Assignment 6: Selected Solutions
Heat Equation
Due Friday, March 14, 2025

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Problem 1 (locally Lipschitz functions) A function $f : (a, b) \to \mathbb{R}$ is said to be **locally Lipschitz** and we write $f \in \text{Lip}_{loc}(a, b)$ if given any $\alpha, \beta \in \mathbb{R}$ with $a < \alpha < \beta < b$, there is some constant $M > 0$ for which

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \qquad \text{for all } x_1, x_2 \in [\alpha, \beta].$$

Show a convex function $f : (a, b) \to \mathbb{R}$ is locally Lipschitz.

Solution: Recall that the main defining property for a convex function like this is

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \quad \text{when} \quad a < x_1, x_2 < b, \quad 0 \leq t \leq 1. \tag{1}$$

Generally speaking one can assume also $x_1 \leq x_2$, because one can change the names of these points. There are two nice consequences of this condition that it may be useful to point out in isolation. Both are related to the following observation about convex combinations:

Given $x_1 \leq x \leq x_2$ with $x_1 < x_2$ there is a unique number

$$t = \frac{x - x_1}{x_2 - x_1}$$

so that $0 \leq t \leq 1$ and

$$x = (1-t)x_1 + tx_2.$$

Note also that

$$1 - t = \frac{x_2 - x}{x_2 - x_1}$$

so what this is saying is

$$\frac{x_2 - x}{x_2 - x_1}x_1 + \frac{x - x_1}{x_2 - x_1}x_2 = x$$

which is easy to just check. In particular, given three points $x_1 < x < x_2$ one has from the basic convexity condition (1)

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2). \tag{2}$$

This expression can be rearranged algebraically in several different ways to obtain various consequences. We have immediately that

$$(x_2 - x_1)f(x) \leq (x_2 - x)f(x_1) + (x - x_1)f(x_2).$$

This means

$$\frac{x_2 - x_1}{x - x_1}f(x) - \frac{x_2 - x}{x - x_1}f(x_1) \leq f(x_2). \tag{3}$$

This gives a lower bound for $f(x_2)$ which can be written as

$$f(x_1) + \frac{x_2 - x_1}{x - x_1}f(x) - \frac{x_2 - x_1}{x - x_1}f(x_1) \leq f(x_2)$$

or

$$f(x_1) + \frac{x_2 - x_1}{x - x_1}[f(x) - f(x_1)] \leq f(x_2).$$

where it may be noticed that the espression on the left gives the value $\ell(x_2)$ of an affine function ℓ evaluated at x with graph passing through $(x_1, f(x_1))$ with slope

$$\frac{f(x) - f(x_1)}{x - x_1}.$$

Alternatively, we can think of x as fixed and write (3) as

$$f(x) + \frac{x_2 - x}{x - x_1}f(x) - \frac{x_2 - x}{x - x_1}f(x_1) \leq f(x_2)$$

or

$$f(x) + \frac{x_2 - x}{x - x_1}[f(x) - f(x_1)] \leq f(x_2). \tag{4}$$

Thus, we have $\ell(x_2) \leq f(x_2)$ where ℓ is the same affine function with graph passing through $(x, f(x))$ and having the same slope

$$\frac{f(x) - f(x_1)}{x - x_1}.$$

We can call the first class of consequences "bounding." For example, in our problem let $a_1 = (a+\alpha)/2$ and $b_1 = (\beta+b)/2$. Notice that

$$a < a_1 < \alpha < \beta < b_1 < b.$$

If $\alpha \le x_1 \le x_2 \le \beta$, then applying the bounding condition (from below) in the form (4) with the three points $\alpha < x_1 \le x_2$ we get

$$f(x_1) + \frac{f(x_1) - f(\alpha)}{x_1 - \alpha}(x_2 - x_1) \le f(x_2). \tag{5}$$

This is illustrated on the left in Figure 1.

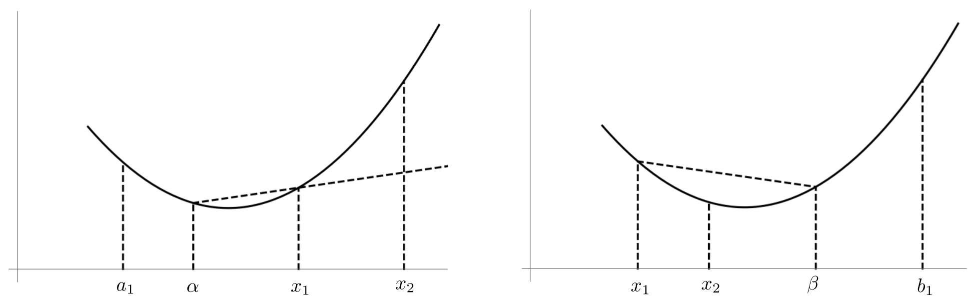

Figure 1: Lower bound for the value $f(x_2)$ (left). Upper bound for the value $f(x_2)$ (right.)

Returning to (2) we can also write

$$f(x) \le \ell(x) = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1} + \frac{f(x_2) - f(x_1)}{x_2 - x_1} x \tag{6}$$

so that $f(x)$ is bounded above by the affine function with graph passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$, that is the secant line on the right in Figure 1.

Taking the three points $x_1 \le x_2 < \beta$, the relation (6) giving a bound from above becomes

$$f(x_2) \le \frac{\beta f(x_1) - x_1 f(\beta)}{\beta - x_1} + \frac{f(\beta) - f(x_1)}{\beta - x_1} x_2 \tag{7}$$

which can also be "read off" from the illustration on the right in Figure 1 in one of the forms

$$f(x_2) \le f(x_1) + \frac{f(\beta) - f(x_1)}{\beta - x_1}(x_2 - x_1)$$

or

$$f(x_2) \le f(\beta) + \frac{f(\beta) - f(x_1)}{\beta - x_1}(x_2 - \beta).$$

Combining (5) and (7) we have

$$f(x_1) + \frac{f(x_1) - f(\alpha)}{x_1 - \alpha}(x_2 - x_1) \le f(x_2) \le f(x_1) + \frac{f(\beta) - f(x_1)}{\beta - x_1}(x_2 - x_1)$$

or

$$\frac{f(x_1) - f(\alpha)}{x_1 - \alpha}(x_2 - x_1) \le f(x_2) - f(x_1) \le \frac{f(\beta) - f(x_1)}{\beta - x_1}(x_2 - x_1)$$

which looks very promising for showing f satisfies some kind of Lipschitz continuity condition. The problem of course is that we need to bound below the quantity

$$m_1 = \frac{f(x_1) - f(\alpha)}{x_1 - \alpha}$$

by some negative constant M_1 and we need to bound from above the quantity

$$m_2 = \frac{f(\beta) - f(x_1)}{\beta - x_1}$$

by some positive constant M_2. We need both bounding constants $M_1 < 0 < M_2$ independent of x_1, and it would be nice to have the relation $-M_1 = M_2 = M$. This latter condition we can always ensure by taking a maximum of two numbers.

To obtain the basic bounds here we return to the general convexity condition (2) and observe a second general consequence which we can call "monotonicity of secant slopes." Specifically, we can rearrange (2) as

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x} \tag{8}$$

Which says that the slope of the secant line determined by the points $(x_1, f(x_1))$ and $(x, f(x))$ is no greater than the slope of the secant line determined by the points $(x, f(x))$ and $(x_2, f(x_2))$. Basically, the slopes of secant lines are increasing on the graph of a convex function. This property is illustrated in Figure 2.

Applying the relation (8) with the three points $a_1 < \alpha < x_1$ we have

$$\frac{f(\alpha) - f(a_1)}{\alpha - a_1} \le \frac{f(x_1) - f(\alpha)}{x_1 - \alpha}.$$

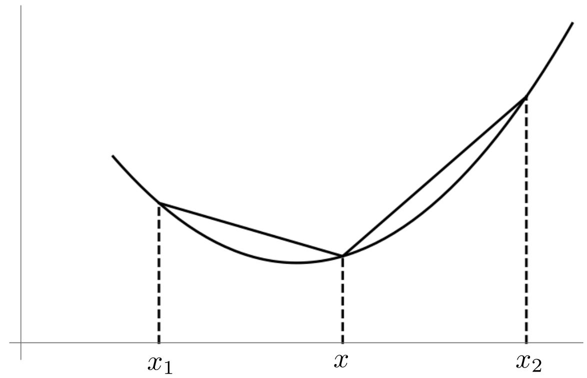

Figure 2: Monotonicity of slopes of secant lines for a convex function.

Note the number on the left is independent of x_1, which is a good thing. This number is not necessarily negative, but we can say

$$-\left|\frac{f(\alpha)-f(a_1)}{\alpha-a_1}\right| \leq \frac{f(\alpha)-f(a_1)}{\alpha-a_1}.$$

The number

$$-\left|\frac{f(\alpha)-f(a_1)}{\alpha-a_1}\right|$$

might still be zero, but we can obtain a negative lower bound as desired by taking

$$M_1 = -\left(\left|\frac{f(\alpha)-f(a_1)}{\alpha-a_1}\right| + 1\right).$$

Notice this is a strictly negative finite valued real number with

$$M_1(x_2-x_1) \leq f(x_2)-f(x_1).$$

Similarly, using the three points $x_2 < \beta < b_1$ we obtain from the monotonicity of slopes

$$\frac{f(\beta)-f(x_2)}{\beta-x_2} \leq \frac{f(b_1)-\beta}{b_1-\beta}$$

and

$$M_2 = \left|\frac{f(b_1)-\beta}{b_1-\beta}\right| + 1$$

is a finite positive real number for which

$$f(x_2) - f(x_1) \leq M_2(x_2 - x_1).$$

Thus, we have

$$M_1(x_2 - x_1) \leq f(x_2) - f(x_1) \leq M_2(x_2 - x_1).$$

Taking $M = \max\{-M_1, M_2\}$ gives

$$|f(x_2) - f(x_1)| \leq M(x_2 - x_1)$$

where M is a fixed number depending only on a, b, α, and β, and the condition holds whenever x_1 and x_2 satisfy $\alpha < x_1 \leq x_2 < \beta$. This is enough to conclude the locally Lipschitz assertion

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

for all x_1 and x_2 with $\alpha < x_1, x_2 < \beta$.

For the problems below let U be a bounded, open, and connected subset of $\mathbb{R}^n$ with ∂U a smooth hypersurface admitting a continuous outward unit normal field $\mathbf{n} : \partial U \to \mathbb{R}^n$.

Please also note the following result:

Theorem 1 (first law of vanishing integrals) If $f \in C^0(U)$ and

$$\int_{B_r(\mathbf{p})} f = 0 \qquad \text{for every ball with } \overline{B_r(\mathbf{p})} \subset U,$$

then $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in U$.

Problem 2 (thermal/heat energy density and thermal flux) Assume $T > 0$ and $\Theta \in C^1(U \times [0,T))$ is a function modeling the density of thermal energy in a spatial region modeled by U at time t so that the total thermal energy within a domain of integration $\Omega \subset\subset U$ at time $t \in [0,T)$ is given by

$$\int_{\mathbf{x}\in\Omega} \Theta(\mathbf{x},t).$$

Assume also that $\vec{\phi} \in C^0(U \times [0,T) \to \mathbb{R}^n)$ is a thermal flux field modeling the flux of thermal energy so that the rate of thermal energy crossing the oriented hypersurface $\mathcal{S}$ in the direction of the unit normal field $\mathbf{n} \in C^0(\mathcal{S} \to \mathbb{R}^n)$ is

$$\int_{\mathbf{x}\in\mathcal{S}} \vec{\phi}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}).$$

(a) Find the physical dimensions of Θ. Hint: The **physical dimensions** of velocity are "length per time," so that if $\dot{x}$ is a velocity, then

$$[\dot{x}] = \frac{L}{T}.$$

The physical dimensions of energy are the dimensions of force times length, and the dimensions of force are mass times acceleration, so if e is an energy

$$[e] = [\text{force}]L = \frac{ML^2}{T^2}.$$

(b) Find the physical dimensions of the thermal flux $\vec{\phi}$.

(c) Assuming the entire change in total thermal energy in a fixed region (modeled by) $\Omega \subset\subset U$ is the result of thermal flux across $\partial\Omega$, state a law of conservation of energy relating Θ and $\vec{\phi}$. Hint: Calculate the rate at which thermal energy exits Ω (in two different ways).

(d) Find the physical dimensions of your equation.

Solution:

(a) Find the physical dimensions of Θ. The thermal energy density is integrated over the medium of dimension n to obtain a total energy. This means

$$L^n[\Theta] = [\text{force}]L = \frac{ML^2}{T^2}$$

or

$$[\Theta] = \frac{M}{T^2 L^{n-2}}.$$

Notice that this means thermal energy density is scale invariant in two spatial dimensions.

(b) Find the physical dimensions of the thermal flux $\vec{\phi}$.

$$L^{n-1}[\vec{\phi}] = \frac{[\text{energy}]}{T} = \frac{ML^2}{T^3}$$

so

$$[\vec{\phi}] = \frac{M}{T^3 L^{n-3}}.$$

(c) Assuming the entire change in total thermal energy in a fixed region (modeled by) $\Omega \subset\subset U$ is the result of thermal flux across $\partial\Omega$, state a law of conservation of energy relating Θ and $\vec{\phi}$. Hint: Calculate the rate at which thermal energy exits Ω (in two different ways).

$$\frac{d}{dt}\int_{\Omega} \Theta = -\int_{\partial\Omega} \vec{\phi} \cdot \mathbf{n}.$$

(d) The dimensions of this equation are those of energy per time:

$$\frac{ML^2}{T^3}.$$

Problem 3 (a first heat equation) Consider generalizing the law of conservation of thermal energy from part **(c)** of Problem 2 above. Assume the change in thermal energy within a region $\Omega \subset\subset U$ is not entirely determined by the thermal flux across $\partial\Omega$ but is also affected by some "bulk" thermal change at a rate

$$\int_{\mathbf{x}\in\Omega} F(\mathbf{x}, t)$$

where $F \in C^0(U \times [0, T))$ and

$$[F] = \frac{[\text{energy}]}{TL^n} = \frac{M}{T^3 L^{n-2}}.$$

Such a function F is called a bulk/internal energy rate density or **thermal forcing**.

(a) Give two examples in which one expects to use a function $F < 0$ to model "bulk" thermal energy change in a region.

(b) Give two examples in which one expects to use a function $F > 0$ to model "bulk" thermal energy change in a region.

(c) Generalize your conservation law from part **(c)** of Problem 2 to account for "bulk" thermal energy change.

(d) Use the first law of vanishing integrals to derive a differential equation modeling thermal energy with a region modeled by U. Hint: Use the divergence theorem to write the thermal flux out of Ω as an integral over Ω.

Solution:

(a) Give two examples in which one expects to use a function $F < 0$ to model "bulk" thermal energy change in a region.

- If there is an endothermic chemical reaction taking place within the conducting material, then thermal energy is removed from the material by a means other than conduction, and this might be modeled by a negative bulk contribution to energy.
- In modeling heat conduction in a medium of space dimension $n = 1$ or $n = 2$ which is viewed as actually positioned in a three-dimensional physical space, the medium is accessible from "outside" and there can be an

extraction of thermal energy through the exposed accessible portion(s) of the medium. For example, if heat conduction in a thin wire is modeled with a heat equation corresponding to spatial dimension $n = 1$ and the wire is located in a bath of cold water or ice, the contact with the cold water can constitute an external extraction of thermal energy not taken account of through "internal" one-dimensional thermal flux.

There is no theoretical restriction on dimension here. For any spatial dimension n, one can model direct thermal extraction of energy from a "lateral" portion of a medium accessible from a higher dimensional ambient space. I do not know of any "real" physical applications of this theoretical observation that fall outside ambient spatial dimension $n+k = 3.$, so $n = 1$ or $n = 2$, but I'll guess there is something like this in general relativity where one can imagine thermal conduction in a three dimensional medium, say a galaxy or nebula, with access from some imagined four dimensional (or higher) ambient space.

(b) Give two examples in which one expects to use a function $F > 0$ to model "bulk" thermal energy change in a region.

- In the presence of an exothermic chemical reaction within the conducting medium being modeled.
- Exchange the bath of ice in the example for $F < 0$ above with a both of "hot" material "outside" the medium being modeled.

(c) Generalize your conservation law from part **(c)** of Problem 2 to account for "bulk" thermal energy change.

$$\frac{d}{dt}\int_\Omega \Theta = -\int_{\partial\Omega} \vec{\phi}\cdot\mathbf{n} + \int_\Omega F.$$

(d) Use the first law of vanishing integrals to derive a differential equation modeling thermal energy with a region modeled by U. Hint: Use the divergence theorem to write the thermal flux out of Ω as an integral over Ω.

$$\int_\Omega \left(\frac{\partial\Theta}{\partial t} + \operatorname{div}\vec{\phi} - F\right) = 0.$$

If this conservation law/relation holds for "nice" subdomains Ω, then

$$\frac{\partial\Theta}{\partial t} = -\operatorname{div}\vec{\phi} + F$$

assuming adequate regularity of course.

Problem 4 Prove the first law of vanishing integrals. Hints:

(a) Assume $f(\mathbf{p}) > 0$ and try to obtain a contradiction.

(b) Use the continuity of f at $\mathbf{p}$ to get an estimate from below on

$$\int_{B_r(\mathbf{p})} f.$$

Solution: If $f(\mathbf{p}) > 0$, then by continuity there is some $r > 0$ so that

$$|f(\mathbf{x}) - f(\mathbf{p})| < \frac{f(\mathbf{p})}{2} \qquad \text{for} \qquad \mathbf{x} \in B_r(\mathbf{p}).$$

This implies that for $\mathbf{x} \in B_r(\mathbf{p})$

$$f(\mathbf{x}) = f(\mathbf{p}) + f(\mathbf{x}) - f(\mathbf{p}) \geq f(\mathbf{p}) - |f(\mathbf{x}) - f(\mathbf{p})| > \frac{f(\mathbf{p})}{2} > 0.$$

Hence

$$\int_{B_r(\mathbf{p})} f \geq \frac{f(\mathbf{p})}{2}\omega_n r^n > 0.$$

This contradicts the hypothesis of the "law."

Thus we know $f(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in U$. Notice the hypothesis of the "law" applies also to the function $g = -f$ because

$$\int_{B_r(\mathbf{p})} g = -\int_{B_r(\mathbf{p})} f.$$

We conclude $g = -f \leq 0$ as well, or $f \geq 0$. This shows $f \equiv 0$ as required.

Problem 5 (A second heat equation) Your answer to part **(d)** of Problem 3 above should be a single partial differential equation for (or relating) two unknown quantities, the thermal energy density Θ and the thermal flux field $\vec{\phi}$.

(a) How many unknown real valued functions are there in your partial differential equation?

(b) The **law of specific heat** asserts that the thermal energy density in a substance is proportional to a physical quantity called **temperature** (or absolute temperature). Modeling the temperature with a function $u : U \times [0, T) \to \mathbb{R}$ and denoting the constant of proportionality[1] by c, the law of specific heat can be expressed as

$$\Theta = cu.$$

The quantity "temperature" is usually considered to have a new fundamental physical dimension denoted by T. Find the physical dimensions of specific heat capacity.

(c) **Fourier's law of heat conduction** asserts that the thermal flux field is proportional to the **spatial gradient** of the temperature. The concept of a spatial gradient applies to functions with no time dependence like solutions of Laplace's equation in which case the spatial gradient is the same as the usual gradient, that is the vector of partial derivatives of the function. For solutions $u : U \times [0, T) \to \mathbb{R}$ of **evolution equations** like the heat equation and wave equation the spatial gradient is the vector of partial derivatives with respect to **only the variables associated with the spatial domain** $U \subset \mathbb{R}^n$. In this context there is a special name for the vector

$$\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t} \right) \in \mathbb{R}^{n+1}.$$

This vector which is not used very often is called the **full space-time gradient**, and usually when one refers to "the gradient" he means the spatial gradient.

The constant of proportionality in Fourier's law of heat conduction is denoted by $-k$ where k is called the **thermal conductivity**.

[1]This constant is called **specific heat capacity**.

(i) Write down Fourier's law of heat conduction.

(ii) Find the physical dimensions of the conductivity, and

(iii) Discuss the sign of the constant of proportionality. In other words, what did Fourier intend to model with this relation?

(d) Substitute expressions involving the temperature u for the thermal energy density and the thermal flux field in your equation from part **(d)** of Problem 3 to obtain a single partial differential equation for the single real valued function $u \in C^2(U \times [0, T))$.

Solution:

(a) Starting with the equation

$$\frac{\partial \Theta}{\partial t} = -\operatorname{div} \vec{\phi} + F,$$

there are n unknown real valued functions $\phi_1, \phi_2, \ldots, \phi_n$ in the thermal flux and one energy density Θ for a total of $n+1$ unknown real valued functions.

(b) **specific heat**. With

$$\Theta = cu$$

and denoting $[u]$ by temp we have

$$[c] = \frac{[\Theta]}{\text{temp}} = \frac{M}{T^2 L^{n-2} \text{temp}}.$$

(c) **Fourier's law of heat conduction**.

(i) Write down Fourier's law of heat conduction.

$$\vec{\phi} = -kDu.$$

(ii) Find the physical dimensions of the conductivity.

$$[k] = \frac{[\vec{\phi}]}{[Du]} = \frac{M}{T^3 L^{n-3}} \bigg/ \frac{\text{temp}}{L} = \frac{M}{T^3 L^{n-4} \text{temp}}.$$

(iii) Discuss the sign of the constant of proportionality. In other words, what did Fourier intend to model with this relation?

Note that $Du(\mathbf{x})$ gives the direction of maximum temperature increase at the point $\mathbf{x}$. Thus, Fourier's law

$$\vec{\phi} = -kDu.$$

states heat flux should be in the direction of maximum decrease in temperature. In other words, at each point $\mathbf{x}$ one assumes for small $h > 0$ the temperature

$$u(\mathbf{x} + h\vec{\phi}(\mathbf{x}))$$

is **lower** than the temperature $u(\mathbf{x})$ so that the **thermal flux is from higher temperature to lower temperature**, and $\vec{\phi}(\mathbf{x})$ is chosen so that

$$\lim_{h \searrow 0} \frac{u(\mathbf{x} + h\vec{\phi}(\mathbf{x})) - u(\mathbf{x})}{h} = Du(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x})$$

takes the minimum or **most negative** value possible. By the Cauchy-Schwarz inequality

$$|Du(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x})| \leq |Du(\mathbf{x})| \, |\vec{\phi}(\mathbf{x})|$$

with equality if and only if $\vec{\phi}(\mathbf{x})$ is parallel to $|Du(\mathbf{x})|$. Thus for a minimum value, one should have

$$\vec{\phi}(\mathbf{x}) = -kDu(\mathbf{x})$$

for some $k > 0$. In principle, at this point one could take the constant k as a function of $\mathbf{x}$. Sometimes that is what one wants to do. One may even want to consider $k = k(\mathbf{x}, t, u)$ to depend on time and temperature as well. We have taken a simple assumption that the thermal conductivity k is constant throughout the medium.

(d) Substitute expressions involving the temperature u for the thermal energy density and the thermal flux field in your equation from part **(d)** of Problem 3 to obtain a single partial differential equation for the single real valued function $u \in C^2(U \times [0, T))$.

With the suggested substitutions the equation reads

$$(cu)_t = - \div (-kDu) + F$$

If c and k are constants, or even if c is time independent and k is spatially independent, then we get

$$cu_t = k\Delta u + F,$$

as always assuming adequate regularity.

Problem 6 (boundary values and special cases of the heat equation) This problem is based on your answer to part **(d)** of Problem 5 above. You should be able to write the equation you have obtained in the form

$$\mathcal{L}u = F \tag{9}$$

where $\mathcal{L} : C^2(U \times [0,T)) \to C^0(U \times [0,T))$ is called the **heat operator**.

(a) A solution $u : U \times [0,T) \to \mathbb{R}$ of the heat equation (9) is said to be **time independent** or an **equilibrium solution** if

$$\frac{\partial u}{\partial t} \equiv 0.$$

What conditions on the internal/bulk forcing function F are required for the existence of an equilibrium solution?

(b) If the conditions on F in part **(a)** above are satisfied and u is an equilibrium solution, what partial differential equation is satisfied by $w : U \to \mathbb{R}$ with $w(\mathbf{x}) = u(\mathbf{x}, t)$?

(c) If u is an equilibrium solution and $F \equiv 0$, what partial differential equation is satisfied by $w : U \to \mathbb{R}$ with $w(\mathbf{x}) = u(\mathbf{x}, t)$?

(d) The heat equation (9) by itself is highly **underdetermined**, i.e., the equation typically has many solutions. The determination of a unique solution usually requires the imposition of both **initial conditions** and **boundary conditions**.

(i) An initial condition for the heat equation (9) takes the form of an **initial temperature profile**

$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

where $u_0 : U \to \mathbb{R}$ is a given function. Find a solution of the initial value problem

$$\begin{cases} Lu = 0, & (\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = \sin(\omega_1 x_1)\sin(\omega_2 x_2)\cdots\sin(\omega_n x_n), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

for the heat equation on all of $\mathbb{R}^n$ with initial condition given by a product of periodic functions. Hint: Look for a solution of the form $u(\mathbf{x}, t) = A(\mathbf{x})B(t)$ for a function $A : \mathbb{R}^n \to \mathbb{R}$ of an appropriate form.

(ii) When U is a bounded open subset of $\mathbb{R}^n$, then a (typical) boundary condition specifies some combination of **temperature and thermal flux**:

$$au(\mathbf{x},t) + bDu(\mathbf{x},T)\cdot\mathbf{n} = g(\mathbf{x},t) \qquad \text{for} \qquad \mathbf{x}\in\partial U$$

where a and b are constants. Let $L > 0$. Solve the initial/boundary value problem for the unforced heat equation:

$$\begin{cases} \mathcal{L}u = 0, & (x,t)\in(0,L)\times(0,\infty) \\ u(x,0) = \sin(2\pi x/L), & 0 < x < L \\ u(0,t) = 0 = u(L,t), & t > 0. \end{cases}$$

Hint: Watch the entertaining video:

`https://tomrocksmaths.com/2022/11/10/`
`oxford-calculus-how-to-solve-the-heat-equation/`

(iii) Let $L > 0$. Solve the initial/boundary value problem for the unforced heat equation:

$$\begin{cases} \mathcal{L}u = 0, & (\mathbf{x},t)\in(0,L)\times(0,\infty) \\ u(\mathbf{x},0) = 2+\cos(2\pi x/a), & 0 < x < L \\ u_x(0,t) = 0 = u_x(L,t), & t > 0. \end{cases}$$

Hint: Subtract from u an appropriate equilibrium solution.

Problem 7 (fundamental solution of Laplace's equation when $n = 2$ and $n = 1$)

(a) Show that if $\Phi : \mathbb{R}^2\backslash\{\mathbf{0}\} \to \mathbb{R}$ is a solution of $\Delta\Phi = 0$ with $\Phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ for some $\phi \in C^2(0,\infty)$, then

$$\lim_{r\searrow 0} \int_{B_r(\mathbf{0})} \Phi = 0.$$

(b) Show that if $\Phi : \mathbb{R}^1\backslash\{0\} \to \mathbb{R}$ is a solution of $\Delta\Phi = 0$ with $\Phi(x) = \phi(|x|)$ for some $\phi \in C^2(0,\infty)$, then

$$\lim_{r\searrow 0} \int_{B_r(0)} \Phi = 0.$$

Solution:

(a) According my solution of Problem 4 of Assignment 4, such a function must have values given by $\Phi(x, y) = c_1 + c_2 \ln\sqrt{x^2 + y^2}$. Therefore,

$$\int_{B_r(\mathbf{0})} \Phi = \pi r^2 c_1 + 2\pi c_2 \int_0^r t \ln t\, dt.$$

Clearly,

$$\lim_{r\searrow 0} \pi r^2 c_1 = 0.$$

We consider then

$$\left| \int_0^r t \ln t\, dt \right| = -\int_0^r t \ln t\, dt$$

for $r < 1$. Recall that

$$\ln t = \int_1^t \frac{1}{x}\, dx \geq -\frac{1-t}{t}$$

for $0 < t < 1$. Therefore,

$$\left| \int_0^r t \ln t\, dt \right| \leq \int_0^r t\frac{1-t}{t}\, dt \leq r.$$

Therefore,

$$\lim_{r\searrow 0} \int_{B_r(\mathbf{0})} \Phi = 0.$$

(b) In this case, we must have $\Phi(x) = c|x|$ for some $c \in \mathbb{R}$, and

$$\left| \int_{B_r(0)} \Phi \right| = |c| \left| \int_{-r}^{r} |x| \, dx \right| = |c| r^2.$$

Clearly then

$$\lim_{r \searrow 0} \int_{B_r(0)} \Phi = 0.$$

Problem 8 (fundamental solution of Laplace's equation when $n = 2$) Let

$$\Phi : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$$

denote a non-constant axially symmetric solution of Laplace's equation with $\Phi(\mathbf{x}) = 0$ when $|\mathbf{x}| = 1$. Find the unique constant $C \in \mathbb{R}$ for which

$$\Delta u = -f \qquad \text{for all } f \in C_c^3(\mathbb{R}^2)$$

where

$$u(\mathbf{x}) = C \int_{\xi\in\mathbb{R}^2} \Phi(\xi) f(\mathbf{x} - \xi).$$

Solution: Since Φ is integrable on all of $\mathbb{R}^2$, we know the convolution $u = \Phi * f$ is differentiable with

$$\Delta u = \Phi * \Delta f.$$

We denote derivatives with respect to a variable of integration with a superscript so

$$\text{div}^\xi D\Phi(\mathbf{x} - \xi) = -\Delta\Phi(\mathbf{x} - \xi) \qquad \text{or} \qquad D^\eta f(\mathbf{x} - \eta) = -Df(\mathbf{x} - \eta).$$

Initially, we start with $\Phi(\mathbf{x}) = c_1 + c_2 \ln |\mathbf{x}|$ in accord with the solution of Problem 4 of Assignment 4. The suggested normalization $\Phi(\mathbf{x}) = 0$ when $|\mathbf{x}| = 1$ gives $c_1 = 0$, so the suggested consideration of the function Ψ with $\Psi(\mathbf{x}) = C\Phi(\mathbf{x}) = Cc_2 \ln |\mathbf{x}|$ may be reduced to the consideration of the case $c_2 = 1$.

It is suggested that there is a unique value of the constant C for which the condition $\Delta u = -f$ holds for all $f \in C_c^3(\mathbb{R}^2)$. As another preliminary observation we can see that the constant $C = 0$ cannot be such a value. Thus, we may assume $C \neq 0$.

We note also that for f fixed, the problem $\Delta u = -f$ on all of $\mathbb{R}^n$ without growth restrictions does not have a unique solution. In particular, given any solution $u \in C^3(\mathbb{R}^2)$, the function with values $u(\mathbf{x}) + \mathbf{v} \cdot \mathbf{x} + b$ is also a solution for any $\mathbf{v} \in \mathbb{R}^2$ and $b \in \mathbb{R}$. And these are not the only solutions simply because the PDE $\Delta u_0 = 0$ on all of $\mathbb{R}$ has many more solutions. For example, $u_0 \in C^\infty(\mathbb{R}^2)$ with

$$u_0(x, y) = a_1 e^x \cos y + a_2 e^x \sin y$$

has $\Delta u_0 \equiv 0$, so the function(s) with values

$$u_0(x, y) = \mathbf{v} \cdot (x, y) + b + a_1 e^x \cos y + a_2 e^x \sin y$$

are also solutions. And one can take the real part of any entire complex differentiable function to find many more solutions.

As in the higher dimensional case, in order to work away from the singularity of Φ we delete a neighborhood $B_r(\mathbf{0})$ of $\mathbf{0}$ and compute

$$A(r) = C \int_{\mathbb{R}^2 \setminus \overline{B_r(\mathbf{0})}} \ln|\xi| \ \Delta f(\mathbf{x} - \xi)$$

with the intention of determining a nice form for the limit

$$C\Phi * \Delta f(\mathbf{x}) = C \int_{\xi \in \mathbb{R}^2} \Phi(\xi) \Delta f(\mathbf{x} - \xi) = \lim_{r \searrow 0} A(r).$$

For a fixed $\mathbf{x}$, the function $h \in C^1(\mathbb{R}^2)$ by $h(\xi) = \Delta f(\mathbf{x} - \xi)$ also has compact support, so there is some $R > 0$ so that

$$A(r) = C \int_{\xi \in B_R(\mathbf{0}) \setminus \overline{B_r(\mathbf{0})}} \ln|\xi| \ \Delta f(\mathbf{x} - \xi).$$

More generally, the functions $f_{\mathbf{x}} \in C^3(\mathbb{R}^2)$ by $f_{\mathbf{x}}(\xi) = f(\mathbf{x} - \xi)$ and $g \in C^2(\mathbb{R}^2 \to \mathbb{R}^2)$ by $g(\xi) = Df(\mathbf{x} - \xi)$ also have compact support which we can assume to be a subset of $B_R(\mathbf{0})$. We will use these conditions on support below.

We proceed with the product rule and the divergence theorem:

$$\begin{aligned}
A(r)/C &= \int_{\xi \in B_R(\mathbf{0}) \setminus \overline{B_r(\mathbf{0})}} \left[-\mathrm{div}^{\xi} \left(\ln|\xi| \ Df(\mathbf{x} - \xi) \right) + \left(\frac{\xi}{|\xi|^2} \cdot Df(\mathbf{x} - \xi) \right) \right] && (10) \\
&= \int_{\xi \in \partial B_r(\mathbf{0})} \ln|\xi| \ Df(\mathbf{x} - \xi) \cdot \frac{\xi}{|\xi|} + \int_{\xi \in B_R(\mathbf{0}) \setminus \overline{B_r(\mathbf{0})}} \frac{\xi}{|\xi|^2} \cdot Df(\mathbf{x} - \xi) \\
&= \ln r \int_{\xi \in \partial B_r(\mathbf{0})} Df(\mathbf{x} - \xi) \cdot \frac{\xi}{r} \\
&\qquad + \int_{\xi \in B_R(\mathbf{0}) \setminus \overline{B_r(\mathbf{0})}} \left[-\mathrm{div}^{\xi} \left(f(\mathbf{x} - \xi) \frac{\xi}{|\xi|^2} \right) + f(\mathbf{x} - \xi) \mathrm{div} \left(\frac{\xi}{|\xi|^2} \right) \right] \\
&= \ln r \int_{\xi \in \partial B_r(\mathbf{0})} Df(\mathbf{x} - \xi) \cdot \frac{\xi}{r} + \frac{1}{r} \int_{\xi \in \partial B_r(\mathbf{0})} f(\mathbf{x} - \xi). && (11)
\end{aligned}$$

We have used in (10) the calculation of the gradient of $\ell(\mathbf{x}) = \ln|\mathbf{x}|$ with

$$D\ell(\mathbf{x}) = \frac{1}{|\mathbf{x}|} \ D|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|^2},$$

and in (11) we used the fact that the vector field $\mathbf{x}/|\mathbf{x}|^2$, that is the radial field of magnitude $1/|\mathbf{x}|$, is divergence free in two dimensions.

Starting from (11) we hae two limits to calculate. Recall that $f, f_{\mathbf{x}} \in C_c^3(\mathbb{R}^3)$ so we can write

$$|Df(\mathbf{x}-\xi)| \leq \|f\|_{C^1(\mathbb{R}^2)}$$

for all $\xi \in \mathbb{R}^2$ and some fixed finite non-negative number $\|f\|_{C^1(\mathbb{R}^2)}$. Observe then for $r < 1$

$$\left| \ln r \int_{\xi \in \partial B_r(\mathbf{0})} Df(\mathbf{x}-\xi) \cdot \frac{\xi}{r} \right| \leq -\ln r \int_{\xi \in \partial B_r(\mathbf{0})} \|f\|_{C^1(\mathbb{R}^2)} = -2\pi r \ln r \|f\|_{C^1(\mathbb{R}^2)}.$$

It follows that the first term/integral in (11) satisfies

$$\lim_{r \searrow 0} \ln r \int_{\xi \in \partial B_r(\mathbf{0})} Df(\mathbf{x}-\xi) \cdot \frac{\xi}{r} = 0.$$

The remaining integral can be expressed in terms of an average value:

$$\frac{1}{r} \int_{\xi \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\xi) = \frac{2\pi}{2\pi r} \int_{\xi \in \partial B_r(\mathbf{0})} f(\mathbf{x}-\xi) = 2\pi \frac{1}{\mathcal{H}^1(\partial B_r(\mathbf{x}))} \int_{B_r(\mathbf{x})} f.$$

This quantity limits to $2\pi f(\mathbf{x})$. Thus, we conclude

$$\lim_{r \searrow 0} \frac{1}{C} A(r) = 2\pi f(\mathbf{x}).$$

That is to say,

$$\Delta u(\mathbf{x}) = \Phi * \Delta f(\mathbf{x}) = \lim_{r \searrow 0} \int_{\mathbb{R}^2 \setminus \overline{B_r(\mathbf{0})}} \ln |\xi| \ \Delta f(\mathbf{x}-\xi) = 2\pi C f(\mathbf{x}).$$

Thus, if we want $\Delta u(\mathbf{x}) = -f(\mathbf{x})$, we should take

$$C = -\frac{1}{2\pi}$$

and observe that the "real" fundamental solution in $n = 2$ is (or should be)

$$\Phi(\mathbf{x}) = -\frac{1}{2\pi} \ln |\mathbf{x}|.$$

While this conclusion is correct and this is the main important piece of information one is intended to "discover" in the problem, there is a little bit of ambiguity in the way the question is posed. We were supposed to find a "unique" value for the constant C, but we made a choice of the constant c_2, namely $c_2 = 1$ which was essentially arbitrary. Thus, technically speaking, according to the strict wording of the problem, one can take any nonzero value for c_2, and then conclude the constant Cc_2 should take the unique value $-1/(2\pi)$. There is no properly unique value for the constant C as the problem is stated, but rather

$$C = -\frac{1}{2\pi c_2}$$

where $\Phi(\mathbf{x}) = c_2 \ln |\mathbf{x}|$ and $c_2 \neq 0$.

Problem 9 Solve the BVP for Laplace's equation

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in B_1(0,0) \subset \mathbb{R}^2 \\ u(\mathbf{x}) = x_1^2, & \mathbf{x} = (x_1, x_2) \in \partial B_1(0,0) \end{cases}$$

using a Green's function.

Solution: In order to find a Green's function one can seek a Green corrector $u_c : B_1(\mathbf{0}) \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u_c = 0, & \text{on } B_1(\mathbf{0}) \\ \\ u_c(\mathbf{x}) = -\dfrac{1}{2\pi} \ln |\mathbf{x} - \xi|, & \mathbf{x} \in \partial B_1(\mathbf{0}). \end{cases}$$

As in the higher dimensional case, one can consider also the fundamental solution with singularity translated to the Kelvin transform $\xi/|\xi|^2$ of ξ at least when $|\xi| \neq 0$. That is,

$$u(\mathbf{x}) = -\frac{1}{2\pi} \ln \left| \mathbf{x} - \frac{\xi}{|\xi|^2} \right|.$$

When $|\mathbf{x}| = 1$, we have

$$\left| \mathbf{x} - \frac{\xi}{|\xi|^2} \right|^2 = \frac{1}{|\xi|^2}(|\xi|^2 + 2\mathbf{x} \cdot \xi + |\mathbf{x}|^2) = \frac{|\mathbf{x} - \xi|^2}{|\xi|^2}.$$

We conclude that for $|\mathbf{x}| = 1$,

$$u(\mathbf{x}) = -\frac{1}{2\pi} \ln \left(\frac{|\mathbf{x} - \xi|}{|\xi|} \right) = \Phi(\mathbf{x} - \xi) + \frac{1}{2\pi} \ln \frac{1}{|\xi|}$$

and the function $u_c \in C^\infty(\mathbb{R}^2 \backslash \{\xi/|\xi|^2\}$ with values

$$u_c(\mathbf{x}) = -\frac{1}{2\pi} \ln \left| \mathbf{x} - \frac{\xi}{|\xi|^2} \right| - \frac{1}{2\pi} \ln |\xi| = -\frac{1}{2\pi} \ln \left| |\xi|\mathbf{x} - \frac{\xi}{|\xi|} \right|$$

is the Green corrector. Thus, the Green's function for $B_1(\mathbf{0})$ satisfies

$$G(\mathbf{x}, \xi) = \Phi(\mathbf{x} - \xi) + \frac{1}{2\pi} \ln \left| |\xi|\mathbf{x} - \frac{\xi}{|\xi|} \right| = \frac{1}{2\pi} \ln \left(\frac{|\ |\xi|^2\mathbf{x} - \xi\ |}{|\xi|\ |\mathbf{x} - \xi|} \right)$$

for $|\xi| \neq 0$. For the purposes of integration on $B_1(\mathbf{0})$, these values are adequate, though one can define also $G(\mathbf{x}, \mathbf{0}) \equiv 0$.

To solve the boundary value problem for u posed above, we consider the alternative problem

$$\begin{cases} \Delta v = -2 = -\Delta u_0, & \mathbf{x} \in B_1(0,0) \subset \mathbb{R}^2 \\ v\big|_{\partial B_1(0,0)} \equiv 0 \end{cases}$$

for Poisson's equation where $u_0(x, y) = x^2$. This problem has solution $v \in C^2(\overline{B_1(\mathbf{0})}$ with

$$v(\mathbf{x}) = 2 \int_{\xi \in B_1(\mathbf{0})} G(\mathbf{x}, \xi) = \frac{1}{\pi} \int_{\xi \in B_1(\mathbf{0})} \ln \left(\frac{|\, |\xi|^2 \mathbf{x} - \xi \,|}{|\xi| \; |\mathbf{x} - \xi|} \right).$$

The solution of the original problem is the function $u \in C^2(\overline{B_1(\mathbf{0})})$ with values $u(x, y) = v(x, y) + x^2$as indicated on the right in Figure 3.

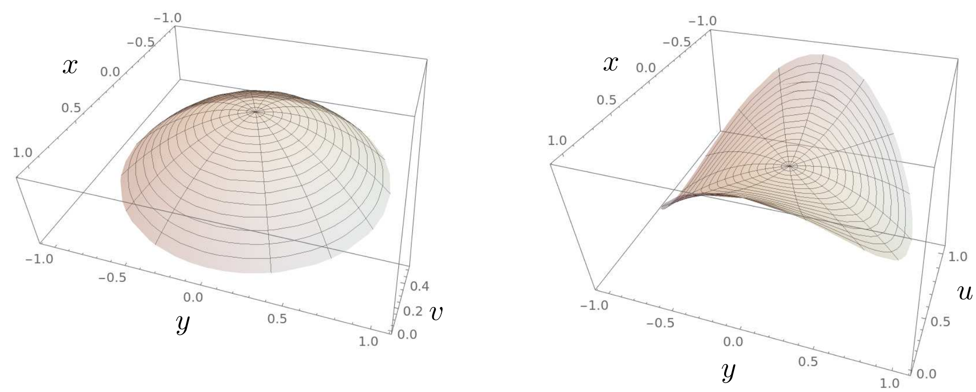

Figure 3: Graph of a harmonic function ($\Delta u = 0$) on the unit disk with $u(x, y) = x^2$ when $x^2 + y^2 = 1$ (right). On the left is the associated solution of Poisson's equation $\Delta v = -2$ with homogeneous boundary values.

Changing variables with $\xi = (r\cos\theta, r\sin\theta)$ and writing the integral defining v in terms of an iterated integral in polar coordinates we find

$$\begin{aligned}
v(x,y) &= \frac{1}{\pi}\int_0^1 r \int_0^{2\pi} \ln\left(\frac{|(rx-\cos\theta, ry-\sin\theta)|}{|(x-r\cos\theta, y-r\sin\theta)|}\right) d\theta\, dr \\
&= \frac{1}{\pi}\int_0^1 r \int_0^{2\pi} \ln\sqrt{\frac{r^2(x^2+y^2)-2r(x\cos\theta+y\sin\theta)+1}{x^2+y^2-2r(x\cos\theta+y\sin\theta)+r^2}}\, d\theta\, dr \\
&= \frac{1}{2\pi}\int_0^1 r \int_0^{2\pi} \ln\left(\frac{r^2(x^2+y^2)-2r(x\cos\theta+y\sin\theta)+1}{x^2+y^2-2r(x\cos\theta+y\sin\theta)+r^2}\right) d\theta\, dr.
\end{aligned}$$

This expression is easier to plot using mathematical software. A plot of the graph of the function $v \in C^2(\overline{B_1(\mathbf{0})})$ satisfying $\Delta v = -2$ with homogeneous boundary conditions is indicated on the left in Figure 3.

Note: The point of this problem is to give a formula for the solution in terms of a Green's function, however, I gave such simple boundary conditions that this particular problem is easy to solve without a Green's function. If one takes a look at the graph of the function v in Figure 3, it looks like an elliptic (circular) paraboloid. One knows, on the other hand, an elliptic paraboloid with circular cross-section and formula $v(x,y) = a(x^2+y^2)$ has $\Delta v = 4a$, so taking $a = -1/2$ gives

$$v(x,y) = -\frac{1}{2}(x^2+y^2)$$

which gives the values of v. Then $u(x,y) = v(x,y) + x^2$, so

$$v(x,y) = \frac{1}{x}(x^2-y^2).$$

Again, this is because I picked the boundary inhomogeneity to be so simple. This kind of thing won't usually work.

Problem 10 (hanging slinky) Note that the tension force at each material point $X(h)$ in the hanging slinky has magnitude $m(h)g$ where g is an appropriate gravitational acceleration constant, e.g., $g = 9.8$ m/s^2 and $m(h)$ is the mass of the portion of the slinky hanging below $X(h)$. Use this observation to model and graphically illustrate the observed physical configuration of the hanging slinky.

Solution: I'm going to use MKS units here as suggested by the gravitational constant above. I'm also going to use some form of my modeling assumption from Problem 10 of Assignment 5 and some measurements associated with Problem 1 of Assignment 4.

From my solution of Problem 10 of Assignment 4 I model the force by

$$F = \gamma(\sigma' - 1)$$

where γ is some material constant I don't have a value for at the moment.

In accord with my solution of Problem 1 of Assignment 3 the mass of the slinky below $\sigma(h)$ should be modeled by $m(h) = \rho_0(h_0 - h)$. Using the suggestion stated in Problem 10 above, I have an ordinary differential equation for σ:

$$\gamma(\sigma' - 1) = \rho_0 g(h_0 - h) \qquad \text{or} \qquad \sigma' = \frac{\rho_0 g}{\gamma} h_0 + 1 - \frac{\rho_0 g}{\gamma} h. \tag{12}$$

This is a very easy ODE to integrate:

$$\sigma(h) = \left(\frac{\rho_0 g}{\gamma} h_0 + 1 \right) h - \frac{\rho_0 g}{2\gamma} h^2.$$

I have taken $\rho_0 = M/h_0$ where M is the total mass and h_0 is the stacked/equilibrium height, so this can also be written as

$$\sigma(h) = \left(\frac{Mg}{\gamma} + 1 \right) h - \frac{Mg}{2h_0\gamma} h^2.$$

I mentioned in my solution of Problem 1 of Assignment 1 that I measured h_0 to be approximately 51 mm = 0.051 m. I do not have a measurement for the exact mass at the moment, but I found a reference suggesting the value should be about 0.218 kg, and this seems about right. I can check that later with a lab scale.

This gives me values for all the constants except the elasticity γ. I do have, however, a measured value for the total length $\sigma(h_0)$ of the hanging slinky at about

$$L = 40 + 5/8 \text{ inches} = 1.032 \text{ m}.$$

That measurement comes from data Nicholas Vellenga and I collected in my office. With this value, we can choose the constant γ for a perfect fit at the endpoint(s) with

$$\gamma(\sigma(h_0) - h_0) = \gamma(L - h_0) = \frac{Mg}{2}h_0 \qquad \text{or} \qquad \gamma \doteq 0.05554.$$

With this value we obtain a formula for σ and we can compare the prediction with the collected data. A comparison in terms length of stretch measured in inches as a function of coil number is indicated in Figure reffig4.

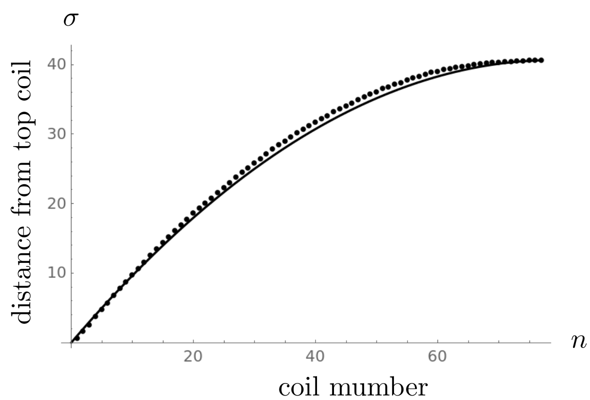

Figure 4: Comparison of stretch for a hanging slinky and a first model prediction.

There are some serious shortcomings of this model, but we can say we have fulfilled some of the features of mathematical modeling in physical science:

1. There is some **understanding** behind the parameters and equations in the model.

2. The model is capable of **prediction**.

3. The prediction has been **compared to measurement** of the physical system.

In short, we are not just "fitting data" which is what often passes for mathematical modeling but is actually something else. Ruling out blind "data fitting," a good standard for mathematical modeling is the following:

> One can say that a mathematical model accurately models a physical phenomenon if the agreement of the prediction of the model is as good as the precision of measurement. That is, the magnitude of the error is at least as small as the known error in measurement of the physical system.

We have clearly not achieved this level of accuracy, which indicates there are fundamental things we either have not incorporated in the model, do not yet understand, or have incorrectly applied in the modeling. In this case all three of these sources of error are probably present making for what one can clearly identify as a "crude" mathematical model. Specifically,

1. We have not taken account of the error associated with the first coil thickness. This is somehthing we know about but have not incorporated.

2. The basic shape of the curve is not correct, indicating that we do not have the constituitive force relation correct. Something more complicated is happening with the force that we do not yet understand.

3. Determination of the constant γ by using the matching of one point is not ideal and probably does not give a very accurate value. Of course, the fact that the basic shape is incorrect strongly suggests one should understand the physical system better—perhaps the introduction of a single elasticity constant is an error—before attempting to improve the determination of such a constant. But if one did have a nominally "correct" model, then using a least squares fit incorporating all the data to determine unknown constants would be a more reasonable thing to do.