Assignment 7:
Partial Differential Equations (heat equation)
Due Wednesday, March 8, 2023

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Problem 1 (A problem in geometric ODEs) This is the second in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Remember Problem 1 of Assignment 6. For this problem, we consider a function $f : (-a, a) \to \mathbb{R}$ for some $a > 0$ with $f \in C^2(-a, a)$ satisfying the initial conditions

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 0.$$

The **arclength** of the graph $\mathcal{G} = \{(x, f(x)) : x \in (-a, a)\}$ of f measured positive along $\mathcal{G}$ from $(0, 0)$ to $(x, f(x))$ for $x > 0$ and negative along $\mathcal{G}$ from $(0, 0)$ to $(x, f(x))$ for $x < 0$ can be expressed in terms of an integral obtained as a limit of the length of polygonal paths as follows: Let $x > 0$ and consider a **partition**

$$\mathcal{P} = \{x_0 = 0, x_1, x_2, \dots, x_k = x\}$$

with $x_0 < x_1 < x_2 < \dots < x_k$. For each such partition consider the polygonal path

$$\Gamma = \bigcup_{j=1}^{k} \{(1 - t)(x_{j-1}, f(x_{j-1})) + t(x_j, f(x_j)) : 0 \le t \le 1\}$$

(a) Draw a picture of the polygonal path Γ.

(b) Express the length of the polygonal path as a sum

$$\sum_{j=1}^{k} \text{length}(\Gamma_j)$$

where $\Gamma_j = \{(1-t)(x_{j-1}, f(x_{j-1})) + t(x_j, f(x_j)) : 0 \le t \le 1\}$ is the line segment connecting $((x_{j-1}, f(x_{j-1}))$ to $(x_j, f(x_j))$ for $j = 1, 2, \ldots, k$.

(c) Introduce appropriate difference quotients in order to obtain the arclength

$$s(x) = \lim_{\|\mathcal{P}\| \to 0\}} \sum_{j=1}^{k} \text{length}(\Gamma_j)$$

along Γ described above as an integral for $x \ge 0$.

(d) Modify your discussion of part **(c)** above to conclude that the same integral gives the negative arclength measured along Γ from $(0,0)$ to $(x, f(x))$ when $x < 0$.

Problem 2 (A problem in geometric ODEs) This is the third in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Assume the signed curvature $k = k(x)$ of the graph $\mathcal{G}$ of f considered in Problem 1 above is equal to the signed arclength $s(x)$.

(a) Find a first order system of ordinary differential equations satisfied by the three real valued functions $f = f(x)$, $s = s(x)$, and $w = f'(x)$.

(b) Combine the system you found in part **(a)** of this problem with the initial conditions from Problem 1 above, and find a numerical approximation of the solution of the resulting initial value problem (IVP). (Use mathematical software like Mathematica (`NDSolve`) or Matlab (`ODE45`).)

(c) Plot your numerical solution/approximation on some interval $(-a, a)$. (Use mathematical software like Mathematica or Matlab.)

(d) What interesting thing do you find? For example, what is the maximum value of the endpoint a you can use? What is

$$\lim_{x \nearrow a} f(x)?$$

Problem 3 (A problem in geometric ODEs) This is the fourth in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Use the existence and uniqueness theorem for ODEs to show the solution $f = f(x)$ you approximated in Problem 2 above has the following properties

(a) f exists and is unique on some maximal interval $(-a, a)$ for some $a > 0$.

(b) The solution f is odd: $f(-x) = -f(x)$.

(c) f is increasing and bounded on $(-a, a)$.

(d) There is a **finite height gradient blow-up** at $x = a$:

$$\lim_{x\nearrow a} f(x) < \infty \qquad \text{and} \qquad \lim_{x\nearrow a} f'(x) = +\infty.$$

Solution: Here we start with the system of ODEs

$$\begin{cases} \dfrac{f''}{(1+f'^2)^{3/2}} = s, & f(0) = f'(0) = 0 \\ s' = \sqrt{1+f'^2}, & s(0) = 0 \end{cases} \tag{1}$$

or the equivalent first order system

$$\begin{cases} f' = w, & f(0) = 0 \\ w' = s(1+w^2)^{3/2)}, & w(0) = 0 \\ s' = \sqrt{1+w^2}, & s(0) = 0. \end{cases} \tag{2}$$

It will be observed that the autonomous vector field

$$\mathbf{v} = \mathbf{v}(f, w, s) = (w, s(1+w^2)^{3/2)}, \sqrt{1+w^2})$$

defining this system satisfies $\mathbf{v} \in C^\infty(\mathbb{R}^3 \to \mathbb{R}^3)$. Thus, we certainly have local existence and uniqueness at every initial point $(x_0, f(x_0), w(x_0), s(x_0)) \in \mathbb{R}^4$. In particular, there exists some $\epsilon > 0$ for which (2) has a well-defined solution (f, w, s) defined on some interval $(-\epsilon, \epsilon)$.

For a moment let ϵ be any positive number for which (2) has a solution $(f, w, s) \in C^1(-\epsilon, \epsilon)$. Note that this implies $f \in C^2(-\epsilon, \epsilon)$ with (f, s) a solution of (1) on $(-\epsilon, \epsilon)$ and

$$\frac{f''}{(1+f'^2)^{3/2)}} = \left(\frac{f'}{\sqrt{1+f'^2}} \right)' = s. \tag{3}$$

Furthermore, the last equation in (2) tells us $s' \geq 1$ so

$$s(x) \geq x \geq 0 \qquad \text{for } 0 \leq x < \epsilon \text{ with strict inequality when } x > 0.$$

Applying this observation to the second equation, we find $w = f'$ is increasing with

$$w'(x) = f''(x) \geq 0 \qquad \text{for } 0 \leq x < \epsilon \text{ with strict inequality when } x > 0$$

and

$$w(x) = f'(x) \geq 0 \qquad \text{for } 0 \leq x < \epsilon \text{ with strict inequality when } x > 0.$$

Finally, applying this last observation to the first equation we have

$$f(x) \geq 0 \qquad \text{for } 0 \leq x < \epsilon \text{ with strict inequality when } x > 0.$$

It follows that each of the limits

$$\lim_{x \nearrow \epsilon} f(x), \quad \lim_{x \nearrow \epsilon} f'(x), \quad \lim_{x \nearrow \epsilon} f''(x), \quad \text{and} \quad \lim_{x \nearrow \epsilon} s(x),$$

exists as an extended real number in $(0, \infty]$. If $w = f'(x)$ has a finite limit

$$\lim_{x \nearrow \epsilon} f'(x) = m \in \mathbb{R},$$

then $f(x) \leq mx$ and $s \leq \sqrt{1 + m^2}$ also have finite limits. Therefore, one may take

$$f(\epsilon) = \lim_{x \nearrow \epsilon} f(x), \quad w(\epsilon) = \lim_{x \nearrow \epsilon} f'(x), \qquad \text{and} \quad s(\epsilon) = \lim_{x \nearrow \epsilon} s(x)$$

as initial conditions for the ODEs in (2) and find for some $\delta > 0$ a extension of the solution to some larger interval $(-\epsilon, \epsilon + \delta)$. In this case also, we may check that the functions

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \leq x < x + \delta \\ -f(-x), & -\delta - \epsilon < x \leq 0, \end{cases}$$

$$\tilde{w}(x) = \begin{cases} w(x), & 0 \leq x < x + \delta \\ w(-x), & -\delta - \epsilon < x \leq 0, \end{cases}$$

and

$$\tilde{s}(x) = \begin{cases} s(x), & 0 \leq x < \epsilon + \delta \\ -s(-x), & -\delta - \epsilon < x \leq 0 \end{cases}$$

are each in $C^1(-\delta-\epsilon,\epsilon+\delta)$ and together give a solution, and hence the unique solution, of (2) on the entire symmetric interval $(-\delta-\epsilon,\epsilon+\delta)$.

Note finally, that since this modified solution $(\tilde{f},\tilde{w},\tilde{s})$ has $\tilde{f}$ and $\tilde{s}$ odd and $\tilde{w}$ even, and this is the unique solution, the original solution must also have component functions with these symmetries.

Letting

$$a=\sup\{\epsilon>0:\text{(2) has a unique solution on }(-\epsilon,\epsilon).\}$$

we obtain a unique maximal interval $(-a,a)$ on which (2) has a unique solution. This solution satisfies $f(-x)=-f(x)$, $w(-x)=w(x)$, and $s(-x)=-s(x)$.

(a) It remains to show $a<\infty$. To this end, let us return to a particular $\epsilon>0$ for which (2) has a unique solution (f,w,s) on the interval $(-\epsilon,\epsilon)$. By restricting to a smaller interval if necessary, we may assume $\lim_{x\nearrow\epsilon}f'(x)<\infty$ so that $(f,w,s)\in C^1[-\epsilon,\epsilon]$ and

$$f(\epsilon)=\lim_{x\nearrow\epsilon}f(x),\quad f'(\epsilon)=\lim_{x\nearrow\epsilon}f'(x),\quad\text{and}\quad s(\epsilon)=\lim_{x\nearrow\epsilon}s(x)$$

with each of these values finite and positive. This situation is indicated in Figure 1. I wish to compare to a tangent (osculating) circle to the graph of f

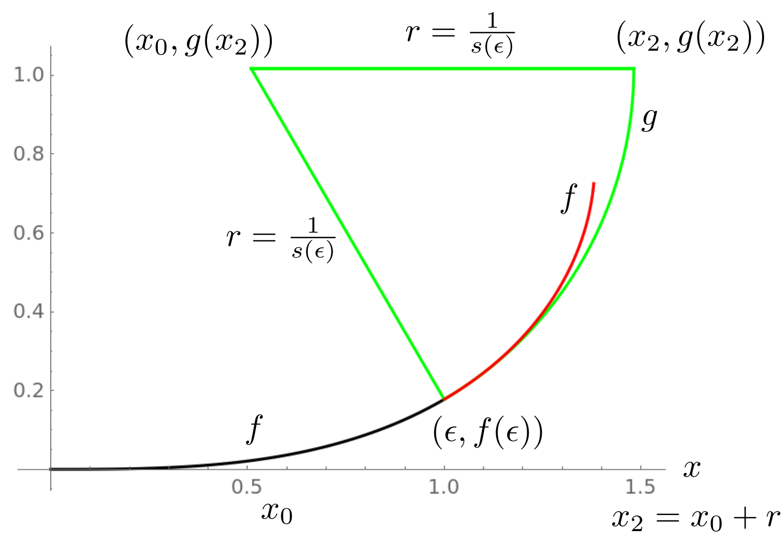

Figure 1: A local regular solution at $x=0$ and a comparison circle.

at the point $(\epsilon,f(\epsilon))$. I will only consider the portion of this circle to the right of $x = \epsilon$ as indicated in Figure 1. The radius of this circle is $r = 1/s(\epsilon) > 0$, and the portion of interest is the graph of a function $g : [\epsilon, x_2] \to \mathbb{R}$ of the form

$$g(x) = f(\epsilon) + \sqrt{r^2 - (\epsilon - x_0)^2} - \sqrt{r^2 - (x - x_0)^2}$$

where x_0 is the x coordinate of the center. We have then

$$g'(x) = \frac{x - x_0}{\sqrt{r^2 - (x - x_0)^2}}$$

and want

$$g'(\epsilon) = \frac{\epsilon - x_0}{\sqrt{r^2 - (\epsilon - x_0)^2}} = f'(\epsilon).$$

This implies $(\epsilon - x_0)^2 = f'(\epsilon)^2[r^2 - (\epsilon - x_0)^2]$ or

$$x_0 = \epsilon - \frac{rf'(\epsilon)}{\sqrt{1 + f'(\epsilon)^2}}.$$

We then have g increasing on $[\epsilon, x_2] = [\epsilon, x_0 + r]$ with $g \in C^\infty[\epsilon, x_2) \cap C^0[\epsilon, x_2]$,

$$g(x_2) = f(\epsilon) + \sqrt{r^2 - (\epsilon - x_0)^2} = f(\epsilon) + \frac{r}{\sqrt{1 + f'(\epsilon)^2}}$$

and

$$\frac{g''}{(1 + g'^2)^{3/2}} \equiv \frac{1}{r} = s(\epsilon).$$

I record a simplified formula for g:

$$g(x) = f(\epsilon) + \frac{r}{\sqrt{1 + f'(\epsilon)^2}} - \sqrt{r^2 - \left(x - \epsilon + \frac{rf'(\epsilon)}{\sqrt{1 + f'(\epsilon)^2}}\right)^2}.$$

Assuming (2) has a solution on the interval $[\epsilon, \epsilon + \delta)$ for some $\delta > 0$, notice that

$$s(x) = s(\epsilon) + \int_\epsilon^x \sqrt{1 + w(t)^2}\, dt \geq s(\epsilon) + (x - \epsilon) \geq s(\epsilon)$$

for $\epsilon \leq x < \epsilon + \delta$ with strict inequality when $x > \epsilon$.

Therefore, if $f, g \in C^1[\epsilon, \epsilon + \delta)$, then

$$s(x) = \frac{f''}{(1 + f'^2)^{3/2}} \geq \frac{g''}{(1 + g'^2)^{3/2}} \equiv s(\epsilon) \tag{4}$$

with strict inequality for $x > \epsilon$. Rewriting this inequality using the identity (3), we get

$$\frac{d}{dx}\left(\frac{f'}{\sqrt{1+f'^2}}\right) \geq \frac{d}{dx}\left(\frac{g'}{\sqrt{1+g'^2}}\right),$$

and integrating

$$\frac{f'}{\sqrt{1+f'^2}} \geq \frac{g'}{\sqrt{1+g'^2}}.$$

Finally, this means

$$f'(x) \geq g'(x) \qquad \text{and} \qquad f''(x) \geq g''(x). \tag{5}$$

All these inequalities are strict for $x > \epsilon$. Having noted above that the maximal interval $[\epsilon, a)$ for the existence of f, if a is finite must be determined by the condition

$$\lim_{x \nearrow a} f'(x) = +\infty,$$

and noting that

$$\lim_{x \nearrow x_2} g'(x) = +\infty,$$

we conclude from (5) that $a \leq x_2$. In particular, $a < \infty$ as was to be shown. It would also be nice to know $a < x_2 = x_0 + r$. We will establish this below.

(b) We have already shown the solution is odd.

(c) We have established in the solution of part **(a)** above that there is some $a \leq x_2$ for which $f \in C^2[\epsilon, a)$, but

$$\lim_{x \nearrow a} f'(x) = +\infty.$$

On this common interval $[\epsilon, a)$ where $f, g \in C^2[\epsilon, a)$ both functions are increasing and have inverses:

$$g^{-1} : [f(\epsilon), g(a)] \to [\epsilon, a] \qquad \text{and} \qquad f^{-1} : \left[f(\epsilon, \lim_{x \nearrow a} f(x)\right) \to [f(\epsilon), a).$$

In fact, g has an inverse $g^{-1} : [f(\epsilon), g(x_2)] \to [\epsilon, x_2]$. At the moment, we still only know $a \leq x_2$. The functions f^{-1} and g^{-1} moreover satisfy

$$g^{-1} \in C^2[f(\epsilon), g(a)] \qquad \text{and} \qquad f^{-1} \in C^2\left[f(\epsilon), \lim_{x \nearrow a} f(x)\right)$$

and have graphs with associated signed curvatures

$$-\frac{\dfrac{d^2g^{-1}}{dy^2}}{\left[1+\left(\dfrac{dg^{-1}}{dy}\right)^2\right]^{3/2}} = s(\epsilon) \qquad \text{and} \qquad -\frac{\dfrac{d^2f^{-1}}{dy^2}}{\left[1+\left(\dfrac{df^{-1}}{dy}\right)^2\right]^{3/2}} = s\circ f^{-1}.$$

Furthermore, the inequality (4) holds in these coordinates so that

$$-\frac{\dfrac{d^2g^{-1}}{dy^2}}{\left[1+\left(\dfrac{dg^{-1}}{dy}\right)^2\right]^{3/2}} \le -\frac{\dfrac{d^2f^{-1}}{dy^2}}{\left[1+\left(\dfrac{df^{-1}}{dy}\right)^2\right]^{3/2}}$$

with strict inequality for $y > f(\epsilon)$ on some common interval $[f(\epsilon), y_1)$ with both f^{-1} and g^{-1} defined and twice differentiable. We know, more precisely, that

$$y_1 = \min\left\{g(x_2), \lim_{x\nearrow a} f(x)\right\}$$

though we can sharpen this characterization later. Just as we used (4) to obtain (5) above, the same reasoning gives

$$0 \le \frac{df^{-1}}{dy} \le \frac{dg^{-1}}{dy} \qquad \text{and} \qquad \frac{d^2f^{-1}}{dy^2} \le \frac{d^2g^{-1}}{dy^2} < 0 \tag{6}$$

for $f(\epsilon) \le y < y_1$ and with strict inequality for $f(\epsilon) < y < y_1$.

If we assume $y_1 = g(x_2)$, then

$$\lim_{y\nearrow y_1} \frac{df^{-1}}{dy} \le \lim_{y\nearrow y_1} \frac{dg^{-1}}{dy} = 0.$$

Therefore,

$$\lim_{y\nearrow y_1} \frac{df^{-1}}{dy} = 0$$

and

$$\lim_{x\nearrow a} f'(x) = \infty \qquad \text{with} \qquad \lim_{x\nearrow x_1} f(x) \ge g(x_2).$$

If

$$\lim_{x \nearrow x_1} f(x) > g(x_2),$$

then

$$\frac{df^{-1}}{dy}(g(x_2)) < \frac{dg^{-1}}{dy}(g(x_2)) = 0$$

which is a contradiction of (6). Thus, we must have

$$\lim_{x \nearrow x_1} f(x) = g(x_2)$$

and

$$a = \lim_{y \nearrow y_1} f^{-1}(y) = f^{-1}(y_1) < g^{-1}(y_1) \leq x_2.$$

Notice that this shows $a < x_2$.

We have shown either

$$f(a) = \lim_{x \nearrow a} f(x) < g(x_2)$$

or

$$f(a) = \lim_{x \nearrow a} f(x) = g(x_2).$$

Either way,

$$f(a) = \lim_{x \nearrow a} f(x) \leq g(x_2) < \infty.$$

This was the main objective of this part.

We have also shown

$$y_1 = \lim_{x \nearrow a} f(x) = f(a).$$

At this point we can also show $s(a) < \infty$. In fact,

$$\begin{aligned}
s(a) &= \lim_{x \nearrow a} \int_0^x \sqrt{1 + f'(\xi)^2}\, d\xi \\
&= \int_0^{\epsilon} \sqrt{1 + f'(\xi)^2}\, d\xi + \lim_{y \nearrow y_1} \int_{f(\epsilon)}^{y} \sqrt{1 + \left(\frac{df^{-1}}{dy}(\eta)\right)^2}\, d\eta \\
&\leq \int_0^{\epsilon} \sqrt{1 + f'(\xi)^2}\, d\xi + \lim_{y \nearrow y_1} \int_{g(\epsilon)}^{y} \sqrt{1 + \left(\frac{dg^{-1}}{dy}(\eta)\right)^2}\, d\eta \\
&\leq \epsilon \sqrt{1 + f'(\epsilon)^2} + \frac{\pi}{2s(\epsilon)} \\
&< \infty.
\end{aligned}$$

(d) These assertions are verified in part **(c)** above.

We have not shown $y_1 = f(a) < g(x_2)$, but this can also be shown.

Exercise 1 *Assume* $y_1 = f(a) = g(x_2)$ *and get a contradiction as follows:*

(a) *Let*

$$t = \max\{g^{-1}(y) - f^{-1}(y) : \epsilon \le y \le g(x_2)\}.$$

Show t *is well-defined and is not attained at either endpoint, but there is some* y_* *with* $\epsilon < y_* < g(x_2)$ *for which*

$$t = g^{-1}(y_*) - f^{-1}(y_*).$$

(b) *Consider the translation* $h : [\epsilon - t, x_2 - t] \to [f(\epsilon), g(x_2)]$ *by*

$$h(x) = g(x + t).$$

Show this function also has an inverse $h^{-1} : [f(\epsilon), g(x_2)] \to [\epsilon - t, x_2 - t]$ *with*

$$h^{-1} \le f^{-1}.$$

(c) *Show* $h^{-1}(y_*) = f^{-1}(y_*)$*, and this implies*

$$-\frac{\dfrac{d^2h^{-1}}{dy^2}(y_*)}{\left[1 + \left(\dfrac{dh^{-1}}{dy}(y_*)\right)^2\right]^{3/2}} \le -\frac{\dfrac{d^2f^{-1}}{dy^2}(y_*)}{\left[1 + \left(\dfrac{df^{-1}}{dy}(y_*)\right)^2\right]^{3/2}},$$

which is a contradiction.

Problem 4 (one dimensional wave equation) Consider the initial/boundary value problem

$$\begin{cases} u_{tt} = u_{xx}, & 0 < t < 3/2,\ -1 < x < 2 \\ u(x,0) = x^2, & -1 \le x \le 3/2 \\ u_t(x,0) = -3x, & -1 \le x \le 3/2 \\ u(-1,t) = 1, & 0 \le t < 3/2 \\ u_x(-1,t) = 1 - t, & 0 < t < 3/2 \qquad \text{(former error } u_x(-1,t) = t - 1\text{)} \\ u(2,t) = 4, & 0 \le t < 3/2 \\ u_x(2,t) = h(t), 0 < t < 3/2. & \end{cases}$$

(a) Use the method of characteristics to find a solution $u \in C^2(W)$ where

$$W = \{(x,t) : 0 < t < 3/2,\ -1 + t < x < 2 - t\}.$$

(b) Use the method of characteristics to find a continuous extension $v \in C^2(W_-)$ of u to W_- where

$$W_- = \{(x,t) : 0 < t < 3/2,\ -1 < x < -1 + t\}$$

and v is a classical solution of the problem away from the singular line $x = -1+t$.

(c) Use the method of characteristics to determine an appropriate function $h = h(t)$ for the Cauchy data along $x = 2$ so that you can find a continuous extension $w \in C^2(W_+)$ of u to W_+ where

$$W_+ = \{(x,t) : 0 < t < 3/2,\ 2 - t < x < 2\}$$

and w is a classical solution of the problem away from the singular line $x = 2-t$.

(d) Plot your solution and the associated evolution of x^2.

Problem 5 (a solution of the heat equation) Consider

$$u(x,t) = e^{-a^2 t}\sin(ax)$$

for some $a > 0$.

(a) Show u satisfies the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

(b) Plot u as a function of t and x on the strip $[0,\pi/a]\times[0,\infty)$.

(c) Plot the evolution of $u_0(x) = \sin(ax)$ for $0 \le x \le x/a$ and $t > 0$.

(d) Use mathematical software to animate the evolution of u_0.

Problem 6 (comparison of qualitative properties of solutions of hyperbolic and parabolic PDE) List three (striking) qualitative differences between the evolution of Problem 5 part **(d)** above and that of Problem 2 part **(d)** of Assignment 6.

Problem 7 (a solution of the heat equation) Taking $a = 1$ in Problem 5 above, consider the initial/boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & (x,t)\in(0,\pi)\times(0,\infty) \\ \\ u(x,0) = \sin(x), & x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0, & t \ge 0 \end{cases}$$

for the heat equation.

(a) Show $u(x,t) = e^{-t}\sin x$ is a solution of this problem.

(b) Compute

$$\frac{d}{dt}\int_0^\pi u(x,t)\,dx.$$

(c) Show u is the unique solution of the problem in $C^2((0,\pi)\times(0,\infty)\cap C^1([0,\pi]\times[0,\infty))$. Hint(s): Let $\tilde{u}$ denote any solution of the problem. Compute

$$\frac{d}{dt}\int_0^\pi (\tilde{u}-u)^2\,dx.$$

Differentiate under the integral sign and use the PDE. Integrate by parts.

Problem 8 (fundamental solution of the heat equation) Consider $\Phi \in C^\infty(\mathbb{R} \times (0, \infty))$ given by

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

(a) Show

$$\Phi_t = \Phi_{xx} \qquad \text{for} \qquad (x,t) \in \mathbb{R} \times (0,\infty).$$

(b) Show

$$\lim_{t \nearrow \infty} \Phi(x,t) = 0.$$

(c) Show

$$\lim_{t \searrow 0} \Phi(x,t) = 0 \qquad \text{for} \qquad x \neq 0.$$

(d) Compute

$$\int_{-\infty}^{\infty} \Phi(x,t)\,dx.$$

Hint(s): Let I be the integral in question. Note that

$$\frac{I^2}{4} = \left(\int_0^\infty \Phi(x,t)\,dx \right) \left(\int_0^\infty \Phi(y,t)\,dy \right).$$

Use polor coordinates.

Problem 9 (Laplacian is of divergence form; Boas Problem 13.1.1) Recall that the **divergence** of a vector field $\mathbf{F} = (F_1, F_2, F_3) : U \to \mathbb{R}^3$ defined on an open subset $U \subset \mathbb{R}^3$ can be defined as

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial x_3}.$$

An **electrostatic field** $\mathbf{E} : U \to \mathbb{R}^3$ is a field with divergence satisfying

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where ρ is a spatially dependent function modeling **charge density** and ϵ_0 is a constant called the **permittivity** of free space. The **electrostatic potential** $\phi : U \to \mathbb{R}$ is defined up to a constant by the relation

$$\mathbf{E} = -D\phi.$$

(a) In general, the electrostatic potential is a solution of **Poisson's equation**, that is, Poisson's partial differential equation. Compute the divergence of a gradient to find the form of Poisson's equation.

(b) If $\rho \equiv 0$, corresponding to the absence of any electrical charges in the region U, then show the electrostatic potential satisfies **Laplace's equation**. Solutions of Laplace's PDE are called **harmonic functions**.

(c) Under what physical circumstances would one expect to model the electrostatic potential in a region by a non-constant solution of Laplace's equation?

Problem 10 (traveling waves; Boas Problem 13.1.2(a)) If $f \in C^2(\mathbb{R})$, then (show) $u(x,t) = f(x - ct)$ solves the wave equation (with the wave speed included).