Assignment 5: Partial Differential Equations (and other topics) Due Wednesday, February 22, 2023

John McCuan

January 31, 2023

Problem 1 Consider the first order linear PDE

$$y\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

Show the ray $\Gamma = \{(x,0) : x > 0\}$ is non-characteristic and solve the PDE on a domain $U = B_1(1,0)$. using the method of characteristics for Cauchy data $u_0(x)$ on Γ.

Problem 2 (continuous differentiability of a function defined on the circle) Let $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. To each real valued function $f : \mathbb{S}^1 \to \mathbb{R}$, associate a corresponding function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(\theta) = f(\cos\theta, \sin\theta).$$

(a) Show $\phi : [0, 2\pi) \to \mathbb{S}^1$ by $\phi(\theta) = (\cos\theta, \sin\theta)$ is one-to-one and onto, i.e., bijective.

(b) Show the function ϕ defined in part **(a)** is continuous but does not have a continuous inverse.

(c) What is the (composition) relation among ϕ, a function $f : \mathbb{S}^1 \to \mathbb{R}$, and the associated function $g : \mathbb{R} \to \mathbb{R}$.

(d) Assume a function $f : \mathbb{S}^1 \to \mathbb{R}$ has the following property:

There exists an open set $\mathcal{N} \subset \mathbb{R}^2$ with $\mathbb{S}^1 \subset \mathcal{N}$ and an extension $F \in C^1(\mathcal{N})$ with

$$F\Big|_{\mathbb{S}^1} = f.$$

Show the function g associated to f by the correspondence above satisfies $g \in C^1(\mathbb{R})$ and g is 2π-periodic. Hint: Use F to compute derivatives of g and show they are continuous.

We denote by $C^1(\mathbb{T})$ the subspace of $C^1(\mathbb{R})$ consisting of 2π-periodic functions.

Problem 3 Let $\phi^{-1} : \mathbb{S}^1 \to [0, 2\pi)$ denote the inverse of the function ϕ defined in part **(a)** of Problem 2 above. If $\theta \in \mathbb{R}$ and $\phi^{-1}(\cos\theta, \sin\theta) = \theta_0$, then what is the relationship between θ and θ_0?

Problem 4 Given $g \in C^1(\mathbb{T})$, show the following:

(a) There exists a well-defined function $f_0 : \mathbb{S}^1 \to \mathbb{R}$ given by

$$f_0(x, y) = g(\theta)$$

where

$$\cos\theta = x \qquad \text{and} \qquad \sin\theta = y.$$

(b) If $f : \mathbb{S}^1 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is the function associated to f by the correspondence of Problem 2, then $f \equiv f_0$.

(c) There exists an open set $\mathcal{N} \subset \mathbb{R}^2$ and a function $F \in C^1(\mathcal{N})$ such that

$$F\Big|_{\mathbb{S}^1} = f.$$

(d) Show the function F given/found in part **(c)** above is not unique.

Problem 5 Note that the assumption $g \in C^1(\mathbb{T})$ was not required to obtain the conclusions of parts **(a)** and **(b)** of Problem 4 above. What more general assumption on g may be used to obtian the conclusions of parts **(a)** and **(b)** of Problem 4?

Problem 6 State precisely the definition of $C^0(\mathbb{S}^1)$. Find a linear isomorphism between $C^0(\mathbb{S}^1)$ and an appropriate subspace of $C^0(\mathbb{R})$. Hint: Look back and think carefully about Problems 2-5 above.

Problem 7 We have not defined the space $C^1(\mathbb{S}^1)$ of continuously differentiable real valued functions with domain the unit circle. How would you define such a space?

Problem 8 Let $U = \mathbb{R}^2 \backslash \{(0,0)\}$ be the punctured plane, and consider the PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0. \tag{1}$$

(a) Show $\Sigma = \{u \in C^1(U) : (1) \text{ holds}\}$ is a vector subspace of $C^1(U)$.

(b) Find the general solution of (1) in $C^1(U)$. This means to give a characterization of the solution set Σ defined in part **(a)** above in terms of a "simpler" collection of functions. Hint: $C^1(\mathbb{S}^1)$ or $C^1(\mathbb{T})$.

Problem 9 Let $U = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ be the right half plane and let $V = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane.

(a) Solve the PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \tag{2}$$

in $C^1(U)$.

(b) Solve (2) in $C^1(V)$.

(c) Solve (2) in $C^1(U \cup V)$.

Problem 10 Solve the inhomogeneous linear first order PDE

$$xu_x - yu_y + (x^2+y^2)u = x^2 - y^2 \qquad \text{on } U = \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$$

where "solve" means "find all possible C^1 solutions."