MATH 6702 Assignment 4 = Exam 2
Due Monday March 22, 2021

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Laplace's Equation and Poisson's Equation

Problem 1 *(Boas 13.1.1) Recall Problem 1 and Problem 4 from Assignment 3. A standard model of electrostatics involves the notions of* **charge density** *and* **electrostatic potential**. *The* **charge density** *is a function $\rho : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^3$ having units*

$$[\rho] = \frac{[\ \textit{charge}\ ]}{L^3}$$

so that the total charge in a region $\Omega \subset \mathcal{U}$ is

$$\int_\Omega \rho.$$

The **electrostatic potential** *is a function $v : \mathcal{U} \to \mathbb{R}$ having, first of all, the property that the* **electric field** *on $\mathcal{U}$ is defined by*

$$\mathbf{E} = -\nabla v = -Dv.$$

The electrostatic potential, furthermore, has units

$$[v] = \frac{[\ \textit{energy}\ ]}{[\ \textit{charge}\ ]}$$

so that ρv defines an **energy density** *on $\mathcal{U}$. Finally, the constant ϵ_0 is called the* **permittivity of free space** *and has units*

$$[\epsilon_0] = \frac{[\ \textit{charge}\ ]^2}{[\ \textit{energy}\ ]\, L}.$$

(a) *Find the units of*

$$\mathcal{D}[v] = \epsilon_0 \int_{\mathcal{U}} |Dv|^2.$$

(b) *Find the units of*

$$\mathcal{E}[v] = \int_{\mathcal{U}} \rho v.$$

(c) *Interpret the extremization of $\mathcal{H} = \mathcal{D} - \mathcal{E}$ in terms of Hamilton's principle.*

(d) *Assume ρ is given and find a PDE satisfied by an extremal $v \in C^2(\mathcal{U})$ of $\mathcal{H}$.*

(e) *Write down a single PDE in the three unknown component functions of the field.*

Problem 2 *(uniqueness) Use the weak maximum principle (Problem 9 of Assignment 3) to prove uniqueness for C^2 solutions of Poisson's equation: Let $\mathcal{U}$ be an open bounded subset of $\mathbb{R}^2$. Let $f \in C^0(\mathcal{U})$ and $g \in C^0(\partial\mathcal{U})$. If $v, \tilde{v} \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$, and both v and $\tilde{v}$ satisfy the boundary value probem*

$$\begin{cases} \Delta u = f, & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g, & \end{cases}$$

then $v \equiv \tilde{v}$. Hint: Consider $w = v - \tilde{v}$ and show w is a solution of a boundary value problem for Laplace's PDE.

Problem 3 *(mean value property) Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and $u \in C^2(\mathcal{U})$ be a solution of Laplace's equation. Assume $\overline{B_r(\mathbf{x})} \subset \mathcal{U}$. For $0 < a \le r$, the mean value of u on the boundary of $B_a(\mathbf{x})$ is defined by*

$$m(\mathbf{x}; a) = \frac{1}{2\pi a} \int_{\partial B_a(\mathbf{x})} u.$$

The mean value of u on $B_r(\mathbf{x})$ is defined by

$$M(\mathbf{x}; r) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{x})} u.$$

(a) *Show*

$$\frac{d}{da} m(\mathbf{x}; a) = 0.$$

Hint: Follow the following steps:

(i) *Change variables to express the integral over the circle $\partial B_a(\mathbf{x})$, either, over a unit circle or over the interval $[0, 2\pi)$, and differentiate under the integral sign.*

(ii) *Change variables back to $\partial B_a(\mathbf{x})$ and use the divergence theorem.*

(b) *Show*

$$M(\mathbf{x}; r) \equiv m(\mathbf{x}; a) \equiv u(\mathbf{x}).$$

Hint(s): Change variables in the integral defining M to "generalized polar coordinates" so that

$$\int_{B_r(\mathbf{x})} u = \int_0^r \left(\int_{\partial B_a(\mathbf{x})} u \right) da.$$

Take a limit as $a \searrow 0$.

This result is called the **mean value property** *for solutions of Laplace's equation.*

Problem 4 *(separation of variables) Let $r, L > 0$ and $f \in C^0(-r, r)$ be given. Let $\mathcal{U} = (-r, r) \times (0, L) \subset \mathbb{R}^2$, and consider the boundary value problem*

$$\begin{cases} \Delta u = 0, & \text{on } \mathcal{U} \\ u\Big|_{x=\pm r} = 0 = u\Big|_{y=L}, & u\Big|_{y=0} = f \end{cases} \tag{1}$$

where $f \in C^0(-r, r)$.

(a) *Assume you have a solution having the special product form $u(x, y) = A(x)B(y)$. Such a solution is called a* **separated variables solution**. *Show that given such a solution the equation $\Delta u = 0$ can be rearranged in the form*

$$\frac{A''}{A} = -\frac{B''}{B}. \tag{2}$$

(b) *Prove that the relation (2) implies both the left and right sides are constant.*

(c) *The constant obtained in the previous part is called a* **separation constant**. *Write down boundary value problems (ODEs) for A and B using the separation constant and the boundary values from (1).*

(d) *One of the ODEs you have written down in the previous part can be used to determine a monotonic infinite sequence*

$$\lambda_1, \lambda_2, \lambda_3, \ldots$$

of permissible values of the separation constant.

(e) *Find all separated variables solutions of the boundary value problem (1).*

Multivariable Calculus

Problem 5 *(differential approximation; Boas4.4.1-3) Use differentials to accomplish the following:*

(a) *Express the approximation formula*

$$\frac{1}{(n+1)^3} - \frac{1}{n^3} \approx -\frac{3}{n^4} \qquad \text{for } n \text{ large}$$

in terms of a differential mapping.

(b) *Express the approximation formula*

$$\sqrt{n+a} - \sqrt{n} \approx \frac{a}{2\sqrt{n}} \qquad \text{for } n \text{ large and } a \text{ small}$$

in terms of a differential mapping.

(c) *Assume $u = u(x, y)$ is determined by the formula*

$$\frac{1}{u} + \frac{1}{x} = \frac{1}{y}.$$

(i) *Find a differential approximation formula*

$$u(x + \delta, y + \epsilon) \qquad \text{for } \delta \text{ and } \epsilon \text{ small}$$

in terms of x and y.

(ii) *Express your approximation formula in terms of only x and $u(x, y)$ (not involving y assuming $\epsilon = 0$.*

Note: Part (c) is corrected from a previous incorrect attempt to "enhance" Problem 3 of Section 4 in Chapter 4 of Boas. Further comments may be found at the end of this assignment.

Problem 6 *(integration and scaling)*

(a) *Let $\Psi : [0, \infty) \times [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by*

$$\Psi(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi).$$

Compute the total derivative $D\Psi$ of Ψ and the associated scaling factor according to the first scaling principle.

(b) *(Boas 5.4.20) Use the change of variables*

$$x = (\xi - \eta)/2$$
$$y = (\xi + \eta)/2$$

to evaluate the integral

$$\int_0^{1/2} \int_x^{1-x} \left(\frac{x-y}{x+y} \right)^2 dy\, dx.$$

(c) *Let $\mathcal{S}$ be the surface parameterized by*

$$X(u, v) = (u \cos v, u \sin v, v)$$

for $0 \le u \le 1$ and $0 \le v \le \pi$. Consider $f : \mathcal{S} \to \mathbb{R}$ by

$$f(p) = \operatorname{dist}(p, L)$$

where $L = \{(0, 0, z) : z \in \mathbb{R}\}$ is the z-axis.

- **(i)** *Sketch (or produce a plot using mathematical software) of the surface $\mathcal{S}$.*
- **(ii)** *Use the second scaling principle to determine the scaling factor associated with the mappint X.*
- **(iii)** *Find $\int_{\mathcal{S}} f$.*

Problem 7 *Assume the following:*

(i) $f, g \in C^0(\mathbb{R})$.

(ii) $f, g \ge 0$.

(iii) $\operatorname{supp}(f) = [a, b]$ *and* $\operatorname{supp}(g) = [c, d]$.

(a) *Define $f * g : \mathbb{R} \to \mathbb{R}$ by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\xi) g(x - \xi)\, d\xi.$$

determine $\operatorname{supp}(f * g)$.

(b) *Assume* $\phi_1 \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}(\phi_1) = [-1, 1]$ *and* $\int \phi_1 = 1$. *Show that for each* $\epsilon > 0$, *the function* $\mu_\epsilon : \mathbb{R}^1 \to \mathbb{R}$ *by*

$$\mu_\epsilon(x) = \frac{1}{\epsilon}\phi_1\left(\frac{x}{\epsilon}\right)$$

satisfies $\operatorname{supp}(\mu_\epsilon) = [-\epsilon, \epsilon]$ *and* $\int \mu_\epsilon = 1$.

(c) *Assume* $\phi_1 \in C_c^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}(\phi_1) = \overline{B_1(\mathbf{0})}$ *and* $\int \phi_1 = 1$. *Show that for each* $\epsilon > 0$, *the function* $\mu_\epsilon : \mathbb{R}^n \to \mathbb{R}$ *by*

$$\mu_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n}\phi_1\left(\frac{\mathbf{x}}{\epsilon}\right)$$

satisfies $\operatorname{supp}(\mu_\epsilon) = \overline{B_\epsilon(\mathbf{0})}$ *and* $\int \mu_\epsilon = 1$.

Problem 8 *Show that if* $\mathcal{U}$ *is a nonempty connected open subset of* $\mathbb{R}^n$ *and* $u : \mathcal{U} \to \mathbb{R}$ *satisfies* $Du(\mathbf{x}) = \mathbf{0}$ *for* $\mathbf{x} \in \mathcal{U}$, *then* u *is a constant function. Show (by example) that the assumption* $\mathcal{U}$ *is connected cannot be relaxed.*

Problem 9 *Let* $\mathcal{U}$ *be an open subset of* $\mathbb{R}^3$ *with* $\mathbf{p} = (p_1, p_2, p_3) \in \mathcal{U}$. *Consider the "epsilon cube" with center* $\mathbf{p}$ *given by*

$$\mathcal{U}_\epsilon = \mathcal{U}_\epsilon(\mathbf{p}) = \{\mathbf{x} = (x_1, x_2, x_3) : |x_j - p_j| < \epsilon \text{ for } j = 1, 2, 3\}.$$

Let $\mathbf{v} = (v_1, v_2, v_3) \in C^1(\mathcal{U} \to \mathbb{R}^3)$ *be a* C^1 *vector field on* $\mathcal{U}$.

(a) *Let* $F = \{(p_1+\epsilon, x_2, x_3) : |x_j - p_j| < \epsilon \text{ for } j = 2, 3\}$ *be the front face of* $\partial\mathcal{U}_\epsilon$. *Define the back face* B *directly opposite* F *on* $\partial\mathcal{U}_\epsilon$ *and use the mean value theorem to show*

$$\int_{F \cup B} \mathbf{v} \cdot \mathbf{n} = 2\epsilon \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{\partial v_1}{\partial x_1}(x_*, p_2 + y, p_3 + z)\, dy\, dz$$

where $x_* = x_*(y, z)$ *is some real number (depending on* y *and* z*) with* $p_1 - \epsilon < x_* < p_1 + \epsilon$.

(b) *Write down similar expressions for the other two pairs of faces (top and bottom, left and right).*

(c) *Compute the limit*

$$\lim_{\epsilon \searrow 0} \frac{1}{\mu(\mathcal{U}_\epsilon)} \int_{\partial\mathcal{U}_\epsilon} \mathbf{v} \cdot \mathbf{n}$$

to obtain the usual formula for $\operatorname{div} \mathbf{v}$ *in standard rectangular coordinates.*

Calculus of Variations

Problem 10 *Let $\mathcal{U}$ be a bounded open subset of $\mathbb{R}^n$. Compute the first variation of the Dirichlet energy $\mathcal{D} : \mathcal{A} \to \mathbb{R}$ by*

$$\mathcal{D}[u] = \frac{1}{2} \int_{\mathcal{U}} |Du|^2$$

where $\mathcal{A} = \{u \in C^1(\mathcal{U}) : u\big|_{\partial\mathcal{U}} = g\}$ and $g \in C^0(\partial\mathcal{U})$ is fixed. Find the Euler-Lagrange PDE satisfied by C^2 extremals.

Notes/Solutions: On Problem 5 Part (c), Boas (almost certainly) intended $u = u(x)$ to be defined by

$$\frac{1}{u} + \frac{1}{x} = \frac{1}{y}$$

and for the differential approximation to be based on

$$-\frac{1}{u^2}\,du - \frac{1}{x^2}\,dx = 0$$

so that

$$du(\delta) = -\frac{u^2}{x^2}\delta$$

and

$$u(x+\delta) \approx u(x) - \frac{u(x)^2}{x^2}\,\delta.$$

Another approach would be to write

$$u(x) = \frac{1}{1/y - \frac{1}{x}} = \frac{xy}{x-y}$$

so that

$$u'(x) = \frac{y}{x-y} - \frac{xy}{(x-y)^2} = \frac{y(x-y) - xy}{(x-y)^2} = -\frac{y^2}{(x-y)^2}$$

and

$$du(\delta) = -\frac{y^2}{(x-y)^2}\,\delta.$$

This leads to the approximation formula

$$u(x+\delta) \approx \frac{xy}{x-y} - \frac{y^2}{(x-y)^2}\,\delta$$

which looks quite different and involves y in particular. It amounts to the same thing, however, which you can see by remembering/noting

$$u(x) = \frac{xy}{x-y} \qquad \text{and} \qquad \frac{y^2}{(x-y)^2} = \frac{1}{x^2}\,u(x)^2.$$

Now, hopefully, it's more or less clear how to do the "enhanced" problem in two different ways. Your answer should involve a differential $du : \mathbb{R}^2 \to \mathbb{R}$.

Problem 7 Part (a) was a little trickier than might have been expected. One should first be careful to understand what it means for a function $f \in C_c^0(\mathbb{R})$ to have $\text{supp}(f) = [a, b]$. Note that such a closed interval $[a, b]$ must be a subset of $\mathbb{R}$, so a and b are (finite) real numbers with, presumably, $a < b$. It may be considered that $a \le b$, but if $a = b$, then $[a, b] = \{a\}$ is a singleton set, and it is impossible for such a set to be the support of a continuous function on $\mathbb{R}$. (Since $f(x) \equiv 0$ for $x \notin \text{supp}(f)$, one would then get $f(a) = 0$ by continuity and, so, $f \equiv 0$ with $\text{supp}(f) = \phi$ (the empty set).

In any case, for this problem we have $a < b$ and $c < d$. The fact that $\text{supp}(f) = [a, b]$ does **not** mean $f(t) \ne 0$ for $t \in (a, b)$. We know, in fact, by continuity that $f(a) = f(b) = 0$. Remember that

$$\text{supp}(f) = \overline{\{t \in \mathbb{R} : f(t) \ne 0\}},$$

and in our case $\{t \in \mathbb{R} : f(t) \ne 0\} = \{t \in \mathbb{R} : f(t) > 0\}$. But the important part to notice is the closure in the definition. What this does mean is that for $t \in \text{supp}(f)$ there must be points ξ arbitrarily close to t with $f(\xi) > 0$. Let us justify this assertion carefully:

If we assume there is some $\delta > 0$ for which $f(\xi) \equiv 0$ for $|\xi - t| < \delta$, then we know

$$\{\xi \in \mathbb{R} : f(\xi) \ne 0\} \subset (-\infty, t-\delta] \bigcup [t+\delta, \infty). \tag{3}$$

In fact, by continuity, we could make the stronger assertion

$$\{\xi \in \mathbb{R} : f(\xi) \ne 0\} \subset (-\infty, t-\delta) \bigcup (t+\delta, \infty),$$

but we are going to take a closure to get $\text{supp}(f)$ so it is of particular interest that the union of intervals in (3) is a closed set. More precisely,

$$\text{supp}(f) = \overline{\{t \in \mathbb{R} : f(t) \neq 0\}} \subset (-\infty, t - \delta] \bigcup [t + \delta, \infty) = A$$

because the closure is the *smallest* closed set containing $\{t \in \mathbb{R} : f(t) \neq 0\}$, i.e., the intersection of all closed sets containing this given set, and A is one particular closed set containing $\{t \in \mathbb{R} : f(t) \neq 0\}$. We conclude from this that $t \notin \text{supp}(f)$ which is a contradiction since we started with $t \in \text{supp}(f)$.

Okay, so now we should understand what it means for t to be in $\text{supp}(f) = [a, b]$. Two more comments about this: (1) We did not use continuity in this discussion, so what we have said applies ot any function $f : \mathbb{R} \to \mathbb{R}$ with $\text{supp}(f) = [a, b]$ and (2) If the point in question is $t = a$ or $t = b$, then clearly the nearby points ξ with $f(\xi) \neq 0$ will have to be on a particular side of t (with $\xi > a$ if $t = a$ and $\xi < b$ if $t = b$).

Of course, all these same comments apply with g in place of f, c in place of a, and d in place of b.

At this point a first observation (the simple direction) is that $\text{supp}(f * g) \subset [a + c, b + d]$. To see this, it is enough to show

$$\{x \in \mathbb{R} : (f * g)(x) \neq 0\} \subset [a, b]$$

or

$$(f * g)(x) = \int_{t \in \mathbb{R}} f(t)g(x - t) = 0 \qquad \text{for} \qquad x \in (-\infty, a + c) \bigcup (b + d, \infty).$$

We can see this as follows: If $x < a + c$, and $t \in (a, b)$, then $x - t < c$, so $g(x - t) = 0$. Similarly, if $x > b + d$, then $x - t > d$, so $g(x - t) = 0$. This shows that the function $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = f(t)g(x - t)$ satisfies

$$h(t)f(t)g(x - t) = 0 \qquad \text{for } t \in (a, b).$$

In fact, for $t \notin (a, b)$, we have $f(t) = 0$, so $h \equiv 0$ or

$$(f * g)(x) = \int_{t \in \mathbb{R}} f(t)g(x - t) = \int_a^b f(t)g(x - t)\, dt = 0$$

for $x \in (-\infty, a + c) \cup (b + d, \infty)$. Consequently, $\text{supp}(f * g) \subset [a + c, b + d]$. The reverse inclusion is the interesting part.

Presumably, it at least occurred to you that $\text{supp}(f * g) = [a + c, b + d]$. A less obvious assertion (and probably the easiest way to see the reverse inclusion) is that

$$(f * g)(x) > 0 \qquad \text{for } x \in (a + c, b + d). \tag{4}$$

To see this it is enough to find, for each fixed $x \in (a+c, b+d)$, a single $t_* \in (a, b)$ for which $f(t_*)g(x - t) > 0$. This is because $h(t) = f(t)g(x - t)$ is a continuous function which is nonnegative and

$$(f * g)(x) = \int_a^b f(t)g(x - t)\, dt = \int_a^b h(t)\, dt.$$

Thus, if $h(t_*) > 0$, then $(f * g)(x) > 0$ and (4) holds.

How do we find such a t_*? At first I thought you could pick any $t \in (a, b)$ for which $f(t) > 0$ and then use the continuity of f and the properties of $\text{supp}(g) = [c, d]$ discussed above to find a point t_* nearby t with $f(t_*) > 0$ by continuity and $g(x-t_*) > 0$ because there are points $x - t_*$ nearby $x - t \in (c, d)$ for which $g(x - t_*) > 0$. The problem with this plan is that if we take any point $t \in (a, b)$ with $f(t) > 0$, this does not mean $x - t \in (c, d)$ (at all). Here is an example where this kind of thing can happen: If $[a, b] = [1, 4]$ and $[c, d] = [5, 6]$, then $[a + c, b + d] = [6, 10]$ and $x = 9 \in (a+c, b+d) = (6, 10)$, but $t = 2 \in (a, b) = (1, 4)$ with $x-t = 9-2 = 7 \notin (c, d) = (5, 6)$.

So we cannot just pick any point t with $f(t) > 0$. We have to be careful about how we pick this first point. Here is a "trick" that helps us pick a point $t \in (a, b)$ that will work: Given $x \in (a + c, b + d)$, there is a unique $\lambda \in (0, 1)$ with

$$x = (1 - \lambda)(a + c) + \lambda(b + d). \tag{5}$$

The expression (5) is called a **convex combination** of $a + c$ and $b + d$. Note that

$$\lambda = \frac{x - (a + c)}{(b + d) - (a + c)} \qquad \text{is uniquely determined in } (0, 1).$$

Thus λ is the ratio of the length of the first segment into which x divides the interval $[a + c, b + d]$ and the entire length of $[a + c, b + d]$. The trick is to take t_1 to be the point in (a, b) determined by the same ratio. That is,

$$t_1 = (1 - \lambda)a + \lambda b.$$

Then $x - t_1 = (1 - \lambda)c + \lambda d$ divides $[c, d]$ into the same ratio and, in particular, is definitely in (c, d). Because $t_1 \in (a, b) \subset \text{supp}(f)$, there is some $t_2 \in (a, b)$ close to t_1

with $f(t_2) > 0$. Now we would also like to have $x - t_2 \in (c, d)$. At this point, there is a second nice "trick" to accomplish what we want. Set

$$\delta_1 = \min\{t_1 - a, b - t_1, (x - t_1) - c, d - (x - t_1)\}.$$

This number $\delta_1 > 0$ and it has the property that if $|t - t_1| < \delta_1$, then $t_1 \in (a, b)$. Also, if $|\eta - (x - t_1)| < \delta_1$, then $\eta \in (c, d)$. Now we pick t_2 more precisely: Since $t_1 \in (a, b) \subset \text{supp}(f)$, there is some $t_2 \in (a, b)$ with

$$|t_2 - t_1| < \delta_1 \qquad \text{and} \qquad f(t_2) > 0.$$

Setting $\eta_1 = x - t_1$ and $\eta_2 = x - t_2$, we see

$$|\eta_2 - \eta_1| = |t_2 - t_1| < \delta_1,$$

so by the property of δ_1 we also know $x - t_2 = \eta_2 \in (c, d)$.

We now have a point $t_2 \in (a, b)$ with $\eta_2 = x - t_2 \in (c, d)$ and $f(t_2) > 0$. Since $\eta_2 = x - t_2 \in (c, d) \subset \text{supp}(g)$ we can use the property of the support of g to find a point η_* nearby η_2 with $g(\eta_*) > 0$. Again, we need to do this somewhat carefully to maintain the conditions we've worked for above. Here we will use the second "trick" above along with the continuity of g. Let

$$\delta_2 = \min\{t_2 - a, b - t_2, \eta_2 - c, d - \eta_2\} > 0.$$

Since f is continuous and $f(t_2) > 0$, we can find $\delta > 0$ with

$$\delta < \delta_2 \qquad \text{and} \qquad f(t) > 0 \text{ for } |t - t_2| < \delta.$$

Let $\eta_* \in (c, d)$ with

$$|\eta_* - \eta_2| < \delta < \delta_2 \qquad \text{and} \qquad g(\eta_*) > 0.$$

We are using the fact that $\eta_2 \in \text{supp}(g)$ here. Now, consider $t_* = x - \eta_*$. Since $|t_* - t_2| = |\eta_* - \eta_2| < \delta$, we know from the continuity of f that

$$f(t_*) > 0.$$

Naturally, this means $t_* \in (a, b)$, but we can also conclude this from the "tricky" property of δ_2, since $\delta < \delta_2$. In any case, we also have $\eta_* = x - t_* \in (c, d)$ so that

$$g(x - t_*) > 0.$$

This means

$$h(t_*) = f(t_*)g(x - t_*) > 0,$$

and consequently $(f*g)(x) > 0$ by the continuity of h for fixed x. Thus, $(f*g)(x) > 0$ for $x \in (a+c, b+d)$ and $\mathrm{supp}(f*g) = [a+c, b+d]$.

Note finally, that this last step uses the continuity of h which follows from the continuity of both f and g, though in the preceeding argument to find the point t_* we only used the continuity of f. Here is an interesting question:

What if the functions f and g are only in $L^1_{loc}(\mathbb{R})$?

Conjecture: If $f, g \in L^1_{loc}(\mathbb{R})$ with $f, g \geq 0$, $\mathrm{supp}(f) = [a, b]$ and $\mathrm{supp}(g) = [c, d]$, then

$$(f*g)(x) = \int_{t \in (a,b)} f(t)g(x-t) > 0 \qquad \text{for } x \in (a+c, b+d).$$