Math 6702, Assignment 7

Laplace's Equation on a Rectangle

1. (Problem 6 on Exam 1) Consider the following boundary value problem for Laplace's equation on the rectangle $U = [0, L] \times [0, M]$ where L and M are positive numbers.

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0,\ u(L,y) = 0,\ u(x,M) = x(x-L),\ u(0,y) = 0 \end{cases} \tag{1}$$

   (a) Find all separated variables solutions $u(x,y) = A(x)B(y)$. Hint: You should obtain solutions of the form $u_j = c_j A_j(x) B_j(y)$ with $A_j'' = -\lambda_j A_j$ and $B_j = \lambda_j B_j$ for some positive increasing sequence

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$

   (b) Find a Fourier expansion of the function $g_3(x) = x(x-L)$ and choose the constants $c_1, c_2, c_3, \ldots$ appropriately so that

$$\sum_{j=1}^{\infty} c_j B_j(M) A_j(x) = g_3(x).$$

   (c) Take the specific values $L = 1$ and $M = 0.5$ and plot enough terms of

$$u(x,y) = \sum_{j=1}^{\infty} c_j A_j(x) B_j(y)$$

   to convince yourself (and me) that you have obtained a series solution for the problem.

Solution: The first thing I would do for this problem is draw the boundary values:

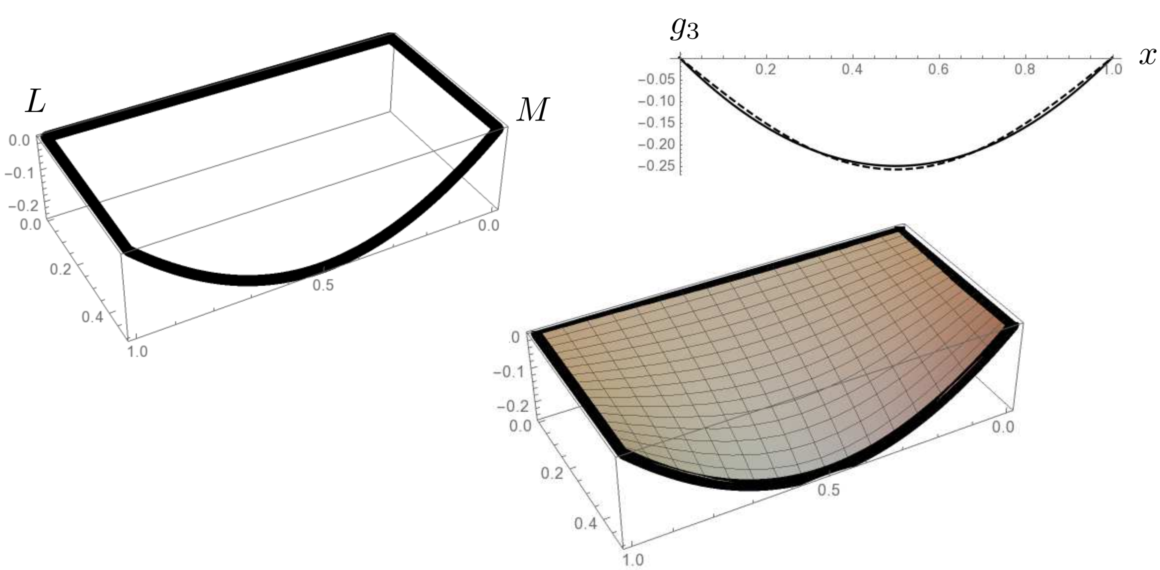

Figure 1: The boundary values (left). The approximation (via one term of the Fourier series) of the values along $y = M$ (right). The approximation of the solution (one term).

(a) Find all separated variables solutions $u(x, y) = A(x)B(y)$. Putting $u = A(x)B(y)$, we get

$$\frac{A''}{A} = -\frac{B''}{B} = -\lambda$$

For boundary conditions, we get $A(0)B(y) = 0$, $A(L)B(y) = 0$, $A(x)B(0) = 0$, and $A(x)B(M) = x(x - L)$ Assuming nontrivial solutions, the first three give $A(0) = 0$, $A(L) = 0$, and $B(0) = 0$. It is unclear what the last one gives, and in fact it does not give anything. This may not be so obvious, but it becomes obvious with experience: The first two conditions $A(0) = 0 = A(L)$ give the crucial boundary conditions for determining separated variables solutions, and **these are the ones you want**. So then you get $A_j(x) = \sin(j\pi x/L)$ for $j = 1, 2, 3, \ldots$ with $\lambda_j = (j\pi/L)^2$. For each λ_j you also get $B_j = \sinh(j\pi y/L)$ and $\tilde{B}_j = \cosh(j\pi y/L)$. Taking account of the last useful boundary condition, you can eliminate $\tilde{B}_j$, so you have separated variables solutions

$$u_j = c_j \sin\frac{j\pi x}{L} \sinh\frac{j\pi y}{L}.$$

Of course, these don't satisfy the last boundary condition, but the idea is that you can handle that in the end by superposition.

(b) Find a Fourier expansion of the function $g_3(x) = x(x-L)$ and choose the constants $c_1, c_2, c_3, \ldots$ appropriately so that

$$\sum_{j=1}^{\infty} c_j B_j(M) A_j(x) = g_3(x).$$

We write (as usual)

$$x(x-L) = \sum_{j=1}^{\infty} b_j \sin\frac{j\pi x}{L}.$$

Multiplying both sided by $\sin(k\pi x/L)$ and integrating from $x=0$ to $x=L$, we get

$$\int_0^L x(x-L)\sin\frac{k\pi x}{L}\,dx = \frac{L}{2}\,b_k.$$

$$\begin{aligned}
\int_0^L x^2 \sin\frac{k\pi x}{L}\,dx &= -x^2\,\frac{L}{k\pi}\cos\frac{k\pi x}{L}\Big|_{x=0}^{L} + \frac{2L}{k\pi}\int_0^L x\cos\frac{k\pi x}{L}\,dx \\
&= -L^2\,\frac{L}{k\pi}\cos(k\pi) + 2x\left(\frac{L}{k\pi}\right)^2 \sin\frac{k\pi x}{L}\Big|_{x=0}^{L} + 2\left(\frac{L}{k\pi}\right)^2\int_0^L \sin\frac{k\pi x}{L}\,dx \\
&= -\frac{L^3}{k\pi}\cos(k\pi) - 2\left(\frac{L}{k\pi}\right)^3 \cos\frac{k\pi x}{L}\Big|_{x=0}^{L} \\
&= -\frac{L^3}{k\pi}\cos(k\pi) - 2\left(\frac{L}{k\pi}\right)^3 (1-\cos(k\pi)),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^L x\sin\frac{k\pi x}{L}\,dx &= -x\,\frac{L}{k\pi}\cos\frac{k\pi x}{L}\Big|_{x=0}^{L} + \frac{2L}{k\pi}\int_0^L \cos\frac{k\pi x}{L}\,dx \\
&= -L\,\frac{L}{k\pi}\cos(k\pi) + \left(\frac{L}{k\pi}\right)^2 \sin\frac{k\pi x}{L}\Big|_{x=0}^{L} \\
&= -\frac{L^2}{k\pi}\cos(k\pi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
b_k &= \frac{2}{L}\left[-\frac{L^3}{k\pi}\cos(k\pi) - 2\left(\frac{L}{k\pi}\right)^3(1-\cos(k\pi)) + \frac{L^3}{k\pi}\cos(k\pi)\right] \\
&= -2\frac{2L^2}{k\pi}\cos(k\pi) - \frac{4L^2}{(k\pi)^3}(1-\cos(k\pi)) + \frac{2L^2}{k\pi}\cos(k\pi),
\end{aligned}$$

and

$$
\begin{aligned}
c_k &= \frac{1}{B_k(M)} b_k \\
&= \frac{1}{\sinh(k\pi M/L)} \left[-\frac{2L^2}{k\pi}\cos(k\pi) - \frac{4L^2}{(k\pi)^3}(1-\cos(k\pi)) + \frac{2L^2}{k\pi}\cos(k\pi) \right].
\end{aligned}
$$

I've plotted the approximation $\sum b_j \sin(j\pi x/L) \sim g_3(x)$ with one term on the right in Figure 1. The function g_3 is plotted with a solid curve, and the approximation is plotted with a dashed curve. Plotting the first three terms (two nonzero) gives an approximation which is quite close.

(c) Take the specific values $L = 1$ and $M = 0.5$ and plot enough terms of

$$u(x, y) = \sum_{j=1}^{\infty} c_j A_j(x) B_j(y)$$

to convince yourself (and me) that you have obtained a series solution for the problem. My plot is shown at the bottom in Figure 1. This is only one term, but it still looks pretty good because I've plotted the boundary with such thick curves.

2. Consider the tent function $T \in \text{Lip}[0, L]$ given by

$$T(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(L-x)/(L-a), & a \le x \le L. \end{cases}$$

Show $T \in W^1(0, L)$ has a weak derivative.

3. Let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be a partition of $[a, b]$. Show that if $f \in C^0[a, b]$ and for each $j = 1, 2, \ldots, k$, there is a function $f_j \in C^1[x_{j-1}, x_j]$ such that

$$f\Big|_{[x_{j-1}, x_j]} = f_j,$$

then $f \in W^1(a, b)$ has a weak derivative. Notice that such a function f also satisfies $f \in \text{Lip}[a, b]$.

Solution: The function $g : [a, b] \to \mathbb{R}$ by $g(x) = f_j'(x)$ for $x_{j-1} < x < x_j$, $j = 1, 2, \ldots, k$ is clearly an integrable function with

$$\int_{(a,b)} |g| = \sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} |f_j'(x)| \, dx.$$

Thus, we may take $\phi \in C_c^\infty(a, b)$ and consider

$$\int_{(a,b)} g\phi = \sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} f_j'(x)\phi(x) \, dx.$$

Integrating by parts on each subinterval, we find

$$\begin{aligned}
\int_{(a,b)} g\phi &= \sum_{j=1}^{k} \left[f_j(x)\phi(x)\Big|_{x=x_{j-1}}^{x_j} - \int_{x_{j-1}}^{x_j} f_j(x)\phi'(x) \, dx \right] \\
&= \sum_{j=1}^{k} [f_j(x_j)\phi(x_j) - f_j(x_{j-1})\phi(x_{j-1})] - \int_{(a,b)} f\phi' \\
&= [f_k(x_k)\phi(x_k) - f_k(x_{k-1})\phi(x_{k-1}) + f_{k-1}(x_{k-1})\phi(x_{k-1}) - \cdots \\
&\qquad\qquad + f_1(x_1)\phi(x_1) - f_1(x_0)\phi(x_0)] - \int_{(a,b)} f\phi' \\
&= [f_k(b)\phi(b) - f_1(a)\phi(a)] - \int_{(a,b)} f\phi' \\
&= -\int_{(a,b)} f\phi'.
\end{aligned}$$

This is what it means for g to be a weak derivative of f.

§4.9-10 Max/Min Problems

4. (4.9.2) Use the method of Lagrange multipliers to maximize the volume of a silo modeled by

$$V = \{(x, y, z) : x^2 + y^2 < r^2 \text{ and } 0 < z < h - m\sqrt{x^2 + y^2}\}$$

given that the total surface area of the structure is a fixed positive number A.

5. (4.10.10) Let $T(x, y, z) = y^2 + xz$ model temperature in the solid unit ball $B_r(0) = \{(x, y, z) : x^2 + y^2 + z^2 < r^2\}$ in $\mathbb{R}^3$ (extending continuously to the closure of the ball).
   (a) Find the highest and lowest temperatures on the circle $y = 0$, $x^2 + z^2 = 1$.
   (b) Find the highest and lowest temperatures on the boundary surface $x^2 + y^2 + z^2 = 1$.
   (c) Find the highest and lowest temperatures on the entire closure of the ball.

§5.3 Physical Quantities Involving Integrals

6. (5.3.1) Prove the parallel axis theorem: The moment of inertia I of a body about a given axis L is $I = I_m + Md^2$ where M is the mass of the body, I_m is the moment of intertia of the body about the axis parallel to L through the center of mass of the body, and d is the distance between the two axes.

7. (5.3.3-4) Let $W = \{(x, 0, 0) : 0 \le x \le \ell\}$ model a thin rod of length ℓ with density $\delta(x) = (1 - x/\ell)a + xb/\ell$ for some positive numbers a and b with $a < b$.
   (a) Find the mass of the rod (according to the model).
   (b) Compute the center of mass $(\bar{x}, 0, 0)$.
   (c) Compute the moment of intertia I_m of the rod about an axis perpendicular to the rod and passing through $(\bar{x}, 0, 0)$.
   (d) Compute the moment of intertia I of the rod about the z-axis.

8. (5.3.31) Consider the volume

$$V = \{(x, y, z) \in \mathbb{R}^3 : 1 \le z \le 1/\sqrt{x^2 + y^2}\}.$$

   (a) Compute

$$\int_V 1.$$

   (b) Show ∂V has infinite area. Hint: Show the area of the lateral portion of ∂V is greater than or equal to

$$\int_1^\infty \frac{1}{y}\, dy = \infty.$$

   (c) Note that $\int_V 1 < \infty$. Evaluate the following "prediction" of this model: If you fill the volume V with a finite amount/volume of paint, and then pour off the excess, you can paint an infinite area with a finite amount of paint.