MATH 6702 Assignment 7 = Final Assignment
Due Wednesday April 30, 2021

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Multivariable Calculus

Green's Theorem (Chapter 6 Section 9 of Boas)

Problem 1 *You know Gauss' theorem (or the divergence theorem) in the plane which says that given a bounded C^1 open domain $U \subset \mathbb{R}^2$ in the domain of a vector field $\mathbf{v}$ we have*

$$\int_U \operatorname{div} \mathbf{v} = \int_{\partial U} \mathbf{v} \cdot \mathbf{n}.$$

Use Gauss' theorem to prove Green's theorem:

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{\partial U} \mathbf{v} \cdot T$$

where $\mathbf{v} = (P, Q)$ and T is the counterclockwise unit normal around ∂U.

Stokes' Theorem (Chapter 6 Section 11 of Boas)

Stokes' theorem states that if $\mathcal{S}$ is an oriented surface in $\mathbb{R}^3$ in the domain of a (differentiable) vector field $\mathbf{v}$ and having C^1 boundary $\partial\mathcal{S}$, then

$$\int_{\mathcal{S}} \operatorname{curl} \mathbf{v} \cdot N = \int_{\partial\mathcal{S}} \mathbf{v} \cdot T$$

where N is the unit normal orienting $\mathcal{S}$ and T is the counterclockwise unit tangent around $\partial\mathcal{S}$ with respect to N.

The following is not the most wonderful problem in the world, but it is kind of fun. If you've been following/picking up on what I've been saying about integration this semester, then it should be way too easy...even sort of juvenile. If it's not like this, then start back with the basics of integration and become an integration ninja.

Problem 2 *(Boas 6.11.16) According to Maxwell's equations (in the potential formulation) any* **magnetic field** $B : U \to \mathbb{R}^3$ *where* U *is a simply connected domain in* $\mathbb{R}^3$ *satisfies*

$$\operatorname{div} B = 0 \qquad \text{and} \qquad B = \operatorname{curl} A$$

where A *is the* **magnetic vector potential***. Observe that*

$$\begin{aligned} 0 &= \int_U \operatorname{div} B \\ &= \int_{\mathcal{S}} B \cdot N \qquad \textit{where } \mathcal{S} = \partial U \textit{ by the divergence theorem} \\ &= \int_{\mathcal{S}} \operatorname{curl} A \cdot N \\ &= \int_{\partial\mathcal{S}} A \cdot T \qquad \textit{by Stokes theorem.} \end{aligned}$$

If for every closed loop $\Lambda = \partial\mathcal{S}$ *we have*

$$\int_{\Lambda} A \cdot T = 0,$$

then A *is conservative. Therefore, there exists a potential function* ψ *with* $A = D\psi$*. Consequently,*

$$B = \operatorname{curl} A = \operatorname{curl} D\psi = \mathbf{0},$$

so **all magnetic fields are zero fields**. *(You can check by calculation that it's always true that the curl of a gradient always vanishes.) Find the error(s) in this lovely "proof." Incidentally, the divergence of a curl always vanishes too. We don't use that here, but it's good to know.*

I think we've pretty much covered (at some level) Chapter 4 (differentiation), Chapter 5 (integration), and Chapter 6 (vector analysis) of Boas. It would have been nice to go over Chapter 13 (PDE) in more detail, but I think with what we did do, there's nothing in Chapter 13 you can't read easily. If you read a couple pages of Boas from time to time, she'll keep you sharp on your applied math, so it's a good book to know about/have.

PDE

Laplace's Equation on a Rectangle

Consider again the boundary value problem for Laplace's equation on the rectangle $U = [0, L] \times [0, M]$ where L and M are positive numbers.

$$\begin{cases} \Delta u = 0, \\ u(x, 0) = 0,\ u(L, y) = 0,\ u(x, M) = x(x - L),\ u(0, y) = 0 \end{cases} \tag{1}$$

Problem 3 **(a)** *Find a function $g \in C^\infty([0, L] \times [0, M])$ such that*

$$g\Big|_{\partial U \setminus \{y=M\}} \equiv 0 \qquad \text{and} \qquad g(x, M) = g_3(x) = x(x - L).$$

Hint: Take a convex combination of $g_1 \equiv 0$ and g_3.

(b) *Let $w = u - g$ and write down the boundary value problem for Poisson's equation satisfied by w.*

(c) *Consider the Fourier basis*

$$\{\phi_{jk}\}_{j,k=1}^{\infty} \qquad \text{with} \qquad \phi_{jk}(x, y) = \sin \frac{j\pi x}{L} \sin \frac{k\pi y}{M}.$$

Expand $-\Delta g$ in a Fourier series

$$-\Delta g = \sum_{j,k=1}^{\infty} a_{jk} \phi_{jk}.$$

(d) *Let* w_{jk} *solve*

$$\begin{cases} \Delta w = \phi_{jk}, \\ w\big|_{\partial U} \equiv 0. \end{cases} \tag{2}$$

Hint: Compute $\Delta\phi_{jk}$.

(e) *Take the specific values* $L = 1$ *and* $M = 0.5$ *and plot enough terms of*

$$u(x, y) = w(x, y) + g(x, y) \qquad \text{where} \qquad w = \sum_{j,k=1}^{\infty} a_{jk} w_{jk}$$

to convince yourself (and me) that you have obtained a series solution for the problem. (Postscript/Note: The plots might look a little better with $L = 1$ *and* $M = 3$. *Or you could do* $L = 2$ *and* $M = 5$ *for example, but these kinds of aspect ratios may be easier for the visualization.)*

First Order Cauchy Problem

Problem 4 (a) *Solve the PDE*

$$xu_x - yu_y + (x^2 + y^2)u = x^2 - y^2 \qquad \text{on } U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}.$$

"Solve" here means "Find all possible C^1 *solutions." Your solution should depend on an arbitrary function which* **you** *will need to introduce. Knowing how to do that is part of the problem. (This is like if someone says: Solve* $x'' = 0$. *Then you know* $x = at + b$ *with two arbitrary constants* a *and* b*.)*

Hint(s): Consider the **characteristic field** $\mathbf{v} = (x, -y)$ *on the first quadrant* U. *Plot it with numerical software if necessary. Choose an appropriate non-characteristic curve.*

(b) *Solve the Cauchy problem:*

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u \\ u(x_1, x_2, 0) = g(x_1, x_2). \end{cases}$$

Problem 5 *(one dimensional wave equation) Solve the initial value problem for the wave equation:*

$$\begin{cases} u_{tt} = u_{xx} \text{ on } \mathbb{R} \times [0, \infty) \\ u(x, 0) = u_0(x) \\ u_t(x, 0) = v_0(x) \end{cases} \tag{3}$$

where $u_0 \in C^2(\mathbb{R})$ *and* $v_0 \in C^1(\mathbb{R})$ *to obtain d'Alembert's solution:*

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

Hint(s): Factor the operator $\Box u = u_{tt} - u_{xx}$ *as either*

$$(u_t - u_x)_t + (u_t - u_x)_x \qquad \text{or} \qquad (u_t + u_x)_t - (u_t + u_x)_x.$$

Then solve two first order PDEs with appropriate Cauchy conditions. Incidentally, the initial conditions in (3) are Cauchy conditions for the wave equation.

The 2D Heat Equation on $\mathcal{U} \subset \mathbb{R}^2$

Problem 6 *Derive the heat equation (carefully and from scratch) as it applies to a laminar domain* $\mathcal{U} \subset \mathbb{R}^2$. *Start by listing/identifying all the quantities you will use with their units. I'll start you out and give you a sort of outline to follow. When I put an ellipsis* ($\cdots$)*, this will mean there are details for you to fill in—probably lots of them.*

quantity	*identification*	*units*
$\theta_2 = \theta_2(x, y, t)$,	*areal or laminar heat energy density*	$[\theta_2] = \frac{[energy]}{L^2}$
$\vdots$		
Incidentally,	*energy has units of work*	$[energy] = [force]L = \frac{ML^2}{T^2}$
$\vdots$		
$\vec{\phi}_2 = \vec{\phi}_2$,	*laminar heat flux field*	$[\vec{\phi}_2] = \dots$
$\vdots$		
$\vdots$		
$u = u(x, y, t)$,	*temperature*	$[u] = [temperature]$
$Du = Du(x, y, t)$,	*temprature gradient*	$[Du] = \dots$
$\sigma = \sigma(x, y, u)$,	*specific heat capacity*	$[\sigma] = \dots$
$K_2 = K_2(x, y, u)$,	*laminar thermal conductivity*	$[K_2] = \dots$
$\vdots$		
$\vdots$		

Accounting of rate of change of total energy

$$\frac{d}{dt}\int_U \theta_2 = -\int_{\partial U} \vec{\phi}_2 \cdot \mathbf{n} + \int_U Q_2$$

...

Law of specific heat ...

Fourier's law ...

$$\frac{\partial}{\partial t}[\sigma \rho_2 u] = \operatorname{div}[K_2 Du] + Q_2.$$

...

Finally, taking $\sigma\rho_2 = K_2$ (constant) and setting $f = Q_2/K_2$,

$$u_t = \Delta u + f.$$

Problem 7 *Consider the heat equation* $u_t = k\Delta u + f$ *(with forcing) on*

$$B_1(0) \times [0, \infty) = \{(x, y, t) : x^2 + y^2 < 1 \text{ and } t \geq 0\}.$$

(a) *Let* $w(\xi, \eta, \tau) = u(\alpha\xi, \alpha\eta, \beta\tau)$ *where* α *and* β *are positive constants. Determine the domain of* w*, and compute*

$$w_\tau(\xi, \eta, \tau) \qquad \text{and} \qquad \Delta w(\xi, \eta, \tau) = w_{\xi\xi} + w_{\eta\eta}.$$

(b) *(scaling in time) Say you know how to solve* $w_\tau - \Delta w = f_0(\xi, \eta, \tau)$ *for any* $f_0 \in C^0(B_1(0) \times [0, \infty))$ *with a particular initial condition*

$$w(x, y, 0) = g_0(x, y)$$

and a homogeneous boundary condition

$$w\Big|_{x^2+y^2=1} = 0 \qquad \textit{for all time } \tau \geq 0.$$

Explain how to solve

$$\begin{cases} u_t - k\Delta u = f & \textit{on } B_1(0) \times [0, \infty) \\ u(x, y, 0) = g_0(x, y), & (x, y) \in B_1(0) \\ u\Big|_{x^2+y^2=1} = 0, & \textit{for all time } t \geq 0 \end{cases}$$

for $k \neq 1$ *by scaling in time. Hint: Use the idea of part (a).*

(c) *(scaling in space) If* $w_t = \Delta w$ *on* $W = B_5(0) \times [0, \infty)$ *find the PDE satisfied by* $u(x, y, t) = w(5x, 5y, t)$ *on* $B_1(0) \times [0, \infty)$.

(d) *(anisotropic heat diffusion) Find an appropriate domain on which to solve the heat equation* $w_t = \Delta w$ *which allows you to solve the anisotropic heat equation*

$$u_t = 3u_{xx} + 2u_{yy} \qquad \textit{on } B_1(0) \times [0, \infty).$$

Hint: Scale in space by different factors in different directions, i.e., anisotropically. If you know

$$u(x, y, 0) = u_0(x, y) \qquad \text{and} \qquad u\Big|_{x^2+y^2=1} = g(x, y, t),$$

then what are the corresponding boundary conditions for w*?*

The Gradient System

Consider the first order system of PDEs

$$\begin{cases} u_x = \phi \\ u_y = \psi \end{cases} \tag{4}$$

for a function $u \in C^1(\mathbb{R}^2)$ where ϕ and ψ are given functions in $C^0(\mathbb{R}^2)$. These are called the **gradient PDEs**. Notice the left side is the same as the Cauchy-Riemann equations.

Problem 8 (a) *How many unknowns are there in (4)? How many equations? What does this suggest from the linear algebra/specification of partials point of view?*

(b) *Show that if $u = u(x, y)$ is a solution of (4) with $u \in C^1(\mathbb{R}^2)$, then*

$$u(x, y) = u_1(x) + \int_0^y \psi(x, \eta)\, d\eta = u_2(y) + \int_0^x \phi(\xi, y)\, d\xi$$

where $u_1 \in C^1(\mathbb{R})$ with $u_1(x) = u(x, 0)$ and $u_2 \in C^1(\mathbb{R})$ with $u_2(y) = u(0, y)$.

(c) *Solve the* **Cauchy problems**

$$\begin{cases} u_x = \phi \\ u(0, y) = g_2(y) \end{cases} \qquad \text{and} \qquad \begin{cases} u_y = \psi \\ u(x, 0) = g_1(x) \end{cases}$$

for $u \in C^2(\mathbb{R}^2)$ where $g_1, g_2 \in C^1(\mathbb{R}^1)$, and show the solutions are unique.

(d) *Show that if $\phi, \psi \in C^1(\mathbb{R}^2)$ and $u \in C^1(\mathbb{R}^2)$ solves (4), then $u \in C^2(\mathbb{R}^2)$ and $\phi_y = \psi_x$. Conclude that*

$$\{Du \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2) : u \in C^1(\mathbb{R}^2)\}$$

the set of all gradients, i.e., pairs $(\phi, \psi) \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2)$ for which (4) is solvable satisfies

$$\{Du \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2) : u \in C^1(\mathbb{R}^2)\} \supset \{(\phi, \psi) \in C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2) : \phi_y = \psi_x\}.$$

Problem 9 *(Boas Chapter 6 sections 7-11) A vector field* $\mathbf{v} = (\phi, \psi)$ *on* $\mathbb{R}^2$ *is a* **gradient field** *or* **exact** *or* **conservative** *if there exists a* **potential function** $u : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\mathbf{v} = Du$.

(a) *Given any vector field* $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$, *extend* $\mathbf{v}$ *to a field* $\overline{\mathbf{v}} : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $\overline{\mathbf{v}}(x, y, z) = (v_1, v_2, 0)$. *Interpret the condition for* $\mathbf{v}$ *to be a gradient field from Problem 8(d) above in terms of the* **curl operator** *applied to* $\overline{\mathbf{v}}$.

(b) *What is a natural domain and codomain for the curl operator?*

(c) *Give a counterexample to the following assertion: If* $\mathbf{v} \in C^1(U)$ *and* $\operatorname{curl} \overline{\mathbf{v}} \equiv \mathbf{0}$ *on* U *then there exists a function* $u \in C^2(U)$ *such that* $Du = \mathbf{v}$. *Notice the domain* U *here is not necessarily all of* $\mathbb{R}^2$. *Hint: If* U *is simply connected, then the assertion does hold. (If this hint doesn't help do a literature/internet search for something like "nonexact curl free field" or "nonconservative curl free field.")*

Energy Estimates and Uniqueness

Problem 10 *In this problem energy estimates/identities are used to prove uniqueness for classical solutions of the three standard linear second order PDE: Laplace's equation, the heat equation, and the wave equation.*

(a) *Show that if* $u \in C^2(\overline{\mathcal{U}})$ *satisfies Laplace's equation, then*

$$\int_{\mathcal{U}} |Du|^2 = \int_{\partial\mathcal{U}} u Du \cdot \mathbf{n}.$$

(b) *Use part (a) to prove classical solutions* $u \in C^2(\overline{\mathcal{U}})$ *of the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g \end{cases}$$

for Poisson's equation with fixed/given inhomogeneities $f, g \in C^0(\overline{\mathcal{U}})$ *are unique. Hint: Let* $v \in C^2(\overline{\mathcal{U}})$ *be another solution and consider* $w = u - v$.

(c) *Use part (a) to prove classical solutions* $u \in C^2(\overline{\mathcal{U}})$ *of the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{U} \\ Du \cdot \mathbf{n}\big|_{\partial\mathcal{U}} = h & \end{cases}$$

for Poisson's equation with fixed/given inhomogeneities $f, h \in C^0(\overline{\mathcal{U}})$ *are unique up to a constant.*

(d) *Show that if* $u \in C^2(\overline{\mathcal{U}} \times [0,T))$ *satisfies the heat equation, then*

$$\frac{d}{dt}\int_{\mathcal{U}} u^2 = 2\int_{\partial\mathcal{U}} uDu \cdot \mathbf{n} - 2\int_{\mathcal{U}} |Du|^2.$$

(e) *Use part (d) to prove classical solutions* $u \in C^2(\overline{\mathcal{U}} \times [0,T))$ *for some* $T > 0$ *of the initial/boundary value problem*

$$\begin{cases} u_t = \Delta u + f & \text{on } \mathcal{U} \times (0,T) \\ u\big|_{t=0} = u_0 & \\ u\big|_{\mathbf{x}\in\partial\mathcal{U}} = g(\mathbf{x},t) & \end{cases}$$

for the forced heat equation with fixed/given inhomogeneities $f, u_0, g \in C^0(\overline{\mathcal{U}} \times [0,T))$ *are unique.*

(f) *Use part (d) to prove classical solutions* $u \in C^2(\overline{\mathcal{U}} \times [0,T))$ *for some* $T > 0$ *of the initial/boundary value problem*

$$\begin{cases} u_t = \Delta u + f & \text{on } \mathcal{U} \times (0,T) \\ u\big|_{t=0} = u_0 & \\ Du \cdot \mathbf{n}\big|_{\mathbf{x}\in\partial\mathcal{U}} = h(\mathbf{x},t) & \end{cases}$$

for the forced heat equation with fixed/given inhomogeneities $f, u_0, h \in C^0(\overline{\mathcal{U}} \times [0,T))$ *are unique.*

(g) *Show that if* $u \in C^2(\overline{\mathcal{U}} \times [0,T))$ *satisfies the wave equation, then*

$$\frac{d}{dt}\int_{\mathcal{U}} (u_t^2 + |Du|^2) = 2\int_{\partial\mathcal{U}} u_t Du \cdot \mathbf{n}.$$

(h) *Use part (g) to prove classical solutions* $u \in C^2(\overline{\mathcal{U}} \times [0,T))$ *for some* $T > 0$ *of the initial/boundary value problem*

$$\begin{cases} u_{tt} = \Delta u + f & \text{on } \mathcal{U} \times (0,T) \\ u\Big|_{t=0} = u_0 \\ u_t\Big|_{t=0} = v_0 \\ u\Big|_{\mathbf{x} \in \partial\mathcal{U}} = g(\mathbf{x}, t) \end{cases}$$

for the forced wave equation with fixed/given inhomogeneities $f, u_0, v_0, g \in C^0(\overline{\mathcal{U}} \times [0,T))$ *are unique.*