Assignment 4:
Partial Differential Equations (and other topics)
Due Wednesday, February 15, 2023

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Problem 1 (Slinky) Make measurements of the hanging slinky (and any other measurements associated with the hanging slinky physical system which you hope to be able to compare to your model function from Problem 1 of Assignment 1. Do the measurements match the qualitative expectations you gave in Problem 10 of Assignment 1?

Problem 2 Find a first order system equivalent to the ODE

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x).$$

Problem 3 Find a system of first order (partial differential) equations equivalent to the hyperbolic PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

Problem 4 (Boas 8.6.28) Consider the following ordinary differential operators on complex valued functions of a real variable:

$$\frac{d}{dt} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \frac{d}{dt} u = u'$$

and

$$\mathrm{id} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \mathrm{id}\, u = u.$$

(a) Expand the linear constant coefficient operator

$$L : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad Lu = \left(\frac{d}{dt} - a\,\mathrm{id}\right)\left(\frac{d}{dt} - b\,\mathrm{id}\right) u$$

where a and b are complex numbers to obtain an expression of the form $Lu = u'' + pu' + qu$ for complex numbers p and q.

(b) Find the general solution of $Lu = ke^{ct}$ where k and c are complex numbers by solving $y' - ay = ke^{ct}$ first and then solving $u' - bu = y$ (as linear first order ODEs) in the three cases:

(i) $c \neq a$ and $c \neq b$.

(ii) $a \neq b$ and $c = a$.

(iii) $a = b = c$.

Problem 5 Find a function $u : \mathbb{R}^2 \to \mathbb{R}$ with

$$\frac{\partial u}{\partial x} \in C^0(\mathbb{R}^2) \qquad \text{and} \qquad \frac{\partial u}{\partial y} \in C^0(\mathbb{R}^2),$$

but $u \notin C^2(\mathbb{R}^2)$.

Problem 6 (directional derivatives) Given a function $f \in C^1(\mathcal{U})$ with $\mathcal{U}$ an open subset of $\mathbb{R}^2$ and a unit vector $\mathbf{u} \in \mathbb{R}^2$ the **directional derivative** of f in the direction $\mathbf{u}$ at $\mathbf{p} \in \mathcal{U}$ is defined by

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h}.$$

Assume $f \in C^1(\mathcal{U})$ as above with $\mathbf{p} \in \mathcal{U}$ and $\mathbf{u}$ a unit vector in $\mathbb{R}^2$. Use the chain rule to show the following

(a) If $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ by $\mathbf{r}(t) = \mathbf{p} + t\mathbf{u}$, then

$$D_{\mathbf{u}} f(\mathbf{p}) = \frac{d}{dt} f \circ \mathbf{r}(t)\Big|_{t=0}.$$

(b) $D_{\mathbf{u}} f(\mathbf{p}) = Df(\mathbf{p}) \cdot \mathbf{u}$.

(g) If $\mathbf{r} : (a,b) \to \mathcal{U}$ is any path in $\mathcal{U}$ satisfying $\mathbf{r}(0) = \mathbf{p}$ and $\mathbf{r}'(0) = \mathbf{u}$, then

$$D_{\mathbf{u}} f(\mathbf{p}) = \frac{d}{dt} f \circ \mathbf{r}(t)\Big|_{t=0}.$$

Problem 7 (Lagrange multipliers) Let c be a constant, and let $u, g \in C^1(\mathcal{U})$ with $\mathcal{U} \subset \mathbb{R}^2$. If $\mathbf{p} \in \mathcal{U}$ is a point for which $g(\mathbf{p}) = c$ and we have $u(\mathbf{p}) \le u(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{U}$ with $g(\mathbf{x}) = c$, then show one of the two following conditions holds:

(i) $Dg(\mathbf{p}) = \mathbf{0}$, or

(ii) There exists some $\lambda \in \mathbb{R}$ with

$$D(u - \lambda g)(\mathbf{p}) = \mathbf{0}.$$

Problem 8 (Boas Problem 4.9.4) Consider a rectangular parallelopiped (i.e., box)

$$\{(x, y, z) \in \mathbb{R}^3 : |x| < a, |y| < b, |z| < x\}$$

with

$$\frac{a^2}{4} + \frac{b^2}{9} + \frac{c^2}{25} = 1.$$

Determine the values (a, b, c) corresponding to the box like this with a/the largest volume.

Problem 9 (Boas Problem 4.9.8) Find

$$(x_0, y_0) \in A = \{(x, y) \in \mathbb{R}^2 : 2x + 3y = 4\}$$

to minimize among $(x, y) \in A$ the sum of the squares of the distances from (x, y) to $(-1, 0)$ and $(1, 0)$.

Problem 10 Assume the first order PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

has a solution $u \in C^1(\mathbb{R}^2)$ with $u(x, 0) = u_0(x)$. Find the solution. Hint: Let $\gamma(t)$ be a curve with $\gamma(0) = (x_0, 0)$, and choose γ so that the PDE tells you $u \circ \gamma(t) \equiv u_0(x_0)$ is constant. (The chain rule.)