \mathbb{R}^{n+1}$, and draw the (typical) set A when $n = 1$.

(c) Assume by way of contradiction that $M > m$.

(i) One possibility is that there is a point $(\mathbf{p}, t_0) \in \Omega \times (0, T)$ with

$$v(\mathbf{p}, t_0) = M > m.$$

Obtain a contradiction in this case.

(ii) Determine the remaining possibility for a point $(\mathbf{p}, t_0) \notin A$ with $v(\mathbf{p}, t_0) = M$, and obtain a contradiction in this case as well. Hint: $v_t(\mathbf{p}, t_0) \geq 0$.

(d) Take the limits as $\epsilon \searrow 0$ of M and m to conclude

$$\max_{(\mathbf{x},t) \in \overline{\Omega} \times [0,T]} u(\mathbf{x}, t) \leq \max_{(\mathbf{x},t) \in A} u(\mathbf{x}, t).$$

This is the weak maximum principle for the heat equation.

Problem 5 (uniqueness for solutions of the heat equation) Let Ω be a bounded open subset of $\mathbb{R}^n$. Use the weak maximum principle of Problem 4 above to show the initial/boundary value problem

$$\begin{cases} u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,\infty) \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega \\ u(\mathbf{x},t) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \end{cases}$$

where $u_0 \in C^2(\Omega)$ and $g \in C^0(\partial\Omega)$ can have at most one solution $u \in C^2(\overline{\Omega} \times [0,\infty))$.

Solution: Let u and $\tilde{u}$ be two solutions and let $T > 0$ be arbitrary. Then $w = \pm(u - \tilde{u})$ satisfies $w \in C^2(\overline{\Omega} \times [0,T])$ and

$$\begin{cases} w_t(\mathbf{x},t) = \Delta w(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T] \\ w(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega \\ w(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

It follows from the weak maximum principle of Problem 4 that

$$\max_{(\mathbf{x},t)\in\overline{\Omega}\times[0,T]} w \le \max_{\mathbf{x}\in A} w(\mathbf{x},t) = 0$$

where $A = \{(\mathbf{x},0) : \mathbf{x} \in \Omega\} \cup (\partial\Omega \times [0,T]$. Thus, $u \le \tilde{u}$ on $\Omega \times (0,T)$ and also on $\Omega \times (0,\infty)$ since $T > 0$ is arbitrary.

Similarly $w \ge 0$ on $\Omega \times (0,\infty)$ and so $\tilde{u} \le u$. Hence $\tilde{u} \equiv u$, and there is a unique solution (if there exists a solution at all).

Problem 6 (nonuniqueness for solutions of the heat equation) Consider $g \in C^\infty(0, \infty)$ by

$$g(t) = e^{-1/t^2}.$$

(a) Compute

$$\frac{d^n}{dt^n} g(t)$$

for $n = 1, 2, 3$.

(b) Show

$$\lim_{t \searrow 0} \frac{d^n}{dt^n} g(t) = 0$$

for $n = 0, 1, 2, 3, \ldots$.

(c) Consider $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ with

$$u(x, t) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{d^n}{dt^n} g(t)\ x^{2n}.$$

Show $w : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ by

$$w(x, t) = \lim_{\tau \searrow t} u(x, \tau)$$

satisfies $w \in C^2(\mathbb{R} \times [0, \infty))$ and is a solution of the initial value problem

$$\begin{cases} w_t(x, t) = \Delta w(x, t), & (x, t) \in \mathbb{R} \times (0, \infty) \\ w(x, 0) \equiv 0, & x \in \mathbb{R}. \end{cases} \tag{6}$$

(d) Show the initial value problem (6) for the heat equation does not have a unique solution.

Solution:

(a) Here we start with

$$g(t) = e^{-1/t^2}$$

and compute

$$g'(t) = \frac{2}{t^3} e^{-1/t^2}.$$

$$g''(t) = \left[\left(\frac{2}{t^3} \right)^2 - \frac{3!}{t^4} \right] e^{-1/t^2}.$$

$$g'''(t) = \left[\left(\frac{2}{t^3} \right)^3 - \frac{(2)3!}{t^7} - \frac{4!}{t^7} + \frac{4!}{t^5} \right] e^{-1/t^2}.$$

(b) From part **(a)** it should be clear that $g \in C^\infty(0, \infty)$ with derivatives having the form

$$g^{(n)}(t) = Q_n \left(\frac{1}{t} \right) e^{-1/t^2} \tag{7}$$

for $n = 1, 2, 3, \ldots$ where Q_n is a polynomial. There are various other observations one can make at this point which like this basic observation should be verified using (mathematical) induction. One of those, for example, is that Q_n should have the form

$$Q^{(n)}(X) = \sum_{j=n+2}^{3n} a_{nj} X^j$$

for some real coefficients $a_{n+2}, \ldots, a_{3n}$. The basic form (7) can be extended to the function g itself, though the polynomial in this case should be recognized separately as the constant $Q_0 \equiv 1$.

We will need to work harder later, but let's make an inductive calculation verifying (7). We have already checked the cases $n = 1, 2, 3$ in part **(a)**, so the case $n = 3$ can serve as the "base" for the induction. The idea of the induction here is that we assume for some $k \geq 3$ the assertion we think (or in the case of $k = 2$ already know) is correct:

$$g^{(k)}(t) = Q_k \left(\frac{1}{t} \right) e^{-1/t^2} \qquad \text{with} \qquad Q_k(X) = \sum_{j=k+2}^{3k} a_{kj} X^j.$$

Differentiating we conclude

$$\begin{aligned} g^{(k+1)}(t) &= Q_k \left(\frac{1}{t} \right) \left(\frac{2}{t^3} \right) e^{-1/t^2} - Q_k' \left(\frac{1}{t} \right) \left(\frac{1}{t^2} \right) e^{-1/t^2} \\ &= \left[Q_k \left(\frac{1}{t} \right) \left(\frac{2}{t^3} \right) - Q_k' \left(\frac{1}{t} \right) \left(\frac{1}{t^2} \right) \right] e^{-1/t^2}. \end{aligned}$$

This means $g^{(k+1)}$ has the form (7) with

$$Q_{k+1}(X) = 2X^3 Q_k(X) - X^2 Q_k'(X). \tag{8}$$

Again we have an elementary derivative calculation,

$$Q_k'(X) = \sum_{j=k+2}^{3k} j a_{kj} X^{j-1},$$

and shift some indices to find

$$\begin{aligned}
Q_{k+1}(X) &= \sum_{j=k+2}^{3k} 2a_{kj} X^{j+3} - \sum_{j=k+2}^{3k} j a_{kj} X^{j+1} \\
&= \sum_{j=k+5}^{3k+3} 2a_{k,j-3} X^j - \sum_{j=k+3}^{3k+1} j a_{k,j-1} X^j \\
&= \sum_{j=k+3}^{k+4} j a_{k,j-1} X^j + \sum_{j=k+5}^{3k+1} [2a_{k,j-3} - j a_{k,j-1}] X^j + \sum_{j=3k+2}^{3k+3} 2a_{k,j-3} X^j.
\end{aligned}$$

Of course, one should check that the "lower" limits of summation here are all actually lower than the "upper" limits of summation, but for $k \geq 3$ this is true. This could be a separate induction, but it's pretty obvious. Note also that we can write

$$Q_{k+1}(X) = \sum_{j=(k+1)+2}^{k+4} j a_{k,j-1} X^j + \sum_{j=k+5}^{3k+1} [2a_{k,j-3} - j a_{k,j-1}] X^j + \sum_{j=3k+2}^{3(k+1)} 2a_{k,j-3} X^j,$$

so

$$Q_{k+1}(X) = \sum_{j=(k+1)+2}^{3(k+1)} a_{k+1,j} X^j$$

where

$$a_{k+1,j} = \begin{cases} j a_{k,j-1}, & (k+1)+2 \leq j \leq k+4 \\ 2a_{k,j-3} - j a_{k,j-1}, & k+5 \leq j \leq 3k+1 \\ 2a_{k,j-3}, & 3k+2 \leq j \leq 3(k+1). \end{cases}$$

This completes the induction.

We could also note (and prove by induction) various other properties of the coefficient polynomials and their coefficients themselves, and we should do this in the end, but let's postpone such details for the moment.

We can see right away that

$$\lim_{t \searrow 0} g(t) = 0,$$

and it's not difficult to imagine the decay of the exponential factor always overcomes any growth present in the polynomial factor so that

$$\lim_{t \searrow 0} g^{(n)}(t) = 0 \tag{9}$$

for $n = 1, 2, 3, \dots$ as well (as claimed). Perhaps a simple way to see this precisely is to consider each term

$$\frac{1}{t^k}\ e^{-1/t^2} \tag{10}$$

separately for $k = 1, 2, 3, \dots$. If all terms[1] of the form (10) satisfy

$$\lim_{t \searrow 0} \frac{1}{t^k}\ e^{-1/t^2} = 0,$$

then we can say each $g^{(n)}$ is a linear combination of such terms and must satisfy (9). Note first that

$$\lim_{t \searrow 0} \frac{1}{t^k}\ e^{-1/t^2} = \lim_{X \nearrow \infty} \frac{X^k}{e^{X^2}}.$$

This is indeterminate of the form ∞/∞, so we can apply L'Hopital's rule. When $k = 1$ or $k = 2$ we get

$$\lim_{X \nearrow \infty} \frac{X^k}{e^{X^2}} = \lim_{X \nearrow \infty} \frac{kX^{k-1}}{2Xe^{X^2}} = \frac{k}{2} \lim_{X \nearrow \infty} \frac{1}{X^{2-k}Xe^{X^2}} = 0.$$

That is, we know

$$\lim_{X \nearrow \infty} \frac{X^k}{e^{X^2}} = 0 \qquad \text{for} \qquad k = 0, 1, 2.$$

Looking back at the basic calculation for L'Hopital's rule:

$$\lim_{X \nearrow \infty} \frac{kX^{k-1}}{2Xe^{X^2}} = \frac{k}{2} \lim_{X \nearrow \infty} \frac{X^{k-2}}{e^{X^2}}$$

[1] Note the use of the symbol k here has nothing to do with the use of the same symbol in the induction argument above.

we can now say

$$\lim_{X\nearrow\infty}\frac{X^k}{e^{X^2}}=0$$

as long as $k-2\le 2$, that is

$$\lim_{X\nearrow\infty}\frac{X^k}{e^{X^2}}=0 \qquad \text{for} \qquad k=3,4$$

as well. Once we have the basic assertion for $k=0,1,2,3,4$, then we get that same assertion for $k=5$ and $k=6$ as well, that is, in cases where $k-2\le 4$. In this way, we obtain the desired assertion

$$\lim_{X\nearrow\infty}\frac{X^k}{e^{X^2}}=0 \qquad \text{for} \qquad k=0,1,2,3,\ldots,$$

and (9) follows.

(c) We have completed parts **(a)** and **(b)** with adequate detail for a full and rigorous solution. The assertions of part **(c)** are rather more technical, and I am going to first outline what is required for a full solution without giving some of the necessary details. Let me begin by reinterpreting the definitions of the functions u and w. We have shown that each of the functions $g^{(n)}$ for $n=0,1,2,3,\ldots$ is a smooth function of t for $t\in(0,\infty)$ and decays rapidly as $t\searrow 0$. Consequently we know that each partial sum

$$u_k(x,t)=\sum_{n=0}^{k}\frac{1}{(2n)!}g^{(n)}(t)x^{2n}=\sum_{n=0}^{k}\left[\frac{1}{(2n)!}Q_n\left(\frac{1}{t}\right)x^{2n}\right]\ e^{-1/t^2}$$

determines a function $u_k\in C^\infty(\mathbb{R}\times(0,\infty))$ with

$$\lim_{t\searrow 0}u_k(x,t)=0.$$

In particular,

$$\frac{\partial u_k}{\partial t}=\sum_{n=0}^{k}\left[\frac{1}{(2n)!}\left[\left(\frac{2}{t^3}\right)Q_n\left(\frac{1}{t}\right)-\left(\frac{1}{t^2}\right)Q_n'\left(\frac{1}{t}\right)\right]x^{2n}\right]\ e^{-1/t^2}.$$

Note that the relation

$$Q_{n+1}(X)=2X^3Q_n(X)-X^2Q_n'(X)$$

expressed in (8) actually holds for all $n = 0, 1, 2, 3, \ldots$, so we can also write

$$\frac{\partial u_k}{\partial t} = \sum_{n=0}^{k} \left[\frac{1}{(2n)!} Q_{n+1}\left(\frac{1}{t}\right) x^{2n} \right] \ e^{-1/t^2} = \sum_{n=0}^{k} \frac{1}{(2n)!} g^{(n+1)}(t) x^{2n}.$$

On the other hand,

$$\frac{\partial u_k}{\partial x} = \sum_{n=1}^{k} \frac{1}{(2n-1)!} g^{(n)}(t) x^{2n-1}$$

and

$$\frac{\partial^2 u_k}{\partial x^2} = \sum_{n=1}^{k} \frac{1}{(2n-2)!} g^{(n)}(t) x^{2n-2} = \sum_{n=0}^{k-1} \frac{1}{(2n)!} g^{(n+1)}(t) x^{2n}.$$

Thus,

$$\frac{\partial u_k}{\partial t} - \frac{\partial^2 u_k}{\partial x^2} = \frac{1}{(2k)!} g^{(k+1)}(t) x^{2k}.$$

Evidently, the objective is to prove this quantity tends to 0 as k tends to $+\infty$ and all the partial sums converge to define functions related to the series

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \ \frac{d^n}{dt^n} g(t) \ x^{2n}$$

appearing in the statement of the problem. In this way we should obtain a solution of the heat equation $w_t = w_{xx}$ for which we know

$$w(0,t) = u(0,t) = g(t) \neq 0 \qquad \text{for} \qquad t > 0.$$

This should give us the nonuniqueness suggested in part **(d)**.

Let me begin by defining u (and w) a little differently. It follows from the observations concerning $g \in C^\infty(0,\infty)$ in part **(b)** that the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} 0, & t \leq 0 \\ g(t), & t > 0 \end{cases}$$

satisfies $h \in C^\infty(\mathbb{R})$. The function h is not real analytic on $\mathbb{R}$. In particular, $h_1 : \mathbb{R} \to \mathbb{R}$ by

$$h_1(t) = \begin{cases} 0, & t = 0 \\ e^{-1/t^2}, & t \neq 0 \end{cases}$$

determines another function $h_1 \in C^\infty(\mathbb{R})$ different from h but also satisfying

$$h_1^{(n)}(0) = h^{(n)}(0) = 0 \qquad \text{for} \qquad n = 0, 1, 2, 3, \ldots.$$

Instead of $u \in C^2(\mathbb{R} \times [0, \infty))$ we consider the function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with values given by the series

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{d^n}{dt^n} h(t)\ x^{2n}. \tag{11}$$

with partial sums

$$u_k(x,t) = \sum_{n=0}^{k} \frac{1}{(2n)!}\ h^{(k)}(t)\ x^{2n}.$$

Rephrasing the discussion above in terms of these values we have $u_k \in C^\infty(\mathbb{R} \times \mathbb{R})$ with

$$u_k(x,t) \equiv 0 \qquad \text{for} \qquad t \le 0$$

and

$$\frac{\partial u_k}{\partial t} - \frac{\partial^2 u_k}{\partial x^2} = \begin{cases} 0, & t \le 0 \\ \dfrac{1}{(2k)!} g^{(k+1)}(t) x^{2k}, & t > 0. \end{cases}$$

I have not given the details yet, but I claim (11) is convergent and has values defining a function

$$u \in C^2(\mathbb{R}^2) \tag{12}$$

with

$$\lim_{k \nearrow \infty} \|u_k - u\|_{C^2(U)} = 0 \tag{13}$$

for every open bounded set $U \subset\subset \mathbb{R}^2$, so that in particular,

$$\lim_{k \nearrow \infty} \frac{\partial u_k}{\partial t} = \frac{\partial u}{\partial t} \qquad \text{and} \qquad \lim_{k \nearrow \infty} \frac{\partial^2 u_k}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

pointwise with

$$\lim_{k \nearrow \infty} \frac{1}{(2k)!} g^{(k+1)}(t) x^{2k} = 0. \tag{14}$$

If (12), (13), and (14) can be established, then the function u given as a series in (11) constitutes a solution of

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) \equiv 0, & x \in \mathbb{R}. \end{cases} \tag{15}$$

which is a generlized version of (6).

(d) Naturally the zero function $z \in C^\infty(\mathbb{R}^2)$ with $z \equiv 0$ is a solution of (15), and as mentioned above

$$u(0,t) = h(t) = g(t) > 0 \qquad \text{for} \qquad t > 0,$$

so the problem (6) certainly does not have a unique solution.

It remains to justify (12), (13), and (14).