Math 6702, Assignment 4

Introduction

1. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be a complex valued function of a complex variable on an open set U in the complex plane $\mathbb{C}$. Assume u and v are partially differentiable at $\mathbf{p} \in \{(x,y) : x+iy \in U\}$ and there exists a limit

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} \in \mathbb{C}.$$

In this case, we say f is **complex differentiable**. The symbol $f'(z)$ has no other meaning than to denote this limit. In technical terms, one can say the following:

> There is a complex number $L \in \mathbb{C}$ such that for every $\epsilon > 0$ there is some $\delta > 0$ such that
>
> $$\left.\begin{array}{l} \zeta \in U \\ 0 < |\zeta - z| < \delta \end{array}\right\} \qquad \Longrightarrow \qquad \left| \frac{f(\zeta) - f(z)}{\zeta - z} - L \right| < \epsilon.$$
>
> When this holds, we denote the limit $L \in \mathbb{C}$ by $f'(z)$.

Note that the "absolute value" here is the **complex modulus** which is essentially the same as the Euclidean norm in $\mathbb{R}^2$:

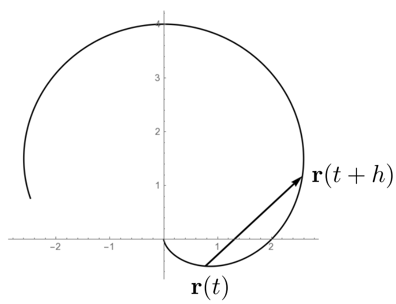
$$|z| = |x+iy| = \sqrt{x^2+y^2}.$$

Show that u and v satisfy the Cauchy-Riemann equations. Hint: Compute the limit as $h \to 0$ and $\zeta = z+h = x+h+iy$. Then compute the limit as $\zeta = z+ih \to z$. Compare the answers you get.

Chapter 4 Partial Derivatives

2. Recall that a **path** connecting two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ is a continuous function $\mathbf{r} : [a,b] \to \mathbb{R}^n$ such that $\mathbf{r}(a) = \mathbf{p}$ and $\mathbf{r}(b) = \mathbf{q}$. Given a path connecting $\mathbf{p}$ to $\mathbf{q}$, prove there is a path $\gamma : [0,1] \to \mathbb{R}^n$ connecting $\mathbf{p}$ to $\mathbf{q}$ with $\gamma(0) = \mathbf{p}$ and $\gamma(1) = \mathbf{q}$.

3. Use mathematical software to produce a drawing like the one below with a secant vector connecting two points on a parameterized curve. Be sure to indicate the formula for the curve you have illustrated.



4. Explain why the derivative

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

if it exists, is a vector with norm giving the instantaneous rate of change of position along the curve with respect to t. Hint: Your answer should have the notion of **average speed** in it somewhere.

5. Recall that a path $\mathbf{r}$ connecting two points is C^1 if the coordinate functions $r_1, r_2, \ldots, r_n$ are C^1. Show that if there is a C^1 path connecting two points, then there is a **unit speed** path connecting the two points. This means, we can assume $|\mathbf{r}'(t)| = 1$ for all t.

6. Let $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0. \end{cases}$$

Show the first order partials of u both exist at the point $(0,0)$, but u is not continuous at $(0,0)$.

7. Let $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = \begin{cases} xy/(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Show the first order partials of u both exist at every point of $\mathbb{R}^2$, but u is not continuous at $(0,0)$.

§4.5 Chain Rule

8. (4.5.1) If $u(x,y) = xe^{-y}$, $\mathbf{r}(t) = (\cosh t, \cos t)$, use the chain rule to find

$$\frac{d}{dt} u \circ \mathbf{r}(t).$$

9. (4.5.5) If $u = u(x,y)$ and $\mathbf{r}(x) = (x, y(x))$, show

$$\frac{d}{dx} u \circ \mathbf{r} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$