Assignment 7:
Partial Differential Equations (heat equation)
Due Wednesday, March 8, 2023

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Problem 1 (A problem in geometric ODEs) This is the second in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Remember Problem 1 of Assignment 6. For this problem, we consider a function $f : (-a, a) \to \mathbb{R}$ for some $a > 0$ with $f \in C^2(-a, a)$ satisfying the initial conditions

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 0.$$

The **arclength** of the graph $\mathcal{G} = \{(x, f(x)) : x \in (-a, a)\}$ of f measured positive along $\mathcal{G}$ from $(0, 0)$ to $(x, f(x))$ for $x > 0$ and negative along $\mathcal{G}$ from $(0, 0)$ to $(x, f(x))$ for $x < 0$ can be expressed in terms of an integral obtained as a limit of the length of polygonal paths as follows: Let $x > 0$ and consider a **partition**

$$\mathcal{P} = \{x_0 = 0, x_1, x_2, \ldots, x_k = x\}$$

with $x_0 < x_1 < x_2 < \ldots < x_k$. For each such partition consider the polygonal path

$$\Gamma = \bigcup_{j=1}^{k} \{(1 - t)(x_{j-1}, f(x_{j-1})) + t(x_j, f(x_j)) : 0 \le t \le 1\}$$

(a) Draw a picture of the polygonal path Γ.

(b) Express the length of the polygonal path as a sum

$$\sum_{j=1}^{k} \text{length}(\Gamma_j)$$

where $\Gamma_j = \{(1-t)(x_{j-1}, f(x_{j-1})) + t(x_j, f(x_j)) : 0 \le t \le 1\}$ is the line segment connecting $((x_{j-1}, f(x_{j-1}))$ to $(x_j, f(x_j))$ for $j = 1, 2, \ldots, k$.

(c) Introduce appropriate difference quotients in order to obtain the arclength

$$s(x) = \lim_{\|\mathcal{P}\| \to 0\}} \sum_{j=1}^{k} \text{length}(\Gamma_j)$$

along Γ described above as an integral for $x \ge 0$.

(d) Modify your discussion of part **(c)** above to conclude that the same integral gives the negative arclength measured along Γ from $(0,0)$ to $(x, f(x))$ when $x < 0$.

Problem 2 (A problem in geometric ODEs) This is the third in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Assume the signed curvature $k = k(x)$ of the graph $\mathcal{G}$ of f considered in Problem 1 above is equal to the signed arclength $s(x)$.

(a) Find a first order system of ordinary differential equations satisfied by the three real valued functions $f = f(x)$, $s = s(x)$, and $w = f'(x)$.

(b) Combine the system you found in part **(a)** of this problem with the initial conditions from Problem 1 above, and find a numerical approximation of the solution of the resulting initial value problem (IVP). (Use mathematical software like Mathematica (`NDSolve`) or Matlab (`ODE45`).)

(c) Plot your numerical solution/approximation on some interval $(-a, a)$. (Use mathematical software like Mathematica or Matlab.)

(d) What interesting thing do you find? For example, what is the maximum value of the endpoint a you can use? What is

$$\lim_{x \nearrow a} f(x)?$$

Problem 3 (A problem in geometric ODEs) This is the fourth in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Use the existence and uniqueness theorem for ODEs to show the solution $f = f(x)$ you approximated in Problem 2 above has the following properties

(a) f exists and is unique on some maximal interval $(-a, a)$ for some $a > 0$.

(b) The solution f is odd: $f(-x) = -f(x)$.

(c) f is increasing and bounded on $(-a, a)$.

(d) There is a **finite height gradient blow-up** at $x = a$:

$$\lim_{x \nearrow a} f(x) < \infty \qquad \text{and} \qquad \lim_{x \nearrow a} f'(x) = +\infty.$$

Problem 4 (one dimensional wave equation) Consider the initial/boundary value problem

$$\begin{cases} u_{tt} = u_{xx}, & 0 < t < 3/2,\ -1 < x < 2 \\ u(x,0) = x^2, & -1 \le x \le 3/2 \\ u_t(x,0) = -3x, & -1 \le x \le 3/2 \\ u(-1,t) = 1, & 0 \le t < 3/2 \\ u_x(-1,t) = 1 - t, & 0 < t < 3/2 \qquad \text{(former error } u_x(-1,t) = t - 1\text{)} \\ u(2,t) = 4, & 0 \le t < 3/2 \\ u_x(2,t) = h(t), 0 < t < 3/2. \end{cases}$$

(a) Use the method of characteristics to find a solution $u \in C^2(W)$ where

$$W = \{(x,t) : 0 < t < 3/2,\ -1 + t < x < 2 - t\}.$$

(b) Use the method of characteristics to find a continuous extension $v \in C^2(W_-)$ of u to W_- where

$$W_- = \{(x,t) : 0 < t < 3/2,\ -1 < x < -1 + t\}$$

and v is a classical solution of the problem away from the singular line $x = -1+t$.

(c) Use the method of characteristics to determine an appropriate function $h = h(t)$ for the Cauchy data along $x = 2$ so that you can find a continuous extension $w \in C^2(W_+)$ of u to W_+ where

$$W_+ = \{(x,t) : 0 < t < 3/2,\ 2 - t < x < 2\}$$

and w is a classical solution of the problem away from the singular line $x = 2-t$.

(d) Plot your solution and the associated evolution of x^2.

Problem 5 (a solution of the heat equation) Consider

$$u(x,t) = e^{-a^2 t}\sin(ax)$$

for some $a > 0$.

(a) Show u satisfies the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

(b) Plot u as a function of t and x on the strip $[0, \pi/a] \times [0, \infty)$.

(c) Plot the evolution of $u_0(x) = \sin(ax)$ for $0 \le x \le \pi/a$ and $t > 0$.

(d) Use mathematical software to animate the evolution of u_0.

Problem 6 (comparison of qualitative properties of solutions of hyperbolic and parabolic PDE) List three (striking) qualitative differences between the evolution of Problem 5 part **(d)** above and that of Problem 2 part **(d)** of Assignment 6.

Problem 7 (a solution of the heat equation) Taking $a = 1$ in Problem 5 above, consider the initial/boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,\infty) \\ \\ u(x,0) = \sin(x), & x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0, & t \ge 0 \end{cases}$$

for the heat equation.

(a) Show $u(x,t) = e^{-t}\sin x$ is a solution of this problem.

(b) Compute

$$\frac{d}{dt}\int_0^\pi u(x,t)\,dx.$$

(c) Show u is the unique solution of the problem in $C^2((0,\pi) \times (0,\infty) \cap C^1([0,\pi] \times [0,\infty))$. Hint(s): Let $\tilde{u}$ denote any solution of the problem. Compute

$$\frac{d}{dt}\int_0^\pi (\tilde{u} - u)^2\,dx.$$

Differentiate under the integral sign and use the PDE. Integrate by parts.

Problem 8 (fundamental solution of the heat equation) Consider $\Phi \in C^\infty(\mathbb{R} \times (0,\infty))$ given by

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

(a) Show

$$\Phi_t = \Phi_{xx} \qquad \text{for} \qquad (x,t) \in \mathbb{R} \times (0,\infty).$$

(b) Show

$$\lim_{t \nearrow \infty} \Phi(x,t) = 0.$$

(c) Show

$$\lim_{t \searrow 0} \Phi(x,t) = 0 \qquad \text{for} \qquad x \neq 0.$$

(d) Compute

$$\int_{-\infty}^{\infty} \Phi(x,t)\, dx.$$

Hint(s): Let I be the integral in question. Note that

$$\frac{I^2}{4} = \left(\int_0^\infty \Phi(x,t)\, dx \right) \left(\int_0^\infty \Phi(y,t)\, dy \right).$$

Use polor coordinates.

Problem 9 (Laplacian is of divergence form; Boas Problem 13.1.1) Recall that the **divergence** of a vector field $\mathbf{F} = (F_1, F_2, F_3) : U \to \mathbb{R}^3$ defined on an open subset $U \subset \mathbb{R}^3$ can be defined as

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

An **electrostatic field** $\mathbf{E} : U \to \mathbb{R}^3$ is a field with divergence satisfying

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where ρ is a spatially dependent function modeling **charge density** and ϵ_0 is a constant called the **permittivity** of free space. The **electrostatic potential** $\phi : U \to \mathbb{R}$ is defined up to a constant by the relation

$$\mathbf{E} = -D\phi.$$

(a) In general, the electrostatic potential is a solution of **Poisson's equation**, that is, Poisson's partial differential equation. Compute the divergence of a gradient to find the form of Poisson's equation.

(b) If $\rho \equiv 0$, corresponding to the absence of any electrical charges in the region U, then show the electrostatic potential satisfies **Laplace's equation**. Solutions of Laplace's PDE are called **harmonic functions**.

(c) Under what physical circumstances would one expect to model the electrostatic potential in a region by a non-constant solution of Laplace's equation?

Problem 10 (traveling waves; Boas Problem 13.1.2(a)) If $f \in C^2(\mathbb{R})$, then (show) $u(x,t) = f(x - ct)$ solves the wave equation (with the wave speed included).