Math 6702, Assignment 7

Laplace's Equation on a Rectangle

1. (Problem 6 on Exam 1) Consider the following boundary value problem for Laplace's equation on the rectangle $U = [0, L] \times [0, M]$ where L and M are positive numbers.

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0, \;\; u(L,y) = 0, \;\; u(x,M) = x(x-L), \;\; u(0,y) = 0 \end{cases} \tag{1}$$

   (a) Find all separated variables solutions $u(x, y) = A(x)B(y)$. Hint: You should obtain solutions of the form $u_j = c_j A_j(x)B_j(y)$ with $A_j'' = -\lambda_j A_j$ and $B_j = \lambda_j B_j$ for some positive increasing sequence

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$

   (b) Find a Fourier expansion of the function $g_3(x) = x(x-L)$ and choose the constants $c_1, c_2, c_3, \ldots$ appropriately so that

$$\sum_{j=1}^{\infty} c_j B_j(M) A_j(x) = g_3(x).$$

   (c) Take the specific values $L = 1$ and $M = 0.5$ and plot enough terms of

$$u(x,y) = \sum_{j=1}^{\infty} c_j A_j(x) B_j(y)$$

   to convince yourself (and me) that you have obtained a series solution for the problem.

Weak Derivatives

2. Consider the tent function $T \in \text{Lip}[0, L]$ given by

$$T(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(L-x)/(L-a), & a \le x \le L. \end{cases}$$

   Show $T \in W^1(0, L)$ has a weak derivative.

3. Let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be a partition of $[a, b]$. Show that if $f \in C^0[a, b]$ and for each $j = 1, 2, \ldots, k$, there is a function $f_j \in C^1[x_{j-1}, x_j]$ such that

$$f\Big|_{[x_{j-1}, x_j]} = f_j,$$

   then $f \in W^1(a, b)$ has a weak derivative. Notice that such a function f also satisfies $f \in \text{Lip}[a, b]$.

§4.9-10 Max/Min Problems

4. (4.9.2) Use the method of Lagrange multipliers to maximize the volume of a silo modeled by
$$V = \{(x, y, z) : x^2 + y^2 < r^2 \text{ and } 0 < z < h - m\sqrt{x^2 + y^2}\}$$
given that the total surface area of the structure is a fixed positive number A.

5. (4.10.10) Let $T(x, y, z) = y^2 + xz$ model temperature in the solid unit ball $B_r(0) = \{(x, y, z) : x^2 + y^2 + z^2 < r^2\}$ in $\mathbb{R}^3$ (extending continuously to the closure of the ball).
   (a) Find the highest and lowest temperatures on the circle $y = 0$, $x^2 + z^2 = 1$.
   (b) Find the highest and lowest temperatures on the boundary surface $x^2+y^2+z^2 = 1$.
   (c) Find the highest and lowest temperatures on the entire closure of the ball.

§5.3 Physical Quantities Involving Integrals

6. (5.3.1) Prove the parallel axis theorem: The moment of inertia I of a body about a given axis L is $I = I_m + Md^2$ where M is the mass of the body, I_m is the moment of intertia of the body about the axis parallel to L through the center of mass of the body, and d is the distance between the two axes.

7. (5.3.3-4) Let $W = \{(x, 0, 0) : 0 \le x \le \ell\}$ model a thin rod of length ℓ with density $\delta(x) = (1 - x/\ell)a + xb/\ell$ for some positive numbers a and b with $a < b$.
   (a) Find the mass of the rod (according to the model).
   (b) Compute the center of mass $(\bar{x}, 0, 0)$.
   (c) Compute the moment of intertia I_m of the rod about an axis perpendicular to the rod and passing through $(\bar{x}, 0, 0)$.
   (d) Compute the moment of intertia I of the rod about the z-axis.

8. (5.3.31) Consider the volume
$$V = \{(x, y, z) \in \mathbb{R}^3 : 1 \le z \le 1/\sqrt{x^2 + y^2}\}.$$
   (a) Compute
$$\int_V 1.$$
   (b) Show ∂V has infinite area. Hint: Show the area of the lateral portion of ∂V is greater than or equal to
$$\int_1^\infty \frac{1}{y}\, dy = \infty.$$
   (c) Note that $\int_V 1 < \infty$. Evaluate the following "prediction" of this model: If you fill the volume V with a finite amount/volume of paint, and then pour off the excess, you can paint an infinite area with a finite amount of paint.