Assignment 4: Selected Solutions
Laplace's equation
Due Friday, February 21, 2025

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Problem 1 (Slinky) Make measurements of the hanging slinky (and any other measurements associated with the hanging slinky physical system which you hope to be able to compare to your model function from Problem 1 of Assignment 1). Do the measurements match the qualitative expectations you gave in Problem 10 of Assignment 1?

Problem 2 (non-uniqueness for ODEs) Consider the IVP

$$\begin{cases} y' = \sqrt{|y|} \\ y(t_0) = y_0 \end{cases} \tag{1}$$

with $t_0, y_0 \in \mathbb{R}$.

(a) What does Theorem 1 of Assignment 3 tell you about the IVP (1)?

(b) Solve the IVP (1).

(c) Find particular values of y_0 and t_0 for which (1) has three distinct solutions $y_1, y_2, y_3 \in C^1(\mathbb{R})$.

Problem 3 (harmonic functions) Any solution of Laplace's equation is called a **harmonic function**. Consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^z$.

(a) Write $z = x + iy$ as in part **(a)** of Problem 8 of Assignment 3 and find the real and imaginary parts u and v of e^z. Hint: Remember Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

(b) Show u and v are harmonic.

(c) Two harmonic functions $u, v : \Omega \to \mathbb{R}$ are said to be **harmonic conjugates** if they satisfy[1] the first order system of partial differential equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show u and v from part **(a)** above are harmonic conjugates.

Problem 4 (axially symmetric solutions of Laplace's equation; polar coordinates) Consider Laplace's equation $\Delta u = 0$ on the **punctured plane** $\mathbb{R}^2 \backslash \{(0,0)\}$. Find all solutions of the form

$$u(\mathbf{x}) = \phi(|\mathbf{x}|)$$

where $\phi : (0, \infty) \to \mathbb{R}$ is a function of one variable with $\phi \in C^2(0, \infty)$. Hint:

$$\frac{\partial u}{\partial x_1} = \phi'(|\mathbf{x}|)\, \frac{x_1}{|\mathbf{x}|}.$$

Find an ODE satisfied by $\phi = \phi(r)$ and then integrate to obtain a solution in terms of two constants $\phi(1)$ and $\phi'(1)$.

Solution:

$$\frac{\partial^2 u}{\partial x_j^2} = \phi''(|\mathbf{x}|)\, \frac{x_j^2}{|\mathbf{x}|^2} + \phi'(|\mathbf{x}|) \left(\frac{1}{|\mathbf{x}|} - \frac{x_j^2}{|\mathbf{x}|^3} \right).$$

Therefore,

$$0 = \Delta u = \phi''(|\mathbf{x}|) + \phi'(|\mathbf{x}|) \left(\frac{2}{|\mathbf{x}|} - \frac{1}{|\mathbf{x}|} \right).$$

[1] This system of two first order PDEs is called the system of **Cauchy-Riemann equations**.

Thus, we arrive at the ODE $\phi'' + \phi'/r = 0$. Setting $v = \phi'$ we have

$$v' + v/r = 0 \qquad \text{or} \qquad -\frac{v'}{v} = \frac{1}{r}$$

at least away from the zero solution $v \equiv 0$. Of course the unique solution with $v(1) = \phi'(1) = 0$ is $v \equiv 0$ corresponding to $\phi(r) \equiv c$ for any constant $c \in \mathbb{R}$. This gives axially symmetric constant solutions on all of $\mathbb{R}^2$.

Integrating $-v'/v = 1/r$ from $r = 1$ we find

$$-\int_1^r \frac{v'(\rho)}{v(\rho)}\, d\rho = \ln r.$$

Changing variables with $\eta = v(\rho)$ so that $d\eta = v'(\rho)\, d\rho$ and assuming for the moment that $\phi'(1) > 0$ we see

$$-\int_{\phi'(1)}^{\phi'} \frac{1}{\eta}\, d\eta = \ln \frac{\phi'(1)}{\phi'} = \ln r.$$

This gives

$$\phi'(r) = \frac{\phi'(1)}{r}$$

and $\phi(r) = \phi(1) + \phi'(1)\ \ln r$. If $\phi'(1) < 0$, then we can set $w(r) = -v(r)$ and have $-w'/w = 1/r$ with $w(1) = -\phi'(1) > 0$. The solution above gives

$$-\phi'(r) = w(r) = \frac{w(1)}{r} = -\frac{\phi'(1)}{r},$$

so this is the same equation for ϕ. The general solution $u : \mathbb{R}^1 \backslash \{(0,0)\} \to \mathbb{R}$ for $n = 2$ has values

$$u(x, y) = \phi(1) + \phi'(1)\ \ln \sqrt{x^2 + y^2}$$

where $\phi(1)$ and $\phi'(1)$ are any constants.

Problem 5 (fundamental solution of Laplace's equation when $n = 2$) The **fundamental solution** of Laplace's equation when $n = 2$ is given by

$$\Phi(x, y) = -\frac{1}{2\pi} \log \sqrt{x^2 + y^2}.$$

(a) Note that the ODE you found for ϕ in Problem 4 above was a second order homogeneous linear ODE. Consequently, the solution set

$$\Sigma = \{\phi \in C^\infty(0, \infty) : \Delta u = 0\}$$

is a two dimensional vector subspace of $C^\infty(0, \infty)$. Thus, the solutions you found should have been expressed as linear combinations $\phi = a\phi_1 + b\phi_2$ of two linearly independent basis solutions ϕ_1 and ϕ_2. Find the linear combination of your solutions that gives the fundamental solution.

(b) Sketch the graphs of the functions ϕ and Φ where ϕ is the solution you found in part **(a)** leading to Φ.

(c) The fundamental solution Φ has a singularity at $\mathbf{x} = (x, y) = (0, 0)$. Show however that Φ is integrable across the singularity so that

$$0 < \int_{B_1(\mathbf{0})} \Phi < \infty.$$

Hint: Integrate in polar coordinates.

Solution:

(a) For the solution $\phi(r) = \phi(1) + \phi'(1)\ \ln r$ we can take $\phi_1(r) = \ln r$ and $\phi_2(r) \equiv 1$. Thus,

$$\phi(r) = \phi'(1)\ \ln r + \phi(1)\ \phi_2(r).$$

With this choice, the coefficients $\phi(1) = 0$ and $\phi'(1) = -1/(2\pi)$ give the fundamental solution.

(b) Plots of Φ and ϕ:

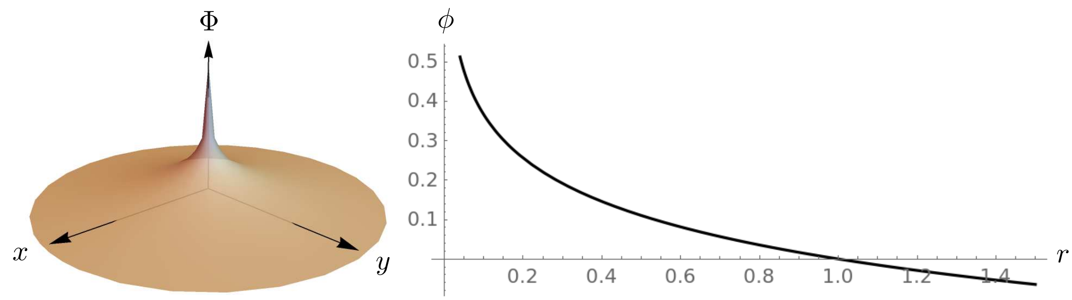

Figure 1: Plot of the graph of the fundamental solution Φ when $n = 2$ (left). Plot of the profile or meridian of the graph (right).

(c) Notice that $\Phi(\mathbf{x}) > 0$ for $|\mathbf{x}| < 1$. Therefore,

$$\int_{B_1(\mathbf{0})} \Phi > 0.$$

Note on the other hand that

$$\int_{B_1(\mathbf{0})} \Phi = \lim_{\epsilon \searrow 0} \int_{B_1(\mathbf{0}) \setminus B_\epsilon(\mathbf{0})} \Phi.$$

We then have for $n = 2$ and $0 < \epsilon < 1$

$$\begin{aligned}
\int_{B_\epsilon(\mathbf{0})} \Phi &= -\frac{1}{2\pi} \int_{\mathbf{x} \in B_1(\mathbf{0}) \setminus B_\epsilon(\mathbf{0})} \ln |\mathbf{x}| \\
&= -\frac{1}{2\pi} \int_0^{2\pi} \int_\epsilon^1 r \ln r \, dr \, d\theta \\
&= -\int_\epsilon^1 r \ln r \, dr \\
&= -\int_\epsilon^1 r \left(\int_1^r \frac{1}{\xi} \, d\xi \right) \, dr \\
&= \int_\epsilon^1 r \left(\int_r^1 \frac{1}{\xi} \, d\xi \right) \, dr \\
&\leq \int_\epsilon^1 r \left(\int_r^1 \frac{1}{r} \, d\xi \right) \, dr \\
&= \int_\epsilon^1 (1 - r) \, dr \\
&= 1 - \epsilon - \frac{1}{2}(1 - \epsilon^2) \\
&< 1/2.
\end{aligned}$$

Alternatively for a more precise answer one can integrate by parts:

$$\begin{aligned}
-\int_\epsilon^1 r \ln r \, dr &= -\frac{1}{2} \int_\epsilon^1 \left(\frac{d}{dr} r^2 \right) \ln r \, dr \\
&= -\frac{1}{2} \left[\left(r^2 \ln r \right)\Big|_\epsilon^1 - \int_\epsilon^1 r \, dr \right] \\
&= \frac{1}{2} \left[\epsilon^2 \ln \epsilon + \frac{1}{2} \left(1 - \epsilon^2 \right) \right] \\
&= \frac{1}{2} \left[\frac{1}{2} - \epsilon^2 \left(\ln \epsilon - \frac{1}{2} \right) \right].
\end{aligned}$$

Here we can take the limit using L'Hopital's rule to see that for $n = 2$

$$-\lim_{\epsilon \searrow 0} \frac{\ln \epsilon}{1/\epsilon} = -\lim_{\epsilon \searrow 0} \epsilon = 0 \qquad \text{and} \qquad \int_{B_1(\mathbf{0})} \Phi = \frac{1}{4}.$$

Problem 6 (separated variables solution) Consider $\Delta u = 0$ on the rectangular domain $\Omega = (0, L) \times (0, M) \subset \mathbb{R}^2$ where L and M are positive real numbers. The boundary of Ω is a rectangle. In this case it is natural to look for a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ where $\overline{\Omega}$ is the closure of Ω or the closed rectangular domain.

Assume $u(x, 0) = \sin(\pi x/L)$ and $u(0, y) = u(L, y)$ for $y \in [0, M]$. Find all solutions of the form $u(x, y) = f(x)\phi(y)$ with $\phi(0) = 1$.

Solution: The PDE gives

$$f''\phi + f\phi'' = 0$$

where the ordinary derivatives on f are with respect to x and the ordinary derivatives on ϕ are with respect to y. For any point $(x, y) \in (0, L) \times (0, M)$ with $f(x) \neq 0$ and $\phi(y) \neq 0$ there holds

$$\frac{f''(x)}{f(x)} = -\frac{\phi''(y)}{\phi(y)}. \tag{2}$$

Fixing one such point $(x_0, y_0) \in (0, L) \times (0, M)$ we may conclude from continuity that there is some $r > 0$ for which the relation (2) will hold for $(x, y) \in B_r(x_0, y_0)$. Differentiating both sides of (2) in this ball with respect to x we see

$$\frac{\partial}{\partial x}\left(\frac{f''(x)}{f(x)}\right) = 0.$$

Thus, the function $g : B_r(x_0, y_0) \to \mathbb{R}$ with values

$$g(x, y) = \frac{f''(x)}{f(x)}$$

is a constant λ on $B_r(x_0, y_0)$. In this way, we obtain two ODEs:

$$f'' = \lambda f \qquad \text{and} \qquad \phi'' = -\lambda\phi \tag{3}$$

for $f = f(x)$ and $\phi = \phi(y)$ holding on $(x, y) \in B_r(x_0, y_0)$ and for $x_0 - r < x < x_0 + r$ and $y_0 - r < y < y_0 + r$ respectively. If the constant λ is negative, then

$$f(x) = a_1 \cos(\mu x) + a_2 \sin(\mu x) \qquad \text{and} \qquad \phi(y) = b_1 \cosh(\mu y) + b_2 \sinh(\mu y)$$

for some real constants a_1, a_2, b_1, b_2 and $\mu = \sqrt{-\lambda}$. Note that any function $w \in C^\infty(B_r(x_0, y_0)$ with values

$$w(x, y) = [a_1 \cos(\mu x) + a_2 \sin(\mu x)]\ [b_1 \cosh(\mu y) + b_2 \sinh(\mu y)]$$

extends to a function $u \in C^\infty(\mathbb{R}^2)$ by the same formula.

If we seek a solution u of the original problem given globally by the formula

$$u(x,y) = [a_1 \cos(\mu x) + a_2 \sin(\mu x)]\ [b_1 \cosh(\mu y) + b_2 \sinh(\mu y)], \tag{4}$$

the auxiliary condition $\phi(0) = 1$ implies $b_1 = 1$, and the condition $u(x,0) = \sin(\pi x/L)$ then requires

$$f(x) = a_1 \cos(\mu x) + a_2 \sin(\mu x) = \sin(\pi x/L) \tag{5}$$

for $0 < x < L$. The condition $u(0,y) = u(L,y)$ is now automatically satisfied. So far we have a potential solution of the form

$$u(x,y) = \sin(\pi x/L)\ [\cosh(\mu y) + b \sinh(\mu y)].$$

Note that with a function u of this form we have

$$\Delta u = \left(-\frac{\pi^2}{L^2} + \mu^2 \right) u.$$

Thus, we must also have $\mu = \pm\pi/L$. Since the hyperbolic cosine is even and the constant b is arbitrary, all of these solutions constitute a one parameter family $u \in C^\infty(\mathbb{R}^2)$ given by the formula(s)

$$u(x,y) = \sin(\pi x/L)\ [\cosh(\pi y/L) + b \sinh(\pi y/L)]. \tag{6}$$

If $\lambda = 0$ the ODEs (3) have solutions

$$f(x) = a_1 x + a_2 \qquad \text{and} \qquad \phi(y) = b_1 y + b_2.$$

Again $w \in C^\infty(\mathbb{R}^2)$ by $w(x,y) = (a_1 x + a_2)(b_1 y + b_2)$ is a four parameter separated variables solution of the PDE $\Delta u = 0$, but if we attempt to find a global solution on $[0, L] \times [0, M]$ satisfying all the conditions of the problem we need the following:

1. $b_2 = 1$ and
2. $a_1 x + a_2 = \sin(\pi x/L)$ for $0 < x < L$.

The second condition cannot hold since taking $x = 0$ gives $a_2 = 0$ and then taking $x = L$ gives $a_1 = 0$.

Finally, we consider the case $\lambda > 0$. Then (3) has (general) solutions

$$f(x) = a_1 \cosh(\mu x) + a_2 \sinh(\mu x) \qquad \text{and} \qquad \phi(y) = b_1 \cos(\mu y) + b_2 \sin(\mu y)$$

for some real constants a_1, a_2, b_1, b_2 and $\mu = \sqrt{\lambda}$. Again, the function $w \in C^\infty(B_r(x_0, y_0))$ with values

$$w(x, y) = [a_1 \cosh(\mu x) + a_2 \sinh(\mu x)]\ [b_1 \cos(\mu y) + b_2 \sin(\mu y)]$$

extends to a function $u \in C^\infty(\mathbb{R}^2)$ by the same formula, and we can seek a solution u of the original problem given globally by the formula

$$u(x, y) = [a_1 \cosh(\mu x) + a_2 \sinh(\mu x)]\ [b_1 \cos(\mu y) + b_2 \sin(\mu y)], \tag{7}$$

the auxiliary condition $\phi(0) = 1$ again implies $b_1 = 1$, and the condition $u(x, 0) = \sin(\pi x/L)$ then requires

$$f(x) = a_1 \cosh(\mu x) + a_2 \sinh(\mu x) = \sin(\pi x/L). \tag{8}$$

for $0 < x < L$. Now taking $x = 0$ gives $a_1 = 0$ and then taking $x = L$ gives $a_2 = 0$, so this approach cannot lead to any new solutions different from those given in (6).

There is a bit of ambiguity here concerning the transition from the local solution w of the PDE to the global solution to which one can apply the auxiliary condition and the boundary conditions, so it is not entirely clear we have found **all** separated variables solutions $u(x, y) = f(x)\phi(y)$. The solutions we have found however are at least the traditional separated variables solutions.

More generally, if Ω is a bounded open subset of $\mathbb{R}^n$ one may consider the boundary value problem (BVP)

$$\begin{cases} \Delta u = 0, & \text{on } \Omega \\ u\Big|_{\partial\Omega} = g \end{cases} \tag{9}$$

where $g : \partial\Omega \to \mathbb{R}$ is a given function. If $g \equiv 0$, then the boundary condition is said to be homogeneous.

Problem 7 (Laplace equation BVP; uniqueness of solutions) Complete the following steps to show the problem (9) has a unique solution when $g \equiv 0$:

(a) Find one solution.

(b) Assume there is a solution u different from the one you found in part **(a)** and $u(\mathbf{p}) > 0$ for some $\mathbf{p} \in \Omega$. Consider $v \in C^\infty(\mathbb{R}^n)$ with values given by

$$v(\mathbf{x}) = u(\mathbf{p}) - \frac{\epsilon}{2}\,|\mathbf{x} - \mathbf{p}|^2,$$

and show that if $\epsilon > 0$ is small enough, then

$$v\Big|_{\partial\Omega} > 0.$$

(c) Show there is some $\alpha \geq 0$ and $\mathbf{q} \in \Omega$ such that $w \in C^\infty(\mathbb{R}^n)$ with values given by

$$w(\mathbf{x}) = v(\mathbf{x}) + \alpha$$

satisfies

(i) $w \geq u$ on $\overline{\Omega}$, and

(ii) $w(\mathbf{q}) = u(\mathbf{q})$.

Hint(s):

$$u(\mathbf{q}) - v(\mathbf{q}) = \max_{\mathbf{x}\in\Omega}[u(\mathbf{x}) - v(\mathbf{x})].$$

(d) Show $\Delta u(\mathbf{q}) \leq \Delta w(\mathbf{q}) < 0$ contradicting the fact that $\Delta u(\mathbf{q}) = 0$.

Problem 8 Explain why the steps in Problem 7 essentially imply uniqueness of solutions for the problem

$$\begin{cases} \Delta u = 0, & \text{on } \Omega \\ u\Big|_{\partial\Omega} \equiv 0 \end{cases}$$

when Ω is bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Problem 9 (Boundary values and Poisson's equation) Consider a special case of the BVP (9) for Laplace's equation in which we assume there is a function $\phi \in C^2(\overline{\Omega})$ with restriction satisfying

$$\phi\Big|_{\partial\Omega} \equiv g.$$

(a) Find a boundary value problem satisfied by $v = u - \phi$.

(b) The equation $\Delta u = f$ where $f : \Omega \to \mathbb{R}$ is a given function is called **Poisson's equation**. Notice that Laplace's equation is the associated homogeneous equation for this PDE. Use Problem 8 to prove the following uniqueness assertion: If Ω is a bounded open subset of $\mathbb{R}^n$ and $v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ both satisfy the BVP

$$\begin{cases} \Delta u = f, & \text{on } \Omega \\ u\Big|_{\partial\Omega} = g \end{cases}$$

then $v \equiv w$.

Problem 10 (Laplace's PDE in one dimension)

(a) State/pose the BVP for Laplace's equation in one dimension. Hint: Take $\Omega = (a, b)$.

(b) Solve the problem from part **(a)**.

(c) Write your solution as a convex combination of $u(a)$ and $u(b)$.