MATH 6702 Assignment 5
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Multivariable Calculus

Problem 1 *(Balls and Spheres) Here we denote the* n*-dimensional Lebesgue measure of* $B_1(\mathbf{0}) \subset \mathbb{R}^n$ *by* ω_n*, so that*

$$\mathcal{L}^n(B_r(\mathbf{p})) = \omega_n r^n.$$

See Part (b) below. You are probably familiar with the formulas

$$\mathcal{L}^2(B_r(\mathbf{p})) = \pi r^2$$

for the area of a disk in $\mathbb{R}^2$ *and*

$$\mathcal{L}^3(B_r(\mathbf{p})) = \frac{4}{3}\pi r^3$$

for the volume of a ball in $\mathbb{R}^3$ *corresponding to* $\omega_2 = \pi$ *and* $\omega_3 = 4\pi/3$*. You may not have thought about it before, but you can easily guess (or figure out)* ω_1*; see Part (a) below. I'll guess you do not know the formula for the four-dimensional volume of* $\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 < r^2\}$*. After completing this problem, you should be able to compute the volumes of all such balls and understand the formula for them. I will give you some of the answers, so you will know you have the answer correct:* $\mathcal{L}^n(B_r(\mathbf{p})) = \omega_n r^n$ *where* $B_r(\mathbf{p}) \subset \mathbb{R}^n$ *and*

$$\omega_4 = \frac{1}{2}\pi^2, \quad \omega_5 = \frac{8}{15}\pi^2, \quad \omega_6 = \frac{1}{6}\pi^3, \quad \omega_7 = \frac{16}{105}\pi^3, \ldots$$

I think this is quite an interesting sequence.

(a) *What is* ω_1 *the one-dimensional (length) measure of the ball of radius* $r = 1$ *in* $\mathbb{R}^1$*?*

(b) *Assuming* $\mathcal{L}^n(B_1(\mathbf{0})) = \omega_n$ *use an appropriate change of variables to prove*

$$\mathcal{L}^n(B_r(\mathbf{p})) = \omega_n r^n.$$

We're going to compute some auxiliary integrals which are important themselves. The first one is the integral of the Gaussian distribution on $\mathbb{R}^n$*.*

(c) *Use the polar coordinates map on* $\mathbb{R}^2$ *to show*

$$\int_{(x,y)\in\mathbb{R}^2} e^{-(x^2+y^2)} = \pi.$$

(d) *Use Fubini's theorem to conclude*

$$\int_{x\in\mathbb{R}} e^{-x^2} = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad \text{and} \qquad \int_{\mathbf{x}\in\mathbb{R}^n} e^{-|\mathbf{x}|^2} = \pi^{n/2}.$$

Next we will find something about the $(n-1)$*-dimensional measure of the boundary of the ball* $B_r(\mathbf{p}) \subset \mathbb{R}^n$*. This is called* **Hausdorff measure***, and you know*

$$\mathcal{H}^1(\partial B_r(\mathbf{p})) = 2\pi r \qquad \textit{for } B_r(\mathbf{p}) \subset \mathbb{R}^2$$

and

$$\mathcal{H}^2(\partial B_r(\mathbf{p})) = 4\pi r^2 \qquad \textit{for } B_r(\mathbf{p}) \subset \mathbb{R}^3.$$

(e) *Use generalized polar coordinates to show*

$$\mathcal{H}^{n-1}(\partial B_1(\mathbf{0})) = n\omega_n \qquad \textit{so that} \qquad \mathcal{H}^{n-1}(\partial B_r(\mathbf{p})) = n\omega_n r^{n-1}.$$

(f) *Compute the integral of the Gaussian distribution using generalized polar coordinates to show*

$$\int_{\mathbf{x}\in\mathbb{R}^n} e^{-|\mathbf{x}|^2} = \frac{n\omega_n}{2}\int_0^{\infty} e^{-t}\, t^{\frac{n}{2}-1}\,dt.$$

We see that combining parts (f) and (d) we obtain a formula for ω_n *in terms of the integral*

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\,dt.$$

This is the other special integral we want to study. It's called the "Gamma function," and we can consider it for $x > 0$*.*

(g) *Show that* $\Gamma(1) = 1$ *and* $\Gamma(1/2) = \sqrt{\pi}$.

(h) *Integrate by parts to show* $\Gamma(x+1) = x\Gamma(x)$.

(i) *Show by induction that* $\Gamma(m) = (m-1)!$ *for* $m = 1, 2, 3, \ldots$ *and*

$$\Gamma\left(\frac{2k+1}{2}\right) = \Gamma\left(\frac{1}{2} + k\right) = \frac{(2k-1)(2k-3)\cdots 1}{2^k}\sqrt{\pi}$$

where $(2k-1)(2k-3)\cdots 1$ *is the product of the first* k *odd natural numbers.*

(j) *Find a general formula for* ω_{2k} *when* $k = 0, 1, 2, \ldots$. *Your answer should be simplified enough so that it involves only an integer power of* π *and a factorial.*

(k) *Find a general formula for* ω_{2k+1} *when* $k = 0, 1, 2, \ldots$. *Your answer should involve an integer power of* 2π *and the product of the first* $k+1$ *odd integers.*

(l) *Check your formulas with the values given for* ω_n *with* $n = 1, 2, \ldots, 7$ *given above.*

(m) *Draw pictures illustrating* $\mathcal{L}^n(B_1(\mathbf{0}))$ *and* $\mathcal{H}^{n-1}(\partial B_1(\mathbf{0}))$ *for the first* **four** *meaningful dimensions. Hint: You should get seven pictures.*

PDE

Weak Solutions

Okay, let's warm up to this idea with ODEs, and we're going to use just about the simplest ODE you can imagine. If $u \in C^1(a,b)$ is a solution of the first order ODE

$$\Phi(x, y, u') = 0,$$

then we say u is a **classical solution**. This, of course, means you can substitute u into the ODE and it satisfies the condition. We want to say what it means for a function $u \in C^0(a,b)$ (or maybe a function with even less regularity) to satisfy an ODE in certain situations.

For example, $u \in C^0(a,b)$ is a **weak solution** of the ODE $u' = 0$ if

$$\int_a^b u(x)\,\phi'(x)\,dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b). \tag{1}$$

Problem 2 **(a)** *Find all classical solutions of the ODE* $u' = 0$.

(b) *Show that if* $u \in C^1(a,b)$ *is a classical solution of* $u' = 0$, *then (1) holds, that is,* u *is a weak solution.*

(c) *Now, we want to show that if* $u \in C^0(a,b)$ *is a weak solution of* $u' = 0$, *then* u *is a classical solution you found in part (a). Complete the following steps carefully to do this.*

Let $\mu \in C_c^\infty(a,b)$ *be fixed and satisfy* $\int \mu = 1$. *Let* $\phi \in C_c^\infty(a,b)$ *and let* $c = \int \phi$.

(i) *Show* $\psi = \phi - c\mu \in C_c^\infty(a,b)$ *and*

$$\int \psi = 0.$$

(ii) *Show there exists some* $\eta \in C_c^\infty(a,b)$ *with*

$$\psi = \eta'.$$

(iii) *Notice that $\psi = \eta'$ is a function which can replace ϕ' in the condition (1) defining what it means for u to be a weak solution. Make this substitution with $\psi = \phi - c\mu$, and use the fundamental lemma of the calculus of variations to determine all weak solutions $u \in C^0(a,b)$ of the ODE $u' = 0$. Hint: This will take some manipulations of the integrals involved, and you'll need to substitute the definition of the constant c. Remember the fundamental lemma.*

(d) *Implicit in part (c) is a characterization of the subspace*

$$N = \{\psi \in C_c^\infty(a,b) : \text{ there exists some } \eta \in C_c^\infty(a,b) \text{ with } \psi = \eta'\}$$

in $C_c^\infty(a,b)$. Show N is the null space of the linear functional $L : C_c^\infty(a,b) \to \mathbb{R}$ given by

$$L\phi = \int \phi.$$

There are several directions we can go from here. One is that we could reduce the regularity in our definition of weak solution even further. We have defined a notion of C^0 weak solution using integration against a smooth test function. Remember, functions do not have to be continuous to be integrable. A function $u : (a,b) \to \mathbb{R}$ is said to be in $L^1_{loc}(a,b)$ if $\int_K |u|$ makes sense and is finite whenever $K \subset\subset (a,b)$. A function $u \in L^1_{loc}(a,b)$ is a weak solution of $u' = 0$ if

$$\int_{(a,b)} u\phi' = 0 \qquad \text{for every } \phi \in C_c^\infty(a,b). \tag{2}$$

Problem 3 **(a)** *Show that the Heaviside function*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not an L^1_{loc} weak solution of $u' = 0$ on $\mathbb{R}$.

(b) *Show that any L^1_{loc} weak solution of $u' = 0$ is a classical solution in the sense that there is some classical solution $u_0 \in C^1(a,b)$ for which*

$$u(x) = u_0(x) \qquad \textbf{for almost every } x \in (a,b).$$

The phrase "for almost every" here has a precise technical meaning. It means

$$\{x \in (a,b) : u(x) \neq u_0(x)\} \qquad \textit{has (one-dimensional Lebesgue)} \textbf{ measure zero}.$$

Hint: Use the strong(er) form of the fundamental lemma of the calculus of variations.

The other natural direction in the consideration of weak solutions of ODEs is to consider the higher order equation $u'' = 0$. As we saw for the first order equation $u' = 0$, there are different possible notions of weak solution for the second order equation $u'' = 0$. We will give **four** of them listed as I, II, III, and IV below. For the second one we will need an auxiliary definition which is also of related interest:

Definition 1 *A function $u \in L^1_{loc}(a,b)$ is said to have a* **weak derivative** $v \in L^1_{loc}(a,b)$ *if*

$$\int v\phi = -\int_{(a,b)} u\phi' \qquad \text{for all } \phi \in C_c^\infty(a,b). \tag{3}$$

The differential operator $D^* : C^\infty(a,b) \to C^\infty(a,b)$ by $D^*\phi = -\phi'$ appearing on the right in (3) is said to be the **adjoint** *of the operator $D : C^1(a,b) \to C^0(a,b)$ by $Du = u'$.*

The collection of all functions $u \in L^1_{loc}(a,b)$ with a weak derivative in $L^1_{loc}(a,b)$ is denoted by $W^1(a,b)$.

(I) A function $u : (a,b) \to \mathbb{R}$ is a C^1 weak solution of $u'' = 0$ if $u \in C^1(a,b)$ and

$$\int_a^b u'(x)\phi'(x)\,dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

(II) A function $u \in W^1(a,b)$ is a W^1 weak solution of $u'' = 0$ if

$$\int_{(a,b)} v\phi'(x) = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b)$$

where $v \in L^1_{loc}(a,b)$ is the weak derivative of u.

(III) A C^0 weak solution of $u'' = 0$ is a function $u \in C^0(a,b)$ with

$$\int_a^b u(x)\phi''(x)\,dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

(IV) A L^1_{loc} weak solution of $u'' = 0$ is a function $u \in L^1_{loc}(a,b)$ with

$$\int_{(a,b)} u\phi''(x) = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

Problem 4 *Find all classical solutions* $u \in C^2(a,b)$ *of* $u'' = 0$ *and show that any classical solution is a weak solution in each of the four senses I-IV given above. (Use integration by parts.)*

Problem 5 **(a)** *Show every weak* C^1 *solution of* $u'' = 0$ *is a classical solution. Hint:* u' *is a* C^0 *weak solution of* $w' = 0$*.*

(b) *Show every weak* W^1 *solution of* $u'' = 0$ *is a classical solution. Hint: This is, in principle, just as easy as the previous part because the weak derivative* v *of a weak* W^1 *solution* u *is an* L^1_{loc} *weak solution of* $w' = 0$*. However, you need to prove a technical lemma in this case:*

Lemma 1 *Given a function* $u \in W^1(a,b)$*, there exists a* **unique** *weak derivative* $v \in L^1_{loc}(a,b)$ *for* u *in the sense that if* $\tilde{v} \in L^1_{loc}(a,b)$ *is another weak derivative of* u*, then* $\tilde{v} = v$ *almost everywhere, i.e.,* $\tilde{v}(x) = v(x)$ *for almost every* $x \in (a,b)$*.*

Hint: Apply the fundamental lemma to the difference $\tilde{v} - v$*.*

(c) *Show every* C^0 *weak solution of* $u'' = 0$ *is a classical solution. Hint: For this you need a characterization of*

$$T = \{\psi \in C_c^\infty(a,b) : \psi = \eta'' \text{ for some } \eta \in C_c^\infty(a,b)\}.$$

If $\psi \in T$*, then* $\int x\psi = 0$*.*

(d) *Show every* L^1_{loc} *weak solution of* $u'' = 0$ *is a classical solution.*

Laplace's Equation and Poisson's Equation

Here is a Theorem.

Theorem 1 *If $\mathcal{U}$ is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and $g \in C^0(\partial\mathcal{U})$, then there exists a unique $u \in C^\infty(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ such that*

$$\begin{cases} \Delta u = 0 & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} \equiv g. \end{cases}$$

This is an existence and uniqueness theorem for Laplace's equation. You may not know what it means for $\partial\mathcal{U}$ to be C^2, so I will tell you: For each point $\mathbf{p} \in \partial\mathcal{U} \subset \mathbb{R}^n$, there is some $\epsilon > 0$ and some C^2 vector valued function

$$\psi \in C^2(B_1(\mathbf{0}) \to B_\epsilon(\mathbf{p}) \cap \partial\mathcal{U})$$

with domain the unit ball $B_1(\mathbf{0}) \subset \mathbb{R}^{n-1}$ and satisfying the following conditions

1. ψ is one-to-one and onto (surjective).
2. $\psi^{-1} \in C^0(B_\epsilon \cap \partial\mathcal{U} \to B_1(\mathbf{0}))$.
3. The total derivative $D\psi : B_1(\mathbf{0}) \to M_{n\times(n-1)}(\mathbb{R})$ where $M_{n\times(n-1)}(\mathbb{R})$ denotes the real valued $n \times (n-1)$ matrices has $Du(\mathbf{q})$ having **full rank**, i.e., rank $n-1$, at each $\mathbf{q} \in B_1(\mathbf{0}) \subset \mathbb{R}^{n-1}$.

It turns out (as you might imagine) this theorem is rather difficult to prove. One approach is to prove the existence of **weak solutions** and then prove that the weak solutions are actually **regular** in the sense of being classical solutions as given in the theorem.

We will not go through the details of the proof of the theorem, but I think it is reasonable for you to understand some of those details. To this end, let me introduce the usual definition of weak solutions for Laplace's equation. These are called H^1 weak solutions. As you might guess the collection $W^1_{loc}(\mathcal{U})$ of functions defined on $\mathcal{U}$ and having weak first partial derivatives is defined as follows: A function $u \in W^1_{loc}(\mathcal{U})$ if $u \in L^1_{loc}(\mathcal{U})$, that is,

$$\int_K |u| \qquad \text{makes sense and is finite whenever } K \subset\subset \mathcal{U},$$

and there exist function $v_1, v_2, \ldots, v_n \in L^1_{loc}(\mathcal{U})$, the weak partial derivatives of u, determined by the condition(s)

$$\int_{\mathcal{U}} v_j \phi = -\int_{\mathcal{U}} u D_j \phi \qquad \text{for all } \phi \in C_c^\infty(\mathcal{U}) \text{ and } j = 1, 2, \ldots, n.$$

The function $u \in W^1_{loc}(\mathcal{U})$ is in $H^1(\mathcal{U})$ if $u \in L^2(\mathcal{U})$, meaning

$$\int_{\mathcal{U}} |u|^2 < \infty,$$

and each of the weak partials $v_1, v_2, \ldots, v_n \in L^2(\mathcal{U})$ as well. The space H^1 is called the **Sobolev space** of functions with weak first partial derivatives in L^2. This space is used primarily because it is an inner product space with inner product

$$\langle u, w \rangle_{H^1} = \int_{\mathcal{U}} \left(uw + \sum_{j=1}^n v_j y_j \right)$$

where $v_1, v_2, \ldots, v_n$ are the weak first partials of u and $y_1, y_2, \ldots, y_n$ are the weak first partials of w.

With these preliminaries, here is the usual definition for weak solutions:

Definition 2 *A function $u \in H^1(\mathcal{U})$ is an H^1* **weak solution** *of the PDE $\Delta u = 0$ if*

$$\sum_{j=1}^n \int_{\mathcal{U}} v_j D_j \phi = 0 \qquad \text{for all } \phi \in C_c^\infty(\mathcal{U}).$$

Problem 6 **(a)** *Show a classical solution $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ of Laplace's equation with first partials $D_j u \in L^2(\mathcal{U})$ for $j = 1, 2, \ldots, n$ is a weak solution. (This requires you to show the classical derivatives are weak first partial derivatives and that the condition in the definition above holds if $\Delta u = 0$.)*

(b) *Show that a C^1 weak extremal $u \in C^1(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ for the Dirichlet energy with first partials $D_j u \in L^2(\mathcal{U})$ for $j = 1, 2, \ldots, n$ is a weak solution of $\Delta u = 0$.*

(c) *Show that an H^1 weak solution $u \in H^1(\mathcal{U}) \cap C^2(\mathcal{U})$ is a classical solution.*

(d) *Show that if a weakly differentiable function $u \in W^1(\mathcal{U})$ has a weak j-th partial derivative and u has a classical j-th partial derivative defined on an open set $U \subset \mathcal{U}$, then*

$$v_j = \frac{\partial u}{\partial x_j} \qquad \text{almost everywhere in } U.$$

Problem 7 *(The Fundamental Solution)*

(a) *Find all axially symmetric (classical) solutions* $u : \mathbb{R}^2 \to \mathbb{R}$ *of Laplace's equation having the form* $u(x, y) = \phi(x^2 + y^2)$.

(b) *Find all axially symmetric (classical) solutions* $u : \mathbb{R}^n \to \mathbb{R}$ *(*$n > 2$*) of Laplace's equation having the form* $u(\mathbf{x}) = \phi(|\mathbf{x}|^2)$.

In each of parts (a) and (b) you should have found a one parameter family of classical solutions $\{a\phi_0 : a \in \mathbb{R}\} \subset C^2(\mathbb{R}^n)$. *When you have identified a basis function* ϕ_0 *with confidence, you may proceed to the next parts.*

(c) *Find a two parameter family*

$$\{a\phi_0 + b\phi_1 : a, b \in \mathbb{R}\} \tag{4}$$

of axially symmetric solutions $\phi \in C^2(\mathbb{R}^n \backslash \{\mathbf{0}\})$ *of* $\Delta u = 0$ *on the punctured Euclidean space* $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

In the remaining parts of this problem, we restrict attention to the case $n > 2$. *You may formulate and solve versions of the parts below also for* $n = 2$.

Take $\phi_0 \in C^2(\mathbb{R}^n \backslash \{\mathbf{0}\})$, $n > 2$ *in part (c) as the restriction of the basis function* ϕ_0 *you found in part (b). Then choose the basis function* ϕ_1 *so that*

$$\lim_{|\mathbf{x}| \to \infty} \phi_1(\mathbf{x}) = 0.$$

We consider a solution of the form $\phi = b\phi_1$ *below. The* **fundamental solution** *of Laplace's equation is a solution of this form with an appropriate scaling constant* $b = b_n$. *The calculations of part (e) below is aimed at identifying the appropriate constant.*

(d) *Show* $\phi_1 \in L^1_{loc}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \backslash \{\mathbf{0}\})$.

(e) *Given* $f \in C^2_c(\mathbb{R}^n)$ *compute*

$$\Delta(\phi_1 * f).$$

If you have trouble with this calculation, you can follow the following steps/hints:

(i) *Write out the convolution integral and differentiate under the integral sign to compute an expression for the Laplacian having the form*

$$\int_{\mathbb{R}^n} f_1.$$

Hint: Write the convolution integral so that the derivatives fall on the function which has them on all of $\mathbb{R}^n$*. Technically, this requires some argument to justify the differentiation under the integral, but you can just assume it works. (Or you can prove it using difference quotients...and the dominated convergence theorem.)*

(ii) *Let* $\epsilon > 0$ *and break up your integral in the form*

$$\int_{\mathbb{R}^n} f_1 = \int_{B_\epsilon(\mathbf{0})} f_1 + \int_{\mathbb{R}^n \backslash B_\epsilon(\mathbf{0})} f_1.$$

At this point, you should make sure the singularity in ϕ_1 *is located at* $\mathbf{0} \in \mathbb{R}^n$ *with respect to the variable of integration, and you can show the first integral limits to zero as* $\epsilon \searrow 0$*. Thus, the strategy is to compute*

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \backslash B_\epsilon(\mathbf{0})} f_1.$$

(iii) *Use the divergence theorem/multivariable integration by parts to write the integral of interest in the form*

$$\int_{\mathbb{R}^n \backslash B_\epsilon(\mathbf{0})} f_1 = \int_{\partial B_\epsilon(\mathbf{0})} \phi_1 \mathbf{v} \cdot \mathbf{n} - \int_{\mathbb{R}^n \backslash B_\epsilon(\mathbf{0})} D\phi_1 \cdot \mathbf{v}$$

for an appropriate vector field $\mathbf{v}$*. Again, show the limit of the first integral is zero:*

$$\lim_{\epsilon \searrow 0} \int_{\partial B_\epsilon(\mathbf{0})} \phi_1 \mathbf{v} \cdot \mathbf{n}.$$

Now you are left with

$$-\int_{\mathbb{R}^n \backslash B_\epsilon(\mathbf{0})} D\phi_1 \cdot \mathbf{v}.$$

(iv) *Integrate by parts again to obtain*

$$\int_{\partial B_\epsilon(\mathbf{0})} \phi_1 \mathbf{v} \cdot \mathbf{n} = \int_{\partial B_\epsilon(\mathbf{0})} \mathbf{w} \cdot \mathbf{n}$$

for an appropriate vector field $\mathbf{w}$*. Hint: A term has vanished here.*

(v) *The vector field* $\mathbf{w}$ *will involve first derivatives of* ϕ_1*, and you'll need to compute those. Final hint: Express the last integral*

$$\int_{\partial B_\epsilon(\mathbf{0})} \mathbf{w} \cdot \mathbf{n}$$

as an **average** *over* $\partial B_\epsilon(\mathbf{p})$ *where* $\mathbf{p}$ *is the point of evaluation for* $\Delta(\phi_1 * f)(\mathbf{p})$*. Take the limit as* $\epsilon \searrow 0$*.*

Separation of Variables; Superposition

Problem 8 *(Assignment 4 Problem 4) Recall that you could not solve the boundary value problem*

$$\begin{cases} \Delta u = 0 \quad \text{on } \mathcal{U} = (-r, r) \times (0, L) \\ u(\pm r, y) = 0, \quad 0 \le y \le L, \quad u(x, L) = 0, \quad u(x, 0) = f(x), \quad |x| \le r, \end{cases} \tag{5}$$

with separated variables solutions $u(x, y) = A(x)B(y)$ *for all functions* $f \in C^2(-r, r)$*, but you can find such a solution for certain choices. For example, if*

$$f(x) = 5 \cos\left(\frac{\pi x}{2r}\right),$$

then

$$u(x, y) = 5 \, \cos\left(\frac{\pi x}{2r}\right) \left[\cosh\left(\frac{\pi y}{2r}\right) - \coth\left(\frac{\pi L}{2r}\right) \sinh\left(\frac{\pi y}{2r}\right)\right]$$

is the (unique) solution.

(a) *Find the solution when*

$$f(x) = 5 \cos\left(\frac{\pi x}{2r}\right) + 3 \sin\left(\frac{5\pi x}{r}\right).$$

(b) *Use mathematical software to plot the solution you found in part (a). (Take* $r = 1$ *and* $L = 3$*.)*

(c) *Adding solutions of a homogeneous linear PDE to obtain other solutions is called* **superposition**. *Often you can add infinitely many such solutions to get another solution as a series. Assuming*

$$f(x) = r^2 - x^2 = \sum_{k=0}^{\infty} a_k \cos\left(\frac{(2k+1)\pi x}{2r}\right)$$

find the (Fourier) coefficients $a_0, a_1, a_2, a_3, \ldots$. *Hint: Show that the set*

$$\left\{\cos\left(\frac{(2k+1)\pi x}{2r}\right)\right\}_{k=0}^{\infty}$$

is an orthogonal set in $L^2(-r, r)$. *(Assignment 3, Problem 11 Part (c))*

Compute the integral

$$\int_{-r}^{r} f(x)\, \cos\left(\frac{(2\ell+1)\pi x}{2r}\right)\, dx$$

assuming the series expansion above.

(d) *Produce a numerical plot of the first few terms of your series compared to the function* $f(x) = r^2 - x^2$. *How many terms are required to obtain an accuracy of* 0.0001*?*

(e) *Obtain the solution of*

$$\begin{cases} \Delta u = 0 \quad \text{on } \mathcal{U} = (-r, r) \times (0, L) \\ u(\pm r, y) = 0, \quad 0 \le y \le L, \quad u(x, L) = 0, \quad u(x, 0) = r^2 - x^2, \quad |x| \le r \end{cases}$$

as a superposition

$$u(x, y) = \sum_{k=0}^{\infty} c_k u_k(x, y)$$

for appropriate constants $c_0, c_1, c_2, c_3, \ldots$ *and separated variables solutions* $u_0, u_1, u_2, u_3, \ldots$.

(f) *Plot the first few terms of your solution series.*

First Order PDE

Problem 9 *Assume the first order PDE*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

has a solution $u \in C^1(\mathbb{R}^2)$ *with* $u(x, 0) = u_0(x)$. *Find the solution. Hint: Let* $\gamma(t)$ *be a curve with* $\gamma(0) = (x_0, 0)$, *and choose* γ *so that the PDE tells you* $u \circ \gamma(t) \equiv u_0(x_0)$ *is constant. (The chain rule.)*

The problem

$$\begin{cases} au_x + bu_y = f & \text{for } (x, y) \in \mathcal{U} \\ u \circ \alpha(t) = g \circ \alpha(t) & \text{for } t \in \mathbb{R} \end{cases}$$

is called the **Cauchy problem** for the first order PDE $au_x + bu_y = f$ in two variables. In this case, the condition $u \circ \alpha(t) = g \circ \alpha(t)$ is called the **Caunchy data** on the curve $\alpha : \mathbb{R} \to \mathcal{U}$. The Cauchy problem is a kind of PDE analogue of the initial value problem from ODEs.

Definition 1 *A curve* $\alpha \in C^1(\mathbb{R} \to \mathcal{U})$, *is said to be* **non-characteristic** *if* $\alpha'(t)$ *is not parallel to the coefficient vector* (a, b).

Problem 10 *Give examples of what can go wrong in the following situations:*

(a) If α *is characteristic. Hint: The curves* γ *in Problem 9 are called* **characteristic curves**.

(b) If $B_a(\mathbf{0}) \subset\subset \mathcal{U} \subset \mathbb{R}^2$ *and* $\alpha(t) = a(\cos t, \sin t)$. *This example shows why the Cauchy problem is very different from a Dirichlet problem.*

Notes and Solutions

Problem 2

(a) Find all classical solutions of the ODE $u' = 0$.

Classically, the solution space for this (linear homogeneous) ODE is the set of all constant functions:

$\{u : (a, b) \to \mathbb{R}$ such that there exists some $c \in \mathbb{R}$ with $u(x) \equiv c$ for all $x \in (a, b)\}$.

For classical solutions, you can just get this from the fundamental theorem of calculus by integrating the equation: Let $x_0 \in (a, b)$, then

$$\int_{x_0}^{x} u'(t)\, dt = 0,$$

so $u(x) - u(x_0) = 0$ or $u(x) \equiv u(x_0)$.

(b) Show that if $u \in C^1(a, b)$ is a classical solution of $u' = 0$, then (1) holds, that is, u is a weak solution.

There are two (obvious) ways to do this. The first is to note that a classical solution is a constant function, so

$$\int_a^b u(x)\phi'(x)\, dx = \int_a^b c\phi'(x)\, dx = c\int_a^b \phi'(x)\, dx = c[\phi(b) - \phi(b)] = 0$$

by the fundamental theorem of calculus. Another way, which is preferable in a certain sense, is to integrate by parts: Let $\tilde{a} = \min \operatorname{supp}(\phi)$ and $\tilde{b} = \max \operatorname{supp}(\phi)$, then $a < \tilde{a} \le \tilde{b} < b$, and

$$\int_a^b u(x)\phi'(x)\, dx = u\phi\Big|_{\tilde{a}}^{\tilde{b}} - \int_a^b u'(x)\phi(x)\, dx = 0$$

since $\phi(\tilde{a}) = \phi(\tilde{b}) = 0$ and $u'(x) = 0$ clasically.

(c) Now, we want to show that if $u \in C^0(a, b)$ is a weak solution of $u' = 0$, then u is a classical solution you found in part (a). Complete the following steps carefully to do this.

Let $\mu \in C_c^\infty(a, b)$ be fixed and satisfy $\int \mu = 1$. Let $\phi \in C_c^\infty(a, b)$ and let $c = \int \phi$.

(i) Show $\psi = \phi - c\mu \in C_c^\infty(a,b)$ and

$$\int \psi = 0.$$

It's clear that $\psi \in C_c^\infty(a,b)$ since $\operatorname{supp}(\psi)$ is a closed subset of $\operatorname{supp}(\phi) \cup \operatorname{supp}(\mu)$ which is a closed set compactly contained in (a,b). Also ψ is infinitely differentiable of course. Furthermore,

$$\int \psi = \int \phi - c \int \mu = \int \phi - c = 0.$$

(ii) Show there exists some $\eta \in C_c^\infty(a,b)$ with

$$\psi = \eta'.$$

Let $\eta : (a,b) \to \mathbb{R}$ by

$$\eta(x) = \int_a^x \psi(t)\,dt.$$

Then $\eta(x) = 0$ for all $\tilde{a} \leq x \geq \tilde{b}$ where $\tilde{a} = \min \operatorname{supp}(\psi)$ and $\tilde{b} = \max \operatorname{supp}(\psi)$. Therefore, $\operatorname{supp}(\eta) \subset \operatorname{supp}(\psi)$. Also, by the fundamental theorem of calculus

$$\eta'(x) = \psi(x),$$

so $\eta \in C_c^\infty(a,b)$ and satisfies the required condition.

(iii) Notice that $\psi = \eta'$ is a function which can replace ϕ' in the condition (1) defining what it means for u to be a weak solution. Make this substitution with $\psi = \phi - c\mu$, and use the fundamental lemma of the calculus of variations to determine all weak solutions $u \in C^0(a,b)$ of the ODE $u' = 0$.

$$\begin{aligned}
0 &= \int_a^b u(x)\eta'\,dx \\
&= \int_a^b u(x)\psi(x)\,dx \\
&= \int_a^b u(x)(\phi(x) - c\mu(x))\,dx \\
&= \int_a^b u(x)\phi(x)\,dx - c\int_a^b u(x)\mu(x)\,dx \\
&= \int_a^b u(x)\phi(x)\,dx - \int_a^b \phi(x)\,dx(\alpha)
\end{aligned}$$

where

$$\alpha = \int_a^b u(x)\mu(x)\,dx.$$

This means

$$\int_a^b [u(x) - \alpha]\phi(x)\,dx \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

By the fundamental lemma $u(x) = \alpha$ for $x \in (a,b)$. That is, u is a constant function.

(d) Implicit in part (c) is a characterization of the subspace

$$N = \{\psi \in C_c^\infty(a,b) : \text{ there exists some } \eta \in C_c^\infty(a,b) \text{ with } \psi = \eta'\}$$

in $C_c^\infty(a,b)$. Show N is the null space of the linear functional $L : C_c^\infty(a,b) \to \mathbb{R}$ given by

$$L\phi = \int \phi.$$

If $\psi \in N$, then

$$L\psi = \int \psi = \int \eta' = \eta(b) - \eta(a) = 0.$$

Thus, ψ is in the null space of L. On the other hand, if ψ is in the null space of L, then $\int \psi = 0$, and the discussion of part (c-ii) applies with $\eta : (a,b) \to \mathbb{R}$ by

$$\eta(x) = \int_a^x \psi(t)\,dt$$

giving a function $\eta \in C_c^\infty(a,b)$ with $\eta' = \psi$. Thus, $\psi \in N$.

Problem 2 Part (c):

If $\psi = \eta'' \in C_c^\infty(a,b)$, then

$$\int_a^b x\psi(x)\,dx = \int_a^b x\eta''(x)\,dx = -\int_a^b \eta'(x)\,dx = 0.$$

Maybe this is not such a comprehensive hint—actually, it's a pretty weak hint. What you really need is a way to "build" a C_c^∞ function ψ in T using an arbitrary C_c^∞

function ϕ. For this it helps to have a characterization of the functions in T, and the hinted condition is not enough. Let's take it a little more slowly. If, first of all, $\psi = \eta''$, then

$$\eta'(x) = \int_a^x \psi(t)\,dt,$$

and this tells us $\int \psi = 0$. Then we come to the hint:

$$\eta(x) = \int_a^x \int_a^s \psi(t)\,dt\,ds.$$

But Fubini's theorem, we can think of the expression on the right as an integral over a triangle in the s,t-plane and change the order of integration:

$$\eta(x) = \int_a^x \int_t^x \psi(t)\,ds\,dt = \int_a^x \psi(t)(x-t)\,dt = x\int_a^x \psi(t)\,dt - \int_a^x t\psi(t)\,dt.$$

Notice that this means, as suggested in the hint, that $\int x\psi = 0$ is a necessary condition for a function to be in T. In fact, we characterize T as the intersection of two null spaces

$$T = \ker(L) \cap \ker(M)$$

where $L : C_c^\infty(a,b) \to \mathbb{R}$ by $L\psi = \int \psi$ and $M : C_c^\infty(a,b) \to \mathbb{R}$ by $M\psi = \int x\psi$. We have established that $T \subset \ker(L) \cap \ker(M)$. Given any $\psi \in \ker(L) \cap \ker(M)$ it is straightforward to see that

$$\eta(x) = \int_a^x \int_a^s \psi(t)\,dt\,ds = x\int_a^x \psi(t)\,dt - \int_a^x t\psi(t)\,dt$$

defines $\eta \in C_c^\infty(a,b)$ with $\eta'' = \psi$. Thus, we have a characterization.

Then is the tricky part: How do you use this characterization to "build" a function in $\psi \in T$ from an arbitrary $\phi \in C_c^\infty(a,b)$? Here's what you can do. Set $\psi = \phi - c\mu - d\nu$ where $mu, \nu \in C_c^\infty(a,b)$ with $\int \mu = \int \nu = 1$. Then the conditions $\int \psi = 0$ and $\int x\psi = 0$ give a pair of linear equations for the constant coefficents c and d, namely,

$$\left(\int \mu\right) c + \left(\int \nu\right) d = \int \phi$$

$$\left(\int x\mu\right) c + \left(\int x\nu\right) d = \int x\phi.$$

That is,

$$\begin{cases} c + d = \int \phi \\ \left(\int x\mu\right) c + \left(\int x\nu\right) d = \int x\phi. \end{cases}$$

In order to get a solution for this equation, we need

$$\int x\nu - \int x\mu = \int x(\nu - \mu) \neq 0.$$

One easy way to accomplish this is to define ν to simply be a small shift of μ. That is, fix $\mu \in C_c^\infty(a,b)$ with $\int \mu = 1$ and then define $\nu \in C_c^\infty(a,b)$ by $\nu(x) = \mu(x - \epsilon)$ where $\epsilon > 0$ is small enough so that $\{x : x - \epsilon \in \text{supp}(\mu)\} \subset\subset (a,b)$. Then

$$\int x\nu - \int x\mu = \int (x+\epsilon)\mu - \int x\mu = \epsilon > 0.$$

Furthermore, we can take

$$c = \frac{1}{\epsilon}\left[\int x\phi - \left(\int x\nu\right)\left(\int \phi\right)\right] \qquad \text{and} \qquad d = \frac{1}{\epsilon}\left[\left(\int x\mu\right)\left(\int \phi\right) - \int x\phi\right]$$

so that $\psi = \phi - c\mu - c\nu = \eta'' \in T$, and

$$\begin{aligned} 0 &= \int u\psi \\ &= \int u\phi - c\int u\mu - d\int u\nu \\ &= \int u\phi - \frac{1}{\epsilon}\int \left(\int u\mu\right) x\phi + \frac{1}{\epsilon}\int \left(\int u\mu\right)\left(\int x\nu\right)\phi \\ &\qquad - \frac{1}{\epsilon}\int \left(\int u\nu\right)\left(\int x\mu\right)\phi + \frac{1}{\epsilon}\int\left(\int u\nu\right) x\phi \\ &= \int [u - \alpha x - \beta]\phi \end{aligned}$$

where

$$\alpha = \frac{1}{\epsilon}\left[\left(\int u\mu\right) - \left(\int u\nu\right)\right] = \frac{1}{\epsilon}\left(\int u(\mu - \nu)\right)$$

and

$$\beta = \frac{1}{\epsilon}\left[\left(\int u\nu\right)\left(\int x\mu\right) - \left(\int u\mu\right)\left(\int x\nu\right)\right].$$

Since this holds for every $\phi \in C_c^\infty(a,b)$, we have $u(x) = \alpha x + \beta$ for every $x \in (a,b)$.