Assignment 6 = Exam 2: Partial Differential Equations (the wave equation) Due Wednesday, March 1, 2023

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Problem 1 (A problem in geometric ODEs) This is the first in a series of problems designed to help you produce a picture of an interesting curve whose (signed) curvature is given by arclength along the curve (and review what you (might) need to know about ODEs).

Let $a, b \in \mathbb{R}$ with $a < b$ and consider a function $f \in C^2(a, b)$. Given a point $x_0 \in (a, b)$,

(a) Find the formula for the tangent line to $\mathcal{G} = \{(x, f(x)) : x \in (a, b)\}$ at the point $(x_0, f(x_0))$ and plot/draw a picture illustrating this tangent line.

(b) Assuming $f'(x_0) = 0$ and $f''(x_0) \neq 0$, find the equation of the **osculating circle** to the graph of f at $(x_0, f(x_0))$. This circle should pass through $(x_0, f(x_0))$ and be given locally near this point by the graph of a function g satisfying

- **(i)** $g \in C^2(x_0 - \epsilon, x_0 + \epsilon)$ for some $\epsilon > 0$,
- **(ii)** $g'(x_0) = f'(x_0)$, and
- **(iii)** $g''(x_0) = f''(x_0)$.

(c) What is the maximum possible value of the tolerance ϵ in part **(b)(i)** above?

(d) Repeat part **(b)** without the assumption $f'(x_0) = 0$. Note that **(b)(ii)** should be replaced with $g'(x_0) = f'(x_0)$. You might also want to replace **(b)(i)** with $g \in C^2(x_0 - \epsilon, x_0 + \delta)$ for some $\epsilon, \delta > 0$ and pose (and answer) an appropriate version of part **(c)**.

Problem 2 (A solution of the wave equation) Consider

$$u(x,t) = \cos(at)\sin(ax)$$

for some $a > 0$.

(a) Show u satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

(b) Plot u as a function of t and x on the strip $[0, \pi/a] \times [0, \infty)$.

(c) Plot the evolution of $u_0(x) = \sin(ax)$ for $0 \leq x \leq \pi/a$ and $t > 0$.

(d) Use mathematical software to **animate** the evolution of u_0.

Problem 3 (A solution of the wave equation) Taking $a = 1$ in Problem 2 above, consider the (incomplete) initial/boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \sin(x), & x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0, & t \geq 0 \end{cases}$$

for the wave equation

(a) Show $w(x,t) = \cos t \sin x$ is a solution of this problem.

(b) Compute

$$\frac{\partial w}{\partial t}(x,0).$$

(c) Find another solution v satisfying

$$\frac{\partial v}{\partial t}(x,0) = 2\sin x.$$

Hint: Modify the time dependent factor in w.

Problem 4 (wave speed) In Chapter 13 Section 4 of Boas the wave equation is given as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $c > 0$ is called the **wave speed**.

(a) Given a solution $u \in C^2((a,b) \times (0,T)) \cap C^1([a,b] \times [0,T))$ of this PDE, define a function v by

$$v(x,\tau) = u(x, \tau/c).$$

Find the PDE satisfied by v.

(b) What is the (natural) domain for the function v defined in part **(a)**?

(c) If you can solve the initial/boundary value problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial \tau^2} = \dfrac{\partial^2 v}{\partial x^2}, & (x,\tau) \in (a,b) \times (0,\infty) \\ v(x,0) = f(x), & x \in [a,b] \\ \dfrac{\partial v}{\partial \tau}(x,0) = v_0(x), & x \in [a,b], \\ v(a,\tau) = v(b,\tau) = 0, & \tau \geq 0, \end{cases}$$

what initial/boundary value problems for (1) can you solve simply by scaling in time, and what are the solutions for those problems in terms of the solutions $v = v(x,\tau)$?

(d) Specialize the construction of part **(c)** above to the case when $(a,b) = (0,\pi)$, $f(x) = \sin x$, and $v_0(x) \equiv 0$. Plot the corresponding solution/evolution of (1) and explain the "physical" significance of the wave speed c.

Problem 5 (one dimensional wave equation on all of $\mathbb{R}$ with smooth Cauchy data) Solve the initial value problem for the wave equation:

$$\begin{cases} u_{tt} = u_{xx} \text{ on } \mathbb{R} \times [0, \infty) \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases} \tag{2}$$

where $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$ to obtain d'Alembert's solution:

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

Hint(s): Factor the wave operator $\Box u = u_{tt} - u_{xx}$ as either

$$(u_t - u_x)_t + (u_t - u_x)_x \qquad \text{or} \qquad (u_t + u_x)_t - (u_t + u_x)_x.$$

Then solve two first order PDEs with appropriate Cauchy conditions. Incidentally, the initial conditions in (2) are Cauchy conditions for the wave equation. The wave operator is also called the d'Alembertian.

Problem 6 (uniqueness of solutions for the wave equation) Show d'Alembert's solution

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi$$

obtained in Problem 5 above is the unique solution u of the initial/boundary value problem (2) when $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$.

Problem 7 (regularity of solutions of the wave equation) Looking (just) at d'Alembert's solution for the wave equation, what regularity on the functions u_0 and v_0 is necessary (or natural) to make sense of the formula?

Problem 8 (non-smooth solutions of the wave equation) Animate the evolution associated with d'Alembert's solution when $u_0(x) = \chi_{(-1,1)}(x)(1 - |x|)$ and $v_0 \equiv 0$.

Problem 9 How would you modify d'Alembert's solution to give a solution of Boas' version (1) of the wave equation with wave speed c?

Problem 10 Solve the initial/boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,\infty) \\[2ex] u(x,0) = \sin(2x) + \sin(3x), & x \in [0,\pi] \\[2ex] \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in [0,\pi], \\[2ex] u(0,t) = u(\pi,t) = 0, & t \geq 0, \end{cases}$$