MATH 6702 Assignment 6
Due Monday April 19, 2021

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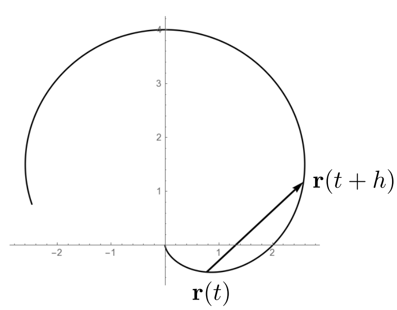
Multivariable Calculus

Problem 1 *Recall that a* **path** *connecting two points* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{R}^n$ *is a continuous function* $\mathbf{r} : [a, b] \to \mathbb{R}^n$ *such that* $\mathbf{r}(a) = \mathbf{p}$ *and* $\mathbf{r}(b) = \mathbf{q}$.

(a) *Given a path connecting* $\mathbf{p}$ *to* $\mathbf{q}$, *prove there is a path* $\gamma : [0, 1] \to \mathbb{R}^n$ *connecting* $\mathbf{p}$ *to* $\mathbf{q}$ *with* $\gamma(0) = \mathbf{p}$ *and* $\gamma(1) = \mathbf{q}$.

(b) *Use mathematical software to produce a drawing like the one below with a secant vector connecting two points on a parameterized curve. Be sure to indicate the formula for the curve you have illustrated.*



(c) *Explain why the derivative*

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h},$$

if it exists, is a vector with norm giving the instantaneous rate of change of position along the curve with respect to t. Hint: Your answer should have the notion of **average speed** *in it somewhere.*

(d) *Recall that a path* $\mathbf{r}$ *connecting two points is* C^1 *if the coordinate functions* $r_1, r_2, \ldots, r_n$ *are* C^1*. Show that if there is a* C^1 *path connecting two points, then there is a* **unit speed** *path connecting the two points. This means, we can assume* $|\mathbf{r}'(t)| = 1$ *for all* t*.*

Problem 2 *Let* $u : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$u(x,y) = \begin{cases} xy/(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Show the first order partials of u *both exist at every point of* $\mathbb{R}^2$*, but* u *is not continuous at* $(0,0)$*.*

Differentiability

A function $u : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^n$ and $\mathbf{p} \in U$ is **differentiable** at $\mathbf{p}$ if there is a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{w} \to \mathbf{0}} \frac{u(\mathbf{p}+\mathbf{w}) - u(\mathbf{p}) - L(\mathbf{w})}{|\mathbf{w}|} = 0. \tag{1}$$

The linear map $L : \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of u at $\mathbf{p}$ and is denoted by $du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$.

Problem 3 *Let* $u : U \to \mathbb{R}$ *be differentiable at* $\mathbf{p} \in U$*.*

(a) *show the first partial derivatives* $D_j u(\mathbf{p})$ *exist for* $j = 1, 2, \ldots, n$*.*

(b) *Express the linear function* $L : \mathbb{R}^n \to \mathbb{R}$ *for which (1) holds in terms of the* **gradient vector**

$$Du(\mathbf{p}) = (D_1 u(\mathbf{p}), D_2 u(\mathbf{p}), \ldots, D_n u(\mathbf{p})).$$

(c) *Let $U = (0,1) \times (0,1) \subset \mathbb{R}^2$ and consider the specific function $u : U \to \mathbb{R}$ by*

$$u(x,\xi) = \begin{cases} x(1-\xi), & 0 \le x \le \xi \\ (1-x)\xi, & \xi \le x \le 1. \end{cases}$$

Determine the points in U at which u is differentiable.

(d) *Let u be the specific function given in the last part of this problem. Reexpress u in the form*

$$u(x,\xi) = \begin{cases} u_1(x,\xi), & 0 \le \xi \le x \\ u_2(x,\xi), & x \le \xi \le 1. \end{cases}$$

(e) *What can you say about the regularity of the specific function u from the previous two parts? Hint: You can start by showing $u \in C^0(\overline{U})$. You can also find some subdomains U_1 and U_2 of U on which the functions u_1 and u_2 are C^∞.*

Note: The funtion u given in the last two parts of this problem is (up to a scaling) the Green's function for the 1-D Laplacian $\Delta u = u''$.

PDE

Green's Function for Laplacian

Assume for the following problem that $\mathcal{U}$ is a bounded open subset of $\mathbb{R}^n$ and **you can solve** for every $f \in C^2(\overline{\mathcal{U}})$ the boundary value problem

$$\begin{cases} \Delta v = f & \text{on } \mathcal{U}, \\ v\big|_{\partial\mathcal{U}} \equiv 0. \end{cases} \tag{2}$$

Problem 4 *Consider the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{on } \mathcal{U}, \\ u\big|_{\partial\mathcal{U}} = g. \end{cases} \tag{3}$$

(a) *If $g \in C^\infty(\mathbb{R}^n)$, find the solution of (3) in terms of a solution of (2).*

(b) *The* **harmonic corrector function** *is a solution* $h(\mathbf{x}) = h(\mathbf{x}, \mathbf{w})$ *of (3) for the particular choice* $g(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{w})$ *where* $\mathbf{w} \in \mathcal{U}$ *and* Φ *is the* **fundamental solution**. *The function* $g : \mathbb{R}^n \backslash \{\mathbf{w}\}$ *given by* $g(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{w})$ *is, of course, not in* $C^\infty(\mathbb{R}^n)$*, so your solution from part (a) does not work immediately to give the harmonic corrector. Nevertheless, use part (a) to find the harmonic corrector in terms of a solution of (2). Hint(s): Remember the proof of the higher regularity theorem.*

Problem 5 *Find the corrector* $h(\mathbf{x}) = \phi(\mathbf{x}, \mathbf{w})$ *and hence the Green's function for the following domains.*

(a) $\mathcal{U} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ and } x_n > 0\}$. *Hint: The fundamental solution* $\Phi(\mathbf{z} - \tilde{\mathbf{w}})$ *with singularity at a point* $\tilde{\mathbf{w}}$ **outside** *the halfspace* $\mathcal{U}$ *has the correct boundary values.*

(b) $\mathcal{U} = B_r(\mathbf{0}) \subset \mathbb{R}^2$. *Initial hint: The fundamental solution* $\Phi(\mathbf{z} - \tilde{\mathbf{w}})$ *with singularity at a point outside the ball has the right boundary values. (Find that point* $\tilde{\mathbf{w}}$*.) Actually, this is not quite true,*[1] *but it's a good starting point. Note the form such a point* $\tilde{\mathbf{w}}$ *would have to have, and then try the following:*

(i) *Given a singularity point* $\tilde{\mathbf{w}}$ *outside* $B_r(\mathbf{0})$*, a function of the form*

$$h(\mathbf{x}) = \Phi(\beta(\mathbf{x} - \tilde{\mathbf{w}}))$$

is also harmonic on $B_r(\mathbf{0})$*. (And you can choose* β *and* $\tilde{\mathbf{w}}$ *so that this function* h *has the correct boundary values.)*

(ii) *That is, you want*

$$\Phi(\beta(\mathbf{x} - \tilde{\mathbf{w}})) = \Phi(\mathbf{x} - \mathbf{w}) \qquad \text{for all } \mathbf{x} \text{ with } |\mathbf{x}| = r.$$

(iii) *Since* Φ *depends on the modulus of the argument, the previous condition means*

$$\beta|\mathbf{x} - \tilde{\mathbf{w}}| = |\mathbf{x} - \mathbf{w}|. \tag{4}$$

Square both sides. You can assume $\beta > 0$*. You should have/know that given* $\mathbf{w}$*,* $\tilde{\mathbf{w}}$ *depends on one positive constant* α *(along with* $\mathbf{w}$*). Choose a relation between* α *and* β *so that the middle terms on the left and right in (4) agree (and cancel each other). This will give you a quadratic equation for* α*. Solve that equation.*

[1] You can't find such a point.

(iv) *At this point, you may think you have found the harmonic corrector and essentially solved the problem. Solving the quadratic equation, however, depends on the assumption* $\mathbf{w} \neq \mathbf{0}$*. If you didn't notice this unfortunate circumstance already, go back and consider the case* $\mathbf{w} = \mathbf{0}$ *as a separate case.*

Problem 6 *Let* $\mathcal{U} = (0, \ln 2) \times (0, \pi)$*. Assume you can solve*

$$\begin{cases} \Delta w = (1 + x/\ln 2)\sin y, \\ w\big|_{\partial\mathcal{U}} \equiv 0. \end{cases} \tag{5}$$

Use the solution of this problem to solve

$$\begin{cases} \Delta u = 0, \\ u\big|_{\partial\mathcal{U}} \equiv (1 + x/\ln 2)\sin y. \end{cases} \tag{6}$$

Other Second Order Linear Operators

Problem 7 *Show each of the following operators is* **linear** *on an appropriate function space of real valued functions.*

(a) *(anisotropic Laplacian)*

$$A[u] = \sum_{j=1}^{n} a_j(x) D^{2\mathbf{e}_j} u.$$

Here, we are using multi-index notation for derivatives and $\mathbf{e}_j$ *is the* j*-th standard unit basis vector. Note: We should also require* $a_j : U \to \mathbb{R}$ *for* $j = 1, \ldots n$ *are given* **positive** *functions on some domain* $U \subset \mathbb{R}^n$*. You may further restrict the coefficients* $a_j = a_j(x)$ *in order to determine/specify the codomain* W *of this operator. The isotropic spacial case* $a_1 = a_2 = \cdots = a_n \equiv 1$ *of this operator gives the Laplacian*

$$\Delta u = \nabla^2 u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

(b) *(heat operator)*

$$H[u] = u_t - k\Delta u.$$

Here the positive constant $k = \alpha^2$ is called the **diffusivity**, *and the operator is also called the* **diffusion operator**.

(c) *(wave operator)*

$$\Box u = u_{tt} - k\Delta u.$$

Here the positive constant $k = c^2$ is called the **square of the propogation speed**. *The wave operator is also sometimes called the* **D'Alembertian** *after Jean D'Alembert.*

Fourier Series

Let $f : [0, L] \to \mathbb{R}$ be a continuous function. A **Fourier sine series** for f has the form

$$f(x) = \sum_{j=1}^{\infty} f_j \sin \frac{j\pi x}{L}. \tag{7}$$

where the **numbers** $f_1, f_2, f_3, \ldots$ are called the **Fourier coefficients** of f and the functions $\sin(\pi x/L)$, $\sin(2\pi x/L)$, $\sin(3\pi x/L)$,... make up what is called the **Fourier sine basis**. We do not need to worry too much about convergence of the series. For a continuous function, if the coefficients are chosen correctly, then the series will converge to $f(x)$ at least on $(0, L)$, and we can manipulate the series, at least as far as integrating term by term, pretty freely.

Problem 8 *This problem is about computing Fourier coefficients.*

(a) *Compute*

$$\int_0^L \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx.$$

Hint: Something special happens when $j = k$ because (you can show)

$$\int_0^L \sin^2 \frac{j\pi x}{L}\, dx = \int_0^L \cos^2 \frac{j\pi x}{L}\, dx \qquad \text{and} \qquad \sin^2 \frac{j\pi x}{L} + \cos^2 \frac{j\pi x}{L} = 1.$$

Something even more special happens when $j \neq k$.

(b) *Multiply both sides of (7) by* $\sin(k\pi x/L)$ *and integrate both sides from* $x = 0$ *to* $x = L$. *Use the result to find a formula for the Fourier coefficients.*

(c) *Consider the specific example* $f : [0, L] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} bx/a, & 0 \le x \le a \\ b(x - L)/(a - L), & a \le x \le L. \end{cases}$$

Draw the graph $\{(x, f(x)) : 0 \le x \le L\}$ *of* f. *What can you say about the regularity of* f*?*

(d) *Let* f *be the specific function from the last part of this problem. Find the Fourier sine series expansion of* f.

(e) *The trigonometric polynomial*

$$P_n(x) = \sum_{j=1}^{n} f_j \sin \frac{j\pi x}{L}$$

is (called) the n**-th Fourier sine approximation of** f. *Use mathematical software (Matlab, Mathematica, Maple, etc.) to plot* $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_{10}(x)$, *and* $P_{100}(x)$ *for the specific example from the last two parts. Suggestion from Ching-Lun Tai: The parameter choice* $(a, b, L) = (3, 1, 10)$ *is a good one (leading to results that are relatively easy to interpret).*

Separation of Variables; Superposition

Consider the following boundary value problem for Laplace's equation on the rectangle $U = [0, L] \times [0, M]$ where L and M are positive numbers.

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0,\ u(L,y) = 0,\ u(x,M) = x(x-L),\ u(0,y) = 0 \end{cases} \tag{8}$$

Problem 9 **(a)** *Find all separated variables solutions* $u(x,y) = A(x)B(y)$. *Hint: You should obtain solutions of the form* $u_j = c_j A_j(x) B_j(y)$ *with* $A_j'' = -\lambda_j A_j$ *and* $B_j'' = \lambda_j B_j$ *for some positive increasing sequence*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$

(b) *Find a Fourier expansion of the function* $g_3(x) = x(x-L)$ *and choose the constants* $c_1, c_2, c_3, \ldots$ *appropriately so that*

$$\sum_{j=1}^{\infty} c_j B_j(M) A_j(x) = g_3(x).$$

(c) *Take the specific values* $L = 1$ *and* $M = 0.5$ *and plot enough terms of*

$$u(x,y) = \sum_{j=1}^{\infty} c_j A_j(x) B_j(y)$$

to convince yourself (and me) that you have obtained a series solution for the problem.

First Order Cauchy Problem

Consider the following initial/boundary value problem:

$$\begin{cases} \dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} = 0 \text{ on } \mathcal{U}, \\ u\Big|_I = g \end{cases} \tag{9}$$

for a function $w = w(x, y)$ defined on an open subset $\mathcal{U}$ of $\mathbb{R}^2$ containing the segment $I = \{(x, 0) : 0 \leq x \leq L\}$.

Problem 10 *Let* $\mathbf{r} : [0, \ell] \to \mathbb{R}^2$ *be a smooth path.*

(a) *Compute*

$$\frac{d}{dt} w(\mathbf{r}(t)) \tag{10}$$

where $w \in C^1(\mathcal{U})$.

(b) *Compare your result from the computation of part (a) to the PDE (9). Choose the path* $\mathbf{r}$ *with* $\mathbf{r}(0) = (x_0, 0) \in I$ *such that the PDE tells you the expression in (10) vanishes (if* w *solves the PDE). What does this tell you about the values of a solution* w *along that path?*

(c) *If* $g \in C^1[0, L]$*, find a solution* $w \in C^1(V)$ *of (9) on some open set* $V \supset I$*. Is it always possible to extend this solution to all of* $\mathcal{U}$*? Why or why not? If there is an extension, is it always unique?*