Math 6702, Assignment 5

Introduction

Consider the first order system of PDEs

$$\begin{cases} u_x = \phi \\ u_y = \psi \end{cases} \tag{1}$$

for a function $u \in C^1(\mathbb{R}^2)$ where ϕ and ψ are given functions in $C^0(\mathbb{R}^2)$. These are called the **gradient PDEs**. Notice the left side is the same as the Cauchy-Riemann equations.

1. How many unknowns are there in (1)? How many equations? What does this suggest from the linear algebra/specification of partials point of view?

2. Show that if $u = u(x, y)$ is a solution of (1) with $u \in C^1(\mathbb{R}^2)$, then

$$u(x, y) = u_1(x) + \int_0^y \psi(x, \eta)\, d\eta = u_2(y) + \int_0^x \phi(\xi, y)\, d\xi$$

where $u_1 \in C^1(\mathbb{R})$ with $u_1(x) = u(x, 0)$ and $u_2 \in C^1(\mathbb{R})$ with $u_2(y) = u(0, y)$.

3. Solve the **Cauchy problems**

$$\begin{cases} u_x = \phi \\ u(0, y) = g_2(y) \end{cases} \qquad \text{and} \qquad \begin{cases} u_y = \psi \\ u(x, 0) = g_1(x) \end{cases}$$

for $u \in C^2(\mathbb{R}^2)$ where $g_1, g_2 \in C^1(\mathbb{R}^1)$, and show the solutions are unique.

4. Show that if $\phi, \psi \in C^1(\mathbb{R}^2)$ and $u \in C^1(\mathbb{R}^2)$ solves (1), then $u \in C^2(\mathbb{R}^2)$ and $\phi_y = \psi_x$. Conclude that

$$\{Du \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2) : u \in C^1(\mathbb{R}^2)\}$$

the set of all gradients, i.e., pairs $(\phi, \psi) \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2)$ for which (1) is solvable satisfies

$$\{Du \in C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2) : u \in C^1(\mathbb{R}^2)\} \supset \{(\phi, \psi) \in C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2) : \phi_y = \psi_x\}.$$

§4.6-7 Chain Rule

5. (4.6.10-11) Let $u(x, y) = xe^y + ye^x$.
   (a) Calculate the gradient Du, and show there is exactly one point (x_0, y_0) for which $Du(x_0, y_0) = (0, 0)$.
   (b) Show $u(x_0, y_0) \neq 0$, but there is at least one point $(x, y) \in \mathbb{R}^2$ for which $u(x, y) = 0$. This means

$$\Gamma = \{(x, y) : u(x, y) = 0\}$$

   is a nonempty C^1 curve.

(c) Use mathematical software to plot the curve Γ and its tangent line at some point. Hint(s): Use the point you found in the previous part. Assume $y = y(x)$ and differentiate the relation $u(x, y) = 0$ to obtain an ordinary differential equation for y. Solve this equation numerically, and then plot the solution. You should run into trouble when $x \approx -0.567$. In fact, I got warnings at $x \approx -0.278$. Find out why. Then Make a rotation by 45 degrees as follows: Let

$$M = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

be the matrix corresponding to clockwise rotation by $\pi/4$. Consider $v(\xi, \eta) = u(M(\xi, \eta)^T)$ in new ξ, η coordinates. Show that the level curve $\{(\xi, \eta) : v(\xi, \eta) = 0\}$ is given explicitly by $\eta = \xi \tanh(\sqrt{2}\,\xi/2)$. Therefore,

$$\xi \mapsto \xi M(1, \tanh(\sqrt{2}\,\xi/2))^T$$

is a global parameterization of the curve.

(d) Use the chain rule to compute the second derivative of $f''(0)$ if $u(x, f(x)) = 0$.

6. (4.7.9) If $(x, y) = \Psi(r, \theta) = (r\cos\theta, r\sin\theta)$ is the polar coordinates transformation/change of variables, compute the partials

$$\frac{\partial r}{\partial x}, \quad \frac{\partial r}{\partial y}, \quad \frac{\partial \theta}{\partial x}, \quad \text{and} \quad \frac{\partial \theta}{\partial y},$$

and express your answers in terms of both x and y and r and θ. See (7.16) in Boas.

7. (4.7.10) If $x^2 + y^2 = 2st - 10$ and $2xy = s^2 - t^2$, find

$$\frac{\partial x}{\partial s}, \quad \frac{\partial x}{\partial t}, \quad \frac{\partial y}{\partial s}, \quad \text{and} \quad \frac{\partial y}{\partial t}$$

at $(x, y, s, t) = (4, 2, 5, 3)$.

First Order Cauchy Problem

Consider the following initial/boundary value problem:

$$\begin{cases} \dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} = 0 \text{ on } U, \\ w\big|_I = g \end{cases} \tag{2}$$

for a function $w = w(x, y)$ defined on an open subset U of $\mathbb{R}^2$ containing the segment $I = \{(x, 0) : 0 \le x \le L\}$.

8. Let $\mathbf{r} : [0, \ell] \to \mathbb{R}^2$ be a smooth path.

(a) Compute

$$\frac{d}{dt}w(\mathbf{r}(t)) \tag{3}$$

where $w \in C^1(U)$.

(b) Compare your result from the computation of part (a) to the PDE (2). Choose the path $\mathbf{r}$ with $\mathbf{r}(0) = (x_0, 0) \in I$ such that the PDE tells you the expression in (3) vanishes (if w solves the PDE). What does this tell you about the values of a solution w along that path?

(c) If $g \in C^1[0, L]$, find a solution $w \in C^2(V)$ of (2) on some open set $V \supset I$. Is it always possible to extend this solution to all of U? Why or why not? If there is an extension, is it always unique?