

Assignment 5: selected solutions (Problems 2(a) and 9)

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Problem 1 Consider the first order linear PDE

$$y \frac{\partial u}{\partial x} = x \frac{\partial u}{\partial y}.$$

Show the ray $\Gamma = \{(x, 0) : x > 0\}$ is non-characteristic and solve the PDE on a domain $U = B_1(1, 0)$. using the method of characteristics for Cauchy data $u_0(x)$ on Γ .

Problem 2 (continuous differentiability of a function defined on the circle) Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. To each real valued function $f : \mathbb{S}^1 \rightarrow \mathbb{R}$, associate a corresponding function $g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g(\theta) = f(\cos \theta, \sin \theta).$$

- (a) Show $\phi : [0, 2\pi) \rightarrow \mathbb{S}^1$ by $\phi(\theta) = (\cos \theta, \sin \theta)$ is one-to-one and onto, i.e., bijective.
- (b) Show the function ϕ defined in part (a) is continuous but does not have a continuous inverse.
- (c) What is the (composition) relation among ϕ , a function $f : \mathbb{S}^1 \rightarrow \mathbb{R}$, and the associated function $g : \mathbb{R} \rightarrow \mathbb{R}$.
- (d) Assume a function $f : \mathbb{S}^1 \rightarrow \mathbb{R}$ has the following property:

There exists an open set $\mathcal{N} \subset \mathbb{R}^2$ with $\mathbb{S}^1 \subset \mathcal{N}$ and an extension $F \in C^1(\mathcal{N})$ with

$$F|_{\mathbb{S}^1} = f.$$

Show the function g associated to f by the correspondence above satisfies $g \in C^1(\mathbb{R})$ and g is 2π -periodic. Hint: Use F to compute derivatives of g and show they are continuous.

We denote by $C^1(\mathbb{T})$ the subspace of $C^1(\mathbb{R})$ consisting of 2π -periodic functions.

Solution: The assertion of part **(a)** may be viewed as very easy or very difficult. On the one hand, most people take the fact that “associated” with each $(x, y) \in \mathbb{S}^2$ there is one and exactly one $\theta \in [0, 2\pi)$ for which $(\cos \theta, \sin \theta) = (x, y)$, i.e., the function ϕ in this problem is a bijection. The “argument” for this point of view is the illustration in Figure 1.

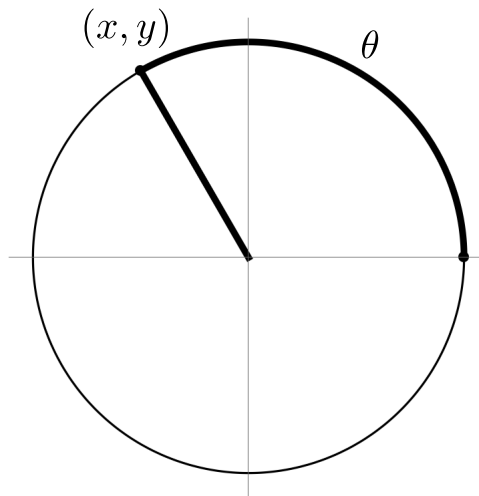


Figure 1: Each angle $\theta \in [0, 2\pi)$ determines a unique point $(x, y) = (\cos \theta, \sin \theta) \in \mathbb{S}^1$ so that the function $\phi : [0, 2\pi) \rightarrow \mathbb{S}^1$ by $\phi(\theta) = (\cos \theta, \sin \theta)$ is one-to-one. Each point $(x, y) \in \mathbb{S}^1$ determines an angle $\theta \in [0, 2\pi)$ for which $\phi(\theta) = (x, y)$, so the function ϕ is also onto. (Proof by picture.)

It was probably an expression of overly high expectations on my part that you would come up with anything more insightful or “rigorous” than this. On the other hand, as Mike said, “It’s better to aim too high and fail, than to aim too low and succeed.” And on the third hand, “proof by picture” is generally frowned upon for good reason, so let me see if I can come up with something (more).

I will take as my basic assumption(s) certain (algebraic and analytic) properties of the well-known trigonometric functions cosine and sine. These properties can

be derived by various means that do not involve the picture in Figure 1 and the triangles with which it may be decorated, and I think/hope (at the end of the day) you might agree I've added some insight/organization to the basic assertion ϕ is bijective associated with Figure 1. You might also think about some details of the basic trigonometric functions you haven't considered before.

Let me begin with some observations about the values of cosine and sine. The values of cosine fall into three classes as illustrated in Figure 2:

- (i) The value $c = 1$. There is precisely one $\theta \in [0, 2\pi)$ with $\cos \theta = 1$, namely $\theta = 0$.
- (ii) The values $-1 < c < 1$. There are precisely two angles θ with $\cos \theta = c$. Moreover, denoting the smaller of these angles by θ_0 we have $0 < \theta_0 < \pi$ and the larger angle is precisely $2\pi - \theta_0$ for which $\pi < 2\pi - \theta_0 < 2\pi$.
- (iii) The value $c = -1$. There is precisely one $\theta \in [0, 2\pi)$ with $\cos \theta = -1$, namely $\theta = \pi$.

Each of the angles of the classes (i) and (iii) and the smaller angle in the case of class (ii) has an analytic expression $\theta_0 = \cos^{-1}(c)$ in terms of the principal branch $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$ of the inverse cosine which I will note is a decreasing function.

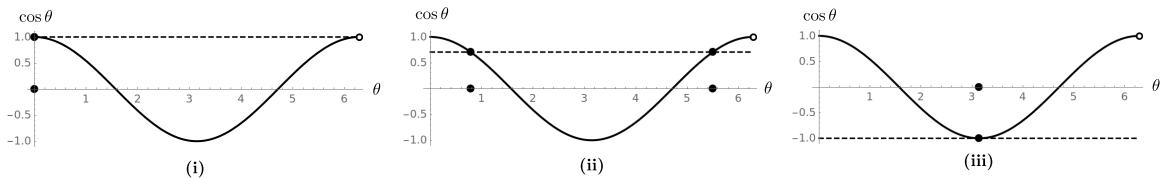


Figure 2: The restriction of cosine to the interval $0 \leq \theta < 2\pi$.

The values of sine, on the other hand, I will organize into four classes which I will leave to you to illustrate:

- (i) The value $c = 1$. There is precisely one $\theta \in [0, 2\pi)$ with $\sin \theta = 1$, namely $\theta = \pi/2$.
- (ii) The values $0 \leq c < 1$. There are precisely two angles θ with $\sin \theta = c$. Denoting the smaller of these angles by θ_0 we have $0 \leq \theta_0 = \sin^{-1}(c) < \pi/2$, and the larger angle is precisely $\pi - \theta_0$ for which $\pi/2 < \pi - \theta_0 \leq \pi$.
- (iii) The values $-1 < c < 0$. There are also precisely two angles θ with $\sin \theta = c$. Denoting the smaller of these angles by θ_0 , we have $\pi < \theta_0 = \pi - \sin^{-1}(c) <$

$3\pi/2$, and the larger angle is precisely $2\pi - (\theta_0 - \pi) = 3\pi - \theta_0 = 2\pi + \sin^{-1}(c)$.
The larger angle satisfies $3\pi/2 < 3\pi - \theta_0 < 2\pi$.

(iv) The value $c = -1$. There is precisely one $\theta \in [0, 2\pi)$ with $\sin \theta = -1$, namely $\theta = 3\pi/2 = 2\pi + \sin^{-1}(-1)$.

In this characterization of the values of sine we have used the principal branch

$$\sin^{-1} : [-1, 1] \rightarrow [-\pi/2, \pi/2].$$

Exercise 1 Illustrate the four classes of values for sine distinguished above. Plot the principal branches of (the real) arccosine and arcsin.

Finally, we note that corresponding to $x^2 + y^2 = 1$ for $(x, y) \in \mathbb{S}^1$, there holds $\cos^2 \theta + \sin^2 \theta = 1$ for every $\theta \in [0, 2\pi)$.

Let us attempt to show $\phi : [0, 2\pi) \rightarrow \mathbb{S}^1$ is surjective. Given $(x, y) \in \mathbb{S}^1$, if $x = 1$, then $(x, y) = (1, 0) = (\cos 0, \sin 0)$. Similarly, if $x = -1$, then $(x, y) = (-1, 0) = (\cos(\pi), \sin(\pi))$. The remaining points $(x, y) \in \mathbb{S}^1$ have first coordinate satisfying $-1 < x < 1$, so that the value x falls into the middle class of values distinguished above for cosine. In particular, we obtain $\theta_0 = \cos^{-1}(x)$ and $2\pi - \cos^{-1}(x)$ as the unique angles $\theta \in [0, 2\pi)$ for which $\cos \theta = x$. If $y \geq 0$, then we claim

$$y = \sqrt{1 - x^2} = \sin \theta_0.$$

In fact, $x = \cos \theta_0$ and since $0 < \theta_0 < \pi$ we also know $\sin \theta_0 > 0$ with

$$\sin \theta_0 = \sqrt{1 - \cos^2 \theta_0} = \sqrt{1 - x^2}.$$

Thus, the claim is established and $\phi(\theta_0) = \phi(\cos^{-1}(x)) = (x, y)$. Otherwise we have $y < 0$ and so $y = -\sqrt{1 - x^2}$. Here we claim

$$y = \sin(2\pi - \theta_0) = \sin(2\pi - \cos^{-1}(x)).$$

To see this, we simply need to realize that since $\pi < 2\pi - \theta_0 < 2\pi$ we must have $\sin(2\pi - \theta_0) < 0$. In particular,

$$\sin(2\pi - \theta_0) = -\sqrt{1 - \cos^2(2\pi - \theta_0)} = -\sqrt{1 - x^2}.$$

Thus, for the final case,

$$\phi(2\pi - \cos^{-1}(x)) = (\cos(2\pi - \theta_0), \sin(2\pi - \theta_0)) = (x, y),$$

and $\phi : [0, 2\pi) \rightarrow \mathbb{S}^1$ is definitely surjective.

We next attempt to verify ϕ is injective. Let us say $(x, y) \in \mathbb{S}^1$ and there are angles θ and $\tilde{\theta}$ with $0 \leq \theta, \tilde{\theta} < 2\pi$ and

$$\phi(\theta) = \phi(\tilde{\theta}) = (x, y).$$

Again, if $|x| = 1$, there can only be one angle $\theta_0 = \cos^{-1}(x) \in [0, 2\pi)$ for which $\cos \theta_0 = x$. Thus, $\theta = \theta_0 = \tilde{\theta}$.

If, on the other hand, $|x| < 1$, then the value x for cosine falls into class **(ii)** of values of cosine. Therefore, there are precisely two angles $\theta_0 = \cos^{-1}(x)$ and $2\pi - \theta_0$ in the interval $[0, 2\pi)$ on which cosine takes the value x . Furthermore,

$$0 < \theta_0 < \pi < 2\pi - \theta_0 < 2\pi$$

so precisely one of the values $\sin \theta_0$ and $\sin(2\pi - \theta_0)$ is positive, and the other is negative. It follows that sine can take the value y on at most one of these angles. That is, only one of the two angles $\theta = \theta_0$ and $\theta = 2\pi - \theta_0$ can satisfy

$$(\cos \theta, \sin \theta) = (x, y).$$

We have established that ϕ is injective. \square

Problem 3 Let $\phi^{-1} : \mathbb{S}^1 \rightarrow [0, 2\pi)$ denote the inverse of the function ϕ defined in part **(a)** of Problem 2 above. If $\theta \in \mathbb{R}$ and $\phi^{-1}(\cos \theta, \sin \theta) = \theta_0$, then what is the relationship between θ and θ_0 ?

Problem 4 Given $g \in C^1(\mathbb{T})$, show the following:

(a) There exists a well-defined function $f_0 : \mathbb{S}^1 \rightarrow \mathbb{R}$ given by

$$f_0(x, y) = g(\theta)$$

where

$$\cos \theta = x \quad \text{and} \quad \sin \theta = y.$$

(b) If $f : \mathbb{S}^1 \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is the function associated to f by the correspondence of Problem 2, then $f \equiv f_0$.

(c) There exists an open set $\mathcal{N} \subset \mathbb{R}^2$ and a function $F \in C^1(\mathcal{N})$ such that

$$F|_{\mathbb{S}^1} = f.$$

(d) Show the function F given/found in part (c) above is not unique.

Problem 5 Note that the assumption $g \in C^1(\mathbb{T})$ was not required to obtain the conclusions of parts (a) and (b) of Problem 4 above. What more general assumption on g may be used to obtain the conclusions of parts (a) and (b) of Problem 4?

Problem 6 State precisely the definition of $C^0(\mathbb{S}^1)$. Find a linear isomorphism between $C^0(\mathbb{S}^1)$ and an appropriate subspace of $C^0(\mathbb{R})$. Hint: Look back and think carefully about Problems 2-5 above.

Problem 7 We have not defined the space $C^1(\mathbb{S}^1)$ of continuously differentiable real valued functions with domain the unit circle. How would you define such a space?

Problem 8 Let $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ be the punctured plane, and consider the PDE

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0. \quad (1)$$

(a) Show $\Sigma = \{u \in C^1(U) : (1) \text{ holds}\}$ is a vector subspace of $C^1(U)$.

(b) Find the general solution of (1) in $C^1(U)$. This means to give a characterization of the solution set Σ defined in part (a) above in terms of a “simpler” collection of functions. Hint: $C^1(\mathbb{S}^1)$ or $C^1(\mathbb{T})$.

Problem 9 Let $U = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ be the right half plane and let $V = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane.

(a) Solve the PDE

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad (2)$$

in $C^1(U)$.

(b) Solve (2) in $C^1(V)$.

(c) Solve (2) in $C^1(U \cup V)$.

Solution:

- (a) The characteristic field is $\mathbf{u} = (x, y)$ which points radially outward. Let $f = f_0(y)$ be a function defined on the line $L = \{(1, y) : y \in \mathbb{R}\}$. Notice this line is a subset of U and has normal $(1, 0)$. Since $(1, 0) \cdot (x, y) = x > 0$ on U , this is a non-characteristic line. Given any $(1, y_0) \in L$, we can solve the characteristic system

$$\begin{cases} \xi' = \xi, & \xi(0) = 1 \\ \eta' = \eta, & \eta(0) = y_0 \end{cases}$$

to obtain a characteristic curve $(\xi(t), \eta(t)) = (1, y_0)e^t$. We can make the computation

$$\begin{aligned} \frac{d}{dt}u(\xi(t), \eta(t)) &= \xi'(t) u_x(\xi(t), \eta(t)) + \eta'(t) u_y(\xi(t), \eta(t)) \\ &= \xi(t) u_x(\xi(t), \eta(t)) + \eta(t) u_y(\xi(t), \eta(t)) \\ &= 0 \end{aligned}$$

to see the value of a C^1 solution u must be constant and have the value $f_0(y_0)$ along the characteristic, but we should have known that already. This is why we chose the characteristic.

It remains to determine y_0 so that the characteristic originating at $(1, y_0)$ passes through a point $(x, y) \in U$. If $(1, y_0)e^t = (x, y)$ for $x > 0$, then $t = \ln x$ and $y_0 = y/x$. Therefore, we have solved the problem in the form

$$u(x, y) = f_0(y/x)$$

where $f_0 \in C^1(\mathbb{R})$. More precisely, the set of all solutions of (2) is

$$\Sigma_0 = \{u \in C^1(U) : \text{there is some } f \in C^1(\mathbb{R}) \text{ with } u(x, y) = f(y/x)\}.$$

Alternatively, we can consider the non-characteristic curve $\mathcal{N} = \mathbb{S}^1 \cap U$ where $\mathbb{S}^1 = \{\mathbf{x} : |\mathbf{x}| = 1\}$ denotes the unit circle as usual. Notice $\mathcal{N} = \mathbb{S}^1 \cap U$ is a non-characteristic curve. For each function $f = f_1(\mathbf{p})$ defined on \mathcal{N} , we can apply the method of characteristics as follows: Given $(x, y) \in U$, there is a characteristic $\gamma : \mathbb{R} \rightarrow U$ with

$$\gamma(0) = \mathbf{p}_0 = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

so that $\gamma(t) = \mathbf{p}_0 e^t$ and $\gamma' = \gamma$. By the PDE (2)

$$\frac{d}{dt}u \circ \gamma(t) = Du \circ \gamma(t) \cdot \gamma'(t) = Du \circ \gamma(t) \cdot \gamma(t) = 0.$$

It is convenient to parameterize $C^1(\mathcal{N})$ by $C^1(-\pi/2, \pi/2)$ so that the alternative solution set is

$$\Sigma_1 = \{u \in C^1(U) : \text{there is some } \phi \in C^1(-\pi/2, \pi/2) \text{ with } u(x, y) = \phi(\tan^{-1}(y/x))\}.$$

Note the relation $f_1(\cos \theta, \sin \theta) = \phi(\theta)$.

Exercise 2 Show Σ_0 and Σ_1 are the same set.

- (b) For the upper half plane V we could use a non-characteristic horizontal line. Anticipating part (c) below, however, it makes more sense to use a circular arc $\mathbb{S}^1 \cap V$. The argument of a point $(x, y) \in V$ is a little trickier to express in this case, but one possibility is

$$\arg(x, y) = \begin{cases} \tan^{-1}(y/x), & x > 0 \\ \pi/2, & x = 0 \\ \pi + \tan^{-1}(y/x), & x < 0. \end{cases}$$

Another clean way to deal with this is to use a little complex analysis and note that $\arg(x, y) = \text{Im} \log(x + iy)$ where \log is the principal branch of the complex logarithm. Either way, the solution set is

$$\Sigma = \{u \in C^1(V) : \text{there is some } \phi \in C^1(0, \pi) \text{ with } u(x, y) = \phi(\arg(x, y))\}.$$

- (c) If you managed to use the circle to get non-characteristic curves in parts (a) and (b), then this part is easy/straightforward/immediate using $\mathbb{S}^1 \cap (U \cup V)$:

$$\Sigma = \{u \in C^1(U \cup V) : \text{there is some } \phi \in C^1(-\pi/2, \pi) \text{ with } u(x, y) = \phi(\arg(x, y))\}.$$

Problem 10 Solve the inhomogeneous linear first order PDE

$$xu_x - yu_y + (x^2 + y^2)u = x^2 - y^2 \quad \text{on } U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$$

where “solve” means “find all possible C^1 solutions.”