**Math 6702, Assignment 6**

First Order Linear PDE

Consider the first order linear PDE

$$au_x + bu_y + cu = f \tag{1}$$

for a function $u \in C^1(U)$ where $U$ is an open subset of $\mathbb{R}^2$ and the coefficients satisfy $a, b, c, f \in C^0(\overline{U})$. We wish to consider several aspects of the study of this equation. For these exercises, let us assume $U$ contains the segment $I = \{(x, 0) : x_1 \leq x \leq x_2\}$ on the $x$-axis.

1. Assume that for each $f_0 \in C^0(\overline{U})$, you can find a function $v \in C^1(\overline{U})$ for which

$$\begin{cases} av_x + bv_y + cv = f_0 \\ v\big|_I \equiv 0. \end{cases} \tag{2}$$

   Show that you can then solve

$$\begin{cases} au_x + bu_y + cu = f \\ u\big|_I \equiv u_0 \end{cases} \tag{3}$$

   for any $f \in C^1(\overline{U})$ and $u_0 = u_0(x)$ with $u_0 \in C^1[x_1, x_2]$. This is called the **Cauchy problem** for (1). Hint(s): Draw a picture showing how the domain $U$ and the segment $I$ might look. Find an extension of $u_0$ to all of $\mathbb{R}$.

2. Let $x_0 \in (x_1, x_2)$ and consider a $C^1$ path $\mathbf{r} : (-1, 1) \to U$ with $\mathbf{r}(0) = (x_0, 0)$.
   (a) Draw a picture of how $U$ and the image of $\mathbf{r}$ might look assuming $\mathbf{r}'(0) \neq 0$.
   (b) Compute the derivative of the composition

$$\frac{d}{dt}[u \circ \mathbf{r}(t)] \qquad \text{where} \qquad u \in C^1(\overline{U}).$$

   You will need to use the chain rule (Chapter 5 section 4 of Boas).
   (c) Compare your result from the computation in part (b) to the PDE (1). Give a condition on the path $\mathbf{r}$ of the form

$$\mathbf{r}'(t) = \underline{\hspace{3cm}} \tag{4}$$

   which allows you to relate the computation in part (b) to the PDE.
   (d) Under what circumstances does there exist a unique solution to (4)?

§4.8 Max/Min Problems

3. (4.8.1) Let $f \in C^2(a, b)$ and $x_0 \in (a, b)$. Use the Taylor approximation formula to show that if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f$ has a local minimum at $x = x_0$.

4. (4.8.2) Let $f \in C^2(U)$ where $U$ is open and $(x_0, y_0) \in U \subset \mathbb{R}^2$.

   (a) Use the second order Taylor approximation formula to show that if $Du(x_0, y_0) = 0$, $f_{xx}(x_0, y_0) > 0$, $f_{yy}(x_0, y_0) > 0$, and $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) > [f_{xy}(x_0 y_0)]^2$, then $f$ has a local minimum at $(x, y) = (x_0, y_0)$.

   (b) Use the second order Taylor approximation formula to show that even if $Du(x_0, y_0) = 0$, $f_{xx}(x_0, y_0) > 0$, and $f_{yy}(x_0, y_0) > 0$, it may be the case that $f$ does not have a local minimum at $(x, y) = (x_0, y_0)$. (Give an explicit example.)

   (c) Is if possible that $Du(x_0, y_0) = 0$, $f_{xx}(x_0, y_0) > 0$, and $f_{yy}(x_0, y_0) > 0$, and $f$ has a local maximum at $(x, y) = (x_0, y_0)$?

   (d) An $n \times n$ matrix $M$ is **positive definite** if

$$M\mathbf{x} \cdot \mathbf{x} \geq 0 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

   and equality holds only if $\mathbf{x} = \mathbf{0} \in \mathbb{R}^n$. This condition on a matrix is also expressed by writing "$M > 0$." The **Hessian matrix** of $u$ is given by

$$D^2u = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

   Show $D^2u$ is positive definite if and only if $f_{xx} > 0$, $f_{yy} > 0$, and $\det D^2 f = f_{xx}f_{yy} - f_{xy}^2 > 0$.

5. (4.8.16) Let $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ be a set of data points which are expected to satisfy an affine relation $y = mx + b$.

   (a) Let

$$\epsilon = \sum_{j=1}^{k} [y_k - (mx_k + b)]^2$$

   be the sum of the squares of the errors associated with each data point. Minimize $\epsilon = \epsilon(m, b)$. The resulting affine function is called the **least squares fit**.

   (b) Find the least squares fit for $(-1, -2)$, $(0, 0)$, and $(1, 3)$.

   (c) Plot the points from part (b) along with the affine least squares fit.

## §5.2 Area and Volume Integrals

6. (5.2.9) Let $A = \{(x, y) \in \mathbb{R}^2 : 0 < y < 2\chi_{(0,\pi/2)}(x) + 4\chi_{(\pi/2,\pi)}(x)\}$. Calculate

$$\int_A \sin x.$$

The function $\chi_E : \mathbb{R}^n \to \mathbb{R}$ is called the **characteristic function** of the set $E \subset \mathbb{R}^n$. It has values given by

$$\chi_E(\mathbf{x}) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

7. (5.2.11) Let $B = \{(x, y) : x^2 < y < 2x + 8\}$. Calculate

$$\int_A x$$

8. (5.2.48,50) Let $V$ be the volume in the first octant bounded by the cone $z^2 = x^2 - y^2$ and the plane $x = 4$.

   (a) Compute

   $$\int_V 1.$$

   (b) If $V$ models a solid object with density $\delta(x, y, z) = z$, determine the mass of the object according to the model.