Assignment 5:
Integration and Green's function
Due Friday, February 28, 2025

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March 13, 2025

Problem 1 (Slinky and forces) Tension forces associated with homogeneous extensions of a spring (like a slinky) are sometimes modeled using a Hooke's constant.

(a) Assume the mapping $\sigma : [0, h_0] \to [0, h_1]$ models a homogeneous extension of a slinky of stacked/equilibrium height h_0 to an extended height $h_1 > h_0$. Find an explicit formula for the model measurement function σ.

(b) Assume this slinky has Hooke's constant k so that an internal tension in the coils of magnitude kx is associated with an extension of length x. If the slinky is cut in half to make two identical slinkys of stacked/equilibrium height $h_0/2$ what is the Hooke's constant associated with one of these two shorter slinkys?

(c) What does the answer to the question in **(b)** about tell you about Hooke's constant?

Solution:

(a) $\sigma(h) = (h_1/h_0)h$.

(b) I assume $\sigma_0(h) \equiv h$, the identity function, is an equilibrium. Then for the homogeneous extension to length h_1, the displacement of the end of the slinky from equilibrium has value $x = h_1 - h_0$. Thus, the corresponding tension force in the coils should be

$$F_1 = k(h_1 - h_0).$$

If I take half of this slinky, then the stacked height should be $h_0/2$, and the same density of coils (and hence the same tension force) should be attained when the slinky is extended to a length $h_1/2$. If I assume the same Hooke's constant k, then the displacement of the half end is $\tilde{x} = h_1/2 - h_0/2 = (h_1 - h_0)/2$, and the force should be

$$\tilde{F} = k\frac{h_1 - h_0}{2},$$

but this is not correct because I just said the density of the coils should be the same as for the initial extension of the original long slinky. This means I should have/get

$$F_{1/2} = k(h_1 - h_0) = (2k)\frac{h_1 - h_0}{2}$$

when the displacement of the half-slinky is $(h_1 - h_0)/2$.

(c) Number one, the Hooke's constant for the half-slinky is not the same as the Hooke's constant for the original slinky even though they are of the same material. The Hooke's constant for the half-slinky is twice that of the original long slinky. This tells me Hooke's constant is not a contant determined by the materials. That is to say Hooke's constant is not a **material constant**.

Problem 2 (hypersurface) Consider $n \in \{2, 3, 4, \ldots\}$ and $U \subset \mathbb{R}^{n-1}$ an open set. A function $X \in C^1(U \to \mathbb{R}^n)$ is said to parameterize an **embedded hypersurface** $\mathcal{S} = X(U)$ if the following hold:

(i) X is one-to-one.

(ii) $X^{-1} \in C^0(\mathcal{S} \to U)$.

(iii) The differential map $dX_{\mathbf{p}} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ with values given by

$$dX_{\mathbf{p}}(\mathbf{v}) = DX(\mathbf{p})\mathbf{v}$$

satisfies $dX_{\mathbf{p}}$ is one-to-one for each $\mathbf{p} \in U$.

(a) Show $X : \mathbb{R}^2 \to \mathbb{R}^3$ by $X(u, v) = (u, v, au + bv)$ parameterizes an embedded hypersurface for every $a, b \in \mathbb{R}$.

(b) Say $f : \mathcal{P} \to \mathbb{R}$ is a real valued function defined on the hypersurface $\mathcal{P} = X(\mathbb{R}^2)$ with X given in **(a)** above. Express

$$\int_A f$$

where A is a domain of integration in $\mathcal{S}$ in terms of an integral over a set in $\mathbb{R}^2$.

(c) Consider $f : \Gamma \to \mathbb{R}$ by $f(x, y) = 1/x^3$ where $\Gamma = \{(x, x^2) : x \in (1, \infty)\}$. Find

$$\int_\Gamma f.$$

Problem 3 (flux integral) Consider a smooth vector field $\mathbf{v} : \Omega \to \mathbb{R}^n$ where Ω is an open subset of $\mathbb{R}^n$. If $\mathcal{S}$ is a smooth hypersurface passing through $\mathbf{p} \in \Omega$ with **orientation field** $\mathbf{n} : \mathcal{S} \to \mathbb{R}^n$ by which we mean $\mathbf{n} \in C^1(\mathcal{S} \to \mathbb{R}^n)$ and

$$dX_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{n} \circ X^{-1}(X(\mathbf{p})) \equiv 0 \qquad \text{for all } \mathbf{p} \in U \text{ and } \mathbf{v} \in \mathbb{R}^{n-1},$$

we define the **flux integral** of $\mathbf{v}$ over $\mathcal{S}$ to be

$$\int_{\mathcal{S}} \mathbf{v} \cdot \mathbf{n}.$$

(a) Express the flux integral as an integral over U.

(b) Imagine a fluid of constant mass density ρ flowing at all points with a constant velocity in a fixed constant direction. This is sometimes called **uniform flow**. Taking the special case $n = 3$ and modeling a constant fluid velocity field by $\mathbf{v} = v\mathbf{e}_1 : \mathbb{R}^3 \to \mathbb{R}^3$ where the constant v has physical dimension given by

$$[v] = \frac{L}{T}.$$

Give an expression for the rate of mass flow through the "window"

$$\mathcal{S} = \{a(-\sin\theta, \cos\theta, 0) + b(-\cos\phi\cos\theta, -\cos\phi\sin\theta, \sin\phi) : |a|, |b| < \epsilon\}$$

(c) Calculate

$$\int_{\mathcal{S}} \mathbf{v} \cdot \mathbf{n}$$

where $\mathbf{n} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

Problem 4 Define the **divergence** of a smooth vector field $\mathbf{v} : \Omega \to \mathbb{R}^2$ to be the function $\operatorname{div} : \Omega \to \mathbb{R}$ with value given by the limit

$$\operatorname{div} \mathbf{v}(\mathbf{p}) = \lim_{\epsilon \searrow 0} \frac{1}{\operatorname{vol}(C_\epsilon(\mathbf{p}))} \int_{\partial C_\epsilon(\mathbf{p})} \mathbf{v} \cdot \mathbf{n}$$

where $C_\epsilon(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : |x_j - p_j| < \epsilon,\ j = 1, 2\}$ is the cube/square of sidelength 2ϵ centered at $\mathbf{p}$.

(a) Express the flux integral around $\partial C_\epsilon(\mathbf{p})$ as the sum of the four flux integrals over the sides.

(b) Group the terms involving integrals over opposite sides.

(c) Use the mean value theorem and take the limit(s).

Problem 5 (product formula) Use the formula obtained in Problem 4 above to establish the product formula for the divergence of a scaled field: If $\mathbf{v} \in C^1(\Omega \to \mathbb{R}^2)$ and $f \in C^1(\Omega)$, then

$$\operatorname{div}(f\mathbf{v}) = Df \cdot \mathbf{v} + f \ \operatorname{div} \mathbf{v}.$$

Solution: Let $\mathbf{v} = (v_1, v_2)$. From the Problem 4

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

This means

$$\operatorname{div}(f\mathbf{v}) = \frac{\partial}{\partial x_1}(fv_1) + \frac{\partial}{\partial x_2}(fv_2).$$

Notice that for $j = 1, 2$

$$\frac{\partial}{\partial x_j}(fv_j) = \frac{\partial f}{\partial x_j} v_j + f \frac{\partial v_j}{\partial x_j}.$$

This is just the product rule for real valued functions from Calculus I. Summing over j gives

$$\begin{aligned}\operatorname{div}(f\mathbf{v}) &= \frac{\partial f}{\partial x_1} v_1 + \frac{\partial f}{\partial x_2} v_2 + f\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) \\ &= Df \cdot \mathbf{v} + f \operatorname{div} \mathbf{v}\end{aligned}$$

as claimed.

There is nothing particularly special about $n = 2$. In rectangular coordinates in $\mathbb{R}^n$ we have for a vector field $\mathbf{v} : \Omega \to \mathbb{R}^n$

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}.$$

Therefore,

$$\begin{aligned}\operatorname{div}(f\mathbf{v}) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(fv_j) \\ &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} v_j + f \frac{\partial v_j}{\partial x_j}\right) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} v_j + f \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \\ &= Df \cdot \mathbf{v} + f \operatorname{div} \mathbf{v}.\end{aligned}$$

Problem 6 (Green's formula) Use the product formula for the divergence and the divergence theorem to show Green's formula in the plane: If $u, v \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain of integration and $\partial\Omega$ is a smooth curve, then

$$\int_{\Omega} (u\Delta v - v\Delta u) = \int_{\partial\Omega} (uDv - vDu) \cdot \mathbf{n}$$

where $\mathbf{n}$ is the outward unit normal field on $\partial\Omega$.

Problem 7 (fundamental solution of Laplace's equation when $n = 2$) Let $\Phi = C \ln |\mathbf{x}|$ with $C < 0$ and consider $f \in C_c^3(\mathbb{R}^2)$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(\mathbf{x}) = \int_{\xi \in \mathbb{R}^2} \Phi(\mathbf{x} - \xi) f(\xi). \tag{1}$$

(a) Change variables to show

$$u(\mathbf{x}) = \int_{\eta \in \mathbb{R}^2} \Phi(\eta)\ f(\mathbf{x} - \eta). \tag{2}$$

State explicitly the chage of variables map and explain/compute the Jacobian scaling factor.

Note: Sometimes the convolution integral considered here is denoted $\Phi * f(\mathbf{x})$. Thus, this exercise establishes the commutativity of this convolution: $\Phi * f = f * \Phi$.

(b) Show u is partially differentiable and find a formula for

$$\frac{\partial u}{\partial x_j}.$$

Hint, use (2) to take a limit of difference quotients.

(c) Find a formula for the Laplacian of u.

Problem 8 What happens if you use the formula (1) instead of (2) in Problem 7 to compute

$$\frac{\partial u}{\partial x_j} \quad ?$$

Problem 9 (fundamental solution $n = 1$) Consider Laplace's equation $u'' = 0$ for $u \in C^\infty(\mathbb{R}\backslash\{0\})$. Find all solutions that are "radial" in the sense that they are even with $u(-x) = u(x)$.

Solution: If $u'' = 0$, then integrating from $r = 1$ we have

$$u'(r) = u'(1) + \int_1^r 0\, dt \equiv u'(1)$$

is constant. Integrating again from $r = 1$ we see

$$u(r) = u(1) + u'(1)(r - 1) = u(1) - u'(1) + u'(1)r.$$

This function is of the form $u(r) = ar + b$ where $a = u'(1)$ and $b = u(1) - u'(1)$. Every solution must have this form for some $a, b \in \mathbb{R}$ and $r > 0$. If the solution is even, then

$$u(x) = a(-x) + b,$$

so in general there is a two parameter family of solutions given by

$$u(x) = \begin{cases} b + ax, & x \geq 0 \\ b - ax, & x \leq 0. \end{cases}$$

Or

$$u(x) = \frac{ax^2}{|x|} + b = a|x|.$$

Note that these solutions satisfy $u \in C^\infty(\mathbb{R}\backslash\{0\})$ and even $u \in C^\omega(\mathbb{R}\backslash\{0\})$. Each extends to a function in $C^0(\mathbb{R}) \cap Lip(\mathbb{R})$, but the extension is in $C^1(\mathbb{R})$ only if $u'(1) = a = 0$. In this case, the radial solution is constant, just like in higher dimensions. Thus, the interesting radial solutions are given by

$$u(x) = C|x|$$

for some nonzero constant C. Finally, the fundamental solution for $n = 1$ returns to the pattern

$$\Phi(\mathbf{x}) = \frac{C}{|\mathbf{x}|^{n-2}}$$

familiar from the case $n \geq 3$. This perhaps suggests consideration of the constant

$$C = \frac{1}{n(n-2)\omega_n} = -\frac{1}{2}$$

when $n = 1$.

Problem 10 (Slinky and forces) Consider again homogeneous extensions $\sigma : [0, h_0] \to [0, h_1]$ of a slinky as discussed in Problem 1 above.

(a) Give an intuitive explanation for why the internal tension force might reasonably depend on the density of coils in the homogeneous extension.

(b) Starting with the Hooke's constant model, express the tension force in terms of the natural quantity used to express the density of coils in terms of σ.

(c) Define a **material constant** with which you can express the tension force and remains the same for the half slinkys of Problem 1 above.

Solution:

(a) The "density" of the coils is directly related to the deformation of the coil from the equilibrium/stacked state. In particular, lower density corresponds to greater deformation from the equilibrium, and should be expected to correspond to greater tension.

(b) From Assignment 3, Problem 1 one has a linear density function $\rho : [0, h_0] \to \mathbb{R}$ given by

$$\rho(x) = \frac{\rho_0}{\sigma'(x)}$$

where ρ_0 is the constant equilibrium density. Also, we have a coil density $\rho_c : [0, h_0] \to \mathbb{R}$ with

$$\rho_c(x) = \frac{1}{m_c}\rho(x)$$

where m_c is the mass of one coil. Clearly, we can use either of these density functions as they are proportional to one another. If we start with a homogeneous deformation for which

$$F_1 = k(h_1 - h_0)$$

as in Problem 1 above and note that $\sigma(h) = (h_1/h_0)h$, then we can write

$$F_1 = kh_0 \left(\frac{h_1}{h_0} - 1 \right) = kh_0(\sigma' - 1).$$

Here of course $\sigma' \equiv h_1/h_0$ is constant, but notice at least that for the half spring

$$F_1 = kh_0(\sigma' - 1)$$

gives the correct force formula also for the deformation $\tilde{\sigma} : [0, h_0/2] \to \mathbb{R}$ of the half slinky with $\tilde{\sigma}(h) = (h_1/h_0)h$ maintaining the same density and density of coils.

(c) This suggests the introduction of a material (elastic) constant γ with value $\gamma = kh_0$ for any slinky of this material, and we can then either derive the force law

$$F = \gamma(\sigma' - 1)$$

for homogeneous deformations or take the general law

$$F = \gamma(\sigma'(x) - 1)$$

for the tension force of **any** deformation as the defining relation for the material constant γ.

One has a few choices at this point, based on this model approach. One can attempt to look more carefully at the physical structure of the slinky to model the value of the material constant γ in some way. One may also take various approaches to use data to obtain an optimal or "best" choice of γ. One of the simplest versions of this involves using one measurement, namely the full length of the hanging slinky.