Assignment 3 = Exam 1:
Solutions 2, 3, 5, 7, 8, 9
Due Friday, February 7, 2025

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Problem 1 (density and the hanging slinky) It was very nicely pointed out in class that the "density of coils" is not constant in a hanging slinky.

(a) Given an initial equilibrium height h_0 of a slinky (sitting on a table for example) define a **constant linear mass density** ρ_0 in terms of the total mass M of the slinky and the height h_0.

(b) Let $\sigma : [0, h_0] \to [0, \infty)$ denote a model measurement (stretch) function based on the height h_0 for the hanging slinky as described in my solution of Problem 1 of Assignment 1. Find a **linear mass density function** $\rho : [0, h_0] \to [0, \infty)$ giving/modeling the mass density at a material point in the hanging slinky corresponding to the stretch value $\sigma(h)$ for $0 \le h \le h_0$. Give your answer in terms of ρ_0 and σ. Hint: Let $\delta > 0$ and consider the portion of the hanging slinky between $X_3(h - \delta) = -\sigma(h - \delta)$ and $X_3(h + \delta) = -\sigma(h + \delta)$. How long is this portion of the hanging slinky and what is its mass?

(c) Formulate a notion of "density of coils" $\rho_c : [0, h_0] \to [0, \infty)$ for the hanging slinky and find a formula for ρ_c in terms of ρ.

Solution:

(a) $\rho_0 = M/h_0$.

(b) Consider the portion of slinky between x and $x + \delta$ in equilibrium. This section should have mass $\delta\rho_0$. Also, when the slinky is stretched, the mass of the stretched portion should be exactly $\delta\rho_0$. However, the length will have changed to $\sigma(x+\delta)-\sigma(x)$. Therefore, the average lineal mass density for this section of stretched spring should be

$$\frac{\delta\rho_0}{\sigma(x+\delta)-\sigma(x)} = \rho_0 \Bigg/ \frac{\sigma(x+\delta)-\sigma(x)}{\delta}.$$

The limit of this quantity should model reasonably well the linear mass density $\rho : [0, h_0] \to \mathbb{R}$ of the slinky stretched into a position modeled by σ, that is

$$\rho(x) = \frac{\rho_0}{\sigma'(x)}.$$

We might worry about the vanishing of σ' here, but in fact, it is physically natural to assume $\sigma'(x) \geq 1$ for all x.

(c) In my solution to Problem 1 of Assignment 1 I noted that there were 78.75 coils, so a natural notion of "density of coils" is

$$\frac{\text{number of coils}}{\text{length of slinky}}$$

with the equilibrium density at $78.75/h_0$. Computing in a manner similar to that above for mass density, we find

$$\rho_c(x) = \frac{78.75}{h_0\sigma'(x)} = \frac{1}{m_c}\,\rho(x)$$

where $m_c = M/78.75 \approx 0.218/78.75 \doteq 0.002768$kg is the mass of one coil.

Recall the existence and uniqueness theorem for a single ordinary differential equation (ODE):

Theorem 1 If $(t_0, y_0) \in (a,b) \times (c,d)$ and $F : (c,d) \times (a,b) \to \mathbb{R}$ satisfies $F \in C^0((c,d) \times (a,b))$ and

$$\frac{\partial F}{\partial y} \in C^0((c,d) \times (a,b)),$$

then there is some $\delta > 0$ for which the initial value problem (IVP)

$$\begin{cases} y' = F(y,t), & t_0 - \delta < t < t_0 + \delta \\ y(t_0) = y_0, \end{cases} \tag{1}$$

has a unique solution $y \in C^1(t_0 - \delta, t_0 + \delta)$.

Problem 2 (finite time blow-up) Consider the IVP

$$\begin{cases} y' = y^2 \\ y(3) = 7. \end{cases} \tag{2}$$

(a) Explain carefully and clearly what Theorem 1 tells you (to expect) about this problem. Clearly identify t_0, y_0 and the intervals (a,b) and (c,d).

(b) According to the statement of Theorem 1, list the quantities upon which you might expect the number δ to depend.

(c) Solve the IVP

$$\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y_0 and t_0 are given real numbers.

Solution:

(a) The initial time is $t_0 = 3$. The initial value is $y_0 = 7$. The structure function is $F(y,t) = y^2$ and the natural domain of regularity for this function is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with $(a,b) = (c,d) = (-\infty, \infty) = \mathbb{R}$. Theorem 1 asserts that there exists some $\delta > 0$ for which the initial value problem

$$\begin{cases} y' = y^2, & 3 - \delta < t < 3 + \delta \\ y(3) = 7 \end{cases}$$

has a unique solution $y : (3 - \delta, 3 + \delta) \to \mathbb{R}$ with $y \in C^1(3 - \delta, 3 + \delta)$.

(b) According to the statement of the theorem, the half-length of the interval of existence might depend on the constants a, b, c, d, t_0, y_0 and the structure function F.

(c) If $y_0 = 0$, then notice $y \in C^\infty(\mathbb{R})$ with $y(t) \equiv 0$ for all t is a solution of the problem.

We can see this is the unique solution of the problem in this case as follows: Assume $y_1 \in C^1(T_1, T_2)$ is a different solution of

$$\begin{cases} y' = y^2, & T_1 < t < T_2 \\ y(t_0) = y_0 \end{cases}$$

for some T_1 and T_2 with $T_1 < t_0 < T_2$. Taking $\delta_1 = \min\{\delta, T_2 - t_0, t_0 - T_1\}$ where δ is the half length of existence and uniqueness given by application of Theorem 1 to the problem, we know $y_1(t) \equiv 0$ for $|t - t_0| < \delta_1$. This is from the uniqueness assertion of Theorem 1. Letting $t_{\max} = \sup\{t > t_0 : y_1(t) \equiv 0\}$, one possibility is that $t_{\max} = T_2$, and we know $y_1(t) \equiv 0$ for $t_0 - \delta_1 < t < T_2$. Let's remember that possibility for later.

The other possibility is $t_{\max} < T_2$. In this case, $y_1(t_{\max})$ is a well-defined number, and we know

$$y_1(t_{\max}) = \lim_{t \nearrow t_{\max}} y_1(t) = 0.$$

Accordingly, we consider the IVP

$$\begin{cases} y' = y^2 \\ y(t_{\max}) = 0. \end{cases}$$

Applying Theorem 1 to this problem, we obtain some $\delta_2 > 0$ for which

$$\begin{cases} y' = y^2, & t_{\max} - \delta_2 < t < t_{\max} + \delta_2 \\ y(t_{\max}) = 0 \end{cases}$$

has a unique solution, and we know that solution satisfies $y(t) \equiv 0$. In particular there must hold

$$y_1(t) \equiv 0 \qquad \text{for} \qquad t_{\max} \le t \le \max\{T_2, t_{\max} + \delta_2\}.$$

Since $\max\{T_2, t_{\max} + \delta_2\} > 0$, notice this contradicts the definition of $t_{\max}$. We conclude the case in which $t_{\max} < T_2$ cannot happen at all.

We have shown $y_1(t) \equiv 0$ for $t_0 - \delta_1 < t < T_2$ where $\delta_1 = \min\{\delta, T_2 - t_0, t_0 - T_1\}$ and δ is the half length of existence and uniqueness given by application of Theorem 1 to the IVP

$$\begin{cases} y' = y^2 \\ y(t_0) = 0. \end{cases}$$

We can make the same argument to the left to show that in fact $y_1(t) \equiv 0$ for $T_1 < t < T_2$.

The final conclusion here is that if $y_0 = 0$, then the unique solution of the problem is $y \equiv 0$.

If $y_0 \neq 0$, then by continuity there is some ϵ with $\epsilon > 0$, for which the unique solution $y \in C^1(t_0 - \delta, t_0 + \delta)$ given by the theorem satisfies $y(t) \neq 0$ at least for $t_0 - \epsilon < t < t_0 + \epsilon$. Notice here there is no problem assuming $\epsilon > 0$ is small enough so that writing

$$y(t) \neq 0 \qquad \text{for} \qquad t_0 - \epsilon < t < t_0 + \epsilon$$

makes sense. In particular, we can assume $\epsilon < \delta$ if we like. In any case for $t_0 - \epsilon < t < t_0 + \epsilon$ we can write

$$\frac{y'(t)}{[y(t)]^2} = 1$$

and integrate both sides from $\tau = t_0$ to t with $|t - t_0| < \epsilon$:

$$\int_{t_0}^{t} \frac{y'(\tau)}{[y(\tau)]^2}\, d\tau = t - t_0.$$

Changing variables in the integral with $u = y(\tau)$, we have $du = y'(\tau)\, d\tau$. Thus,

$$\int_{y_0}^{y(t)} \frac{1}{u^2}\, du = -\frac{1}{y(t)} + \frac{1}{y_0} = t - t_0.$$

This means

$$y(t) = \frac{1}{1/y_0 + t_0 - t}.$$

Notice the singularity at $1/y_0 + t_0 \neq t_0$. The location of this singularity is somewhat difficult to predict from looking at the original IVP (3).

Applying this general solution in the special case $t_0 = 3$ and $y_0 = 7$ goes something like this:

Since $y(3) = 7$ we know by continuity there is some $\epsilon > 0$ for which $y(t) \neq 0$ in some interval $3 - \epsilon < t < 3 + \epsilon$. Starting on this interval we can write

$$\frac{y'(t)}{[y(t)]^2} = 1$$

and integrate both sides from $\tau = 3$ to t with $|t - 3| < \epsilon$:

$$\int_3^t \frac{y'(\tau)}{[y(\tau)]^2}\, d\tau = t - 3.$$

Changing variables in the integral with $u = y(\tau)$, we have $du = y'(\tau)\, d\tau$. Thus,

$$\int_7^{y(t)} \frac{1}{u^2}\, du = -\frac{1}{y(t)} + \frac{1}{7} = t - 3.$$

This means

$$y(t) = \frac{1}{22/7 - t}.$$

Notice the singularity at $22/7 \approx \pi > 3$. In particular, the largest possible value of δ given by the theorem is $\delta_{\max} = 1/7$.

Problem 3 Let F satisfy the conditions of Theorem 1. If $T_1 < T_2$ and the IVP

$$\begin{cases} y' = F(y,t), & T_1 < t < T_2 \\ y(t_0) = y_0 \end{cases} \tag{4}$$

has a unique solution $y \in C^1(T_1, T_2)$ and for every $\delta > 0$ neither of the initial value problems

$$\begin{cases} y' = F(y,t), & T_1 - \delta < t < T_2 \\ y(t_0) = y_0 \end{cases} \quad \text{or} \quad \begin{cases} y' = F(y,t), & T_1 < t < T_2 + \delta \\ y(t_0) = y_0 \end{cases}$$

has a unique solution, then we say (T_1, T_2) is a **maximal interval for existence and uniqueness**.

(a) According to an application of Theorem 1 to the problem (3) list the quantities upon which you might expect the maximal interval for existence to depend. Hint: Your list should be shorter than the list you gave in part **(b)** of Problem 2. Which quantities are missing?

(b) Find the maximal interval of existence for the IVP (3).

Solution: Note first that the definition of **maximal interval of existence and uniqueness** given above is not quite correct. There is a problem with this definition when a maximum interval of existence and uniqueness extends to $-\infty$ or $+\infty$. Here[1] is how the definition is corrected: Denote the IVP

$$\begin{cases} y' = F(y,t), & T_1 - \delta < t < T_2 \\ y(t_0) = y_0 \end{cases}$$

by (A) and the IVP

$$\begin{cases} y' = F(y,t), & T_1 < t < T_2 + \delta \\ y(t_0) = y_0 \end{cases}$$

by (B). The definition should require that the IVP (A) does not have a unique solution whenever $\delta > 0$ **and** $-\infty < T_1 < t_0 < T_2$, and the IVP (B) does not have a unique solution whenever $\delta > 0$ **and** $T_1 < t_0 < T_2 < \infty$.

(a) In the application of Theorem 1 to the general IVP (3) in Problem 2 one might expect dependence in some way on the constants t_0 and y_0, and of course the structure function with $F(y,t) = y^2$. Notice that in this application $a = c = -\infty$ and

[1] I think.

$b = d = +\infty$, so these are not quantities upon which the value of δ or the maximum possible value of δ will depend. Put another way, there is no singularity in the structure function F, so there is no immediately obvious way to see any specific time where the solution might have a singularity. The structure function, however, is **nonlinear** so the point of this problem is to illustrate that nonlinearity in the ODE can have potentially somewhat unexpected consequences for solutions of the problem.

(b) The maximal interval of existence for (3) depends on the value of y_0. If $y_0 = 0$, then the maximal interval of existence is, as shown in detail in the solution of Problem 2 above, $\mathbb{R} = (-\infty, \infty)$.

If $y_0 < 0$, then the singularity occurs at $t_0 + 1/y_0 < t_0$, and the maximum interval of existence is

$$(T_1, T_2) = (t_0 + 1/y_0, +\infty).$$

Notice the argument showing uniqueness of the zero solution in part **(c)** of the solution of Problem 2 above may be applied to show the solution given by

$$y(t) = \frac{1}{1/y_0 + t_0 - t}.$$

when $y_0 < 0$ also given in the solution of part **(c)** of Problem 2 above is the unique solution in this case. Since the left endpoint $t_0 + 1/y_0$ gives the points where the interesting singular behavior

$$\lim_{t \searrow t_0 - 1/y_0} y(t) = -\infty$$

occurs and the analysis of this behavior requires "arguing to the left" as suggested in the solution of Problem 2 above, here are some of the details:

Consider a solution $y_1 \in C^1(T_1, T_2)$ of

$$\begin{cases} y' = y^2, & T_1 < t < T_2 \\ y(t_0) = y_0 \end{cases}$$

for some T_1 and T_2 with $T_1 < t_0 < T_2$. Application of the existence and uniqueness theorem tells us

$$y_1(t) \equiv \frac{1}{1/y_0 + t_0 - t} \qquad \text{for} \qquad \max\left\{\frac{1}{y_0} + t_0, T_1, t_0 - \delta\right\} < t < \min\{T_2, t_0 + \delta\}$$

for some $\delta > 0$. Thus, arguing to the left, we set

$$t_{\min} = \min \left\{ t : y_1(t) \equiv \frac{1}{1/y_0 + t_0 - t} \right\} \qquad (*).$$

If

$$t_{\min} > \max \left\{ T_1, \frac{1}{y_0} + t_0 \right\} \qquad (**)$$

then $y_1(t_{\min})$ is a well-defined real number and by continuity we know

$$y_1(t_{\min}) = \lim_{t \searrow t_{\min}} \frac{1}{1/y_1 + t_0 - t} = \frac{1}{1/y_0 + t_0 - t_{\min}} < 0.$$

We may apply the existence and uniqueness theorem to the IVP

$$\begin{cases} y' = y^2 \\ y(t_{\min}) = y_1(t_{\min}) \end{cases}$$

to obtain some $\delta_2 > 0$ for which

$$\begin{cases} y' = y^2, & t_{\min} - \delta_2 < t < t_{\min} + \delta_2 \\ y(t_{\min}) = y_1(t_{\min}) \end{cases}$$

has a unique solution $y_2 \in C^1(t_{\min} - \delta_2, t_{\min} + \delta_2)$. Accordingly, we know this unique solution must satisfy

$$y_2(t) \equiv \frac{1}{1/y_0 + t_0 - t}$$
$$\text{for} \qquad \max \left\{ t_{\min} - \delta_2, T_1, \frac{1}{y_0} + t_0 \right\} < t < \min \left\{ t_{\min} + \delta_2, T_2 \right\}.$$

In particular, we must have also

$$y_1(t) \equiv \frac{1}{1/y_0 + t_0 - t}$$
$$\text{for} \qquad \max \left\{ t_{\min} - \delta_2, T_1, \frac{1}{y_0} + t_0 \right\} < t < \min \left\{ t_{\min} + \delta_2, T_2 \right\}.$$

Since we know from (**)

$$\max \left\{ t_{\min} - \delta_2, T_1, \frac{1}{y_0} + t_0 \right\} < t_{\min}$$

this contradicts the definition of $t_{\min}$ given in (*). Thus we know (**) cannot hold, and the alternative is

$$t_{\min} = \max\left\{T_1, \frac{1}{y_0} + t_0\right\}.$$

If we assume now that

$$T_1 < \frac{1}{y_0} + t_0,$$

then

$$y_1\left(\frac{1}{y_0} + t_0\right)$$

should have a well-defined real value. On the other hand, that value must satisfy

$$y_1\left(\frac{1}{y_0} + t_0\right) = \lim_{t \searrow 1/y_0 + t_0} \frac{1}{1/y_0 + t_0 - t} = -\infty.$$

This again is a contradiction from which we conclude

$$T_1 \geq \frac{1}{y_0} + t_0$$

and

$$y_1(t) \equiv \frac{1}{1/y_0 + t_0 - t} \qquad \text{for} \qquad T_1 < t < \min\{t_{\min} + \delta_2, T_2\}.$$

Arguing again to the right it follows also that

$$y_1(t) \equiv \frac{1}{1/y_0 + t_0 - t} \qquad \text{for} \qquad T_1 < t < T_2.$$

In particular, this shows the IVP

$$\begin{cases} y' = y^2, & T_1 - \delta < t < T_2 \\ y(t_0) = y_0 \end{cases}$$

with $T_1 = 1/y_0 + t_0$ has no solution $y \in C^1(T_1 - \delta, T_2)$ if $\delta > 0$.

Notice here that the IVP

$$\begin{cases} y' = y^2, & T_1 = 1/y_0 + t_0 < t < T_2 + \delta \\ y(t_0) = y_0 \end{cases}$$

does have a unique solution when $T_2 = +\infty$ since in that case $T_2 + \delta = T_2 = +\infty$. This is why the original definition required correction.

The final case to consider is when $y_0 > 0$. Then the singularity occurs at $T_2 = 1/y_0 + t_0 > t_0$, and the maximum interval of existence (and uniqueness) is

$$(T_1, T_2) = \left(-\infty, t_0 + \frac{1}{y_0}\right).$$

The inital condition $y(3) = 7$ of (2) falls into this case with maximal interval of existence $(-\infty, 22/7)$, and the unique solution $y \in C^1(-\infty, 22/7)$ satisfies

$$y_1(t) = \frac{1}{22/7 - t}.$$

Problem 4 (existence and uniqueness) Draw a picture illustrating the assertion of Theorem 1. Include a representation of the intervals (a, b) and (c, d), the point (t_0, y_0), and (the role played by) the number δ.

Recall the sweeping generalization of Theorem 1 which says roughly that any reasonable ODE always has a unique local solution:

Theorem 2 If

(i) Ω is an open subset of $\mathbb{R}^n$,

(ii) $(t_0, \mathbf{p}) \in (a,b) \times \Omega$, and

(iii) $\mathbf{F} = (F_1, F_2, \ldots, F_n) : \Omega \times (a,b) \to \mathbb{R}^n$ satisfies $\mathbf{F} \in C^0(\Omega \times (a,b) \to \mathbb{R}^n)$ and

$$\frac{\partial F_i}{\partial y_j} \in C^0(\Omega \times (a,b)) \qquad \text{for} \qquad i, j = 1, 2, \ldots, n,$$

then there is some $\delta > 0$ for which the initial value problem (IVP)

$$\begin{cases} \mathbf{y}' = F(\mathbf{y}, t), & t_0 - \delta < t < t_0 + \delta \\ \mathbf{y}(t_0) = \mathbf{p}, \end{cases} \tag{5}$$

has a unique solution $\mathbf{y} = (y_1, y_2, \ldots, y_n) : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n$ satisfying[2] $y_j \in C^1(t_0 - \delta, t_0 + \delta)$ for $j = 1, 2, \ldots, n$.

Problem 5 What does Theorem 2 tell you about the **second order** IVP

$$\frac{u''}{(1 + u'^2)^{3/2}} = u, \qquad u(x_0) = u_0, \qquad u'(x_0) = u_0'$$

where $x_0, u_0, u_0' \in \mathbb{R}$? Hint: Set $y_1 = u$ and $y_2 = u'$.

Solution: In order to apply Theorem 2 to the second order ODE here, we consider an equivalent system:

$$\begin{cases} y_1' = y_2, & y_1(x_0) = u_0 \\ y_2' = y_1(1 + y_2^2)^{3/2}, & y_2(x_0) = u_0'. \end{cases}$$

Writing this system in vector notation with $\mathbf{y} = (y_1, y_2)^T$ we have

$$\begin{cases} \mathbf{y}' = \mathbf{F}(\mathbf{y}) \\ \mathbf{y}(x_0) = (u_0, u_0')^T \end{cases}$$

[2] We denote this regularity conclusion also by writing simply $\mathbf{y} \in C^1((t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n)$.

where $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(y_1, y_2) = \begin{pmatrix} y_2 \\ y_1(1 + y_2^2)^{3/2} \end{pmatrix}.$$

Note that $\mathbf{F} \in C^\infty(\mathbb{R}^2 \to \mathbb{R}^2)$. The existence and uniqueness theorem stated as Theorem 2 above asserts that there exists some $\delta > 0$ for which the equivalent system has a unique solution $\mathbf{y} \in C^1((x_0 - \delta, x_0 + \delta) \to \mathbb{R}^2)$.

Setting $u(t) = y_1(t)$, we know $u'(t) = y_1'(t) = y_2(t)$. Since $y_2 \in C^1(x_0 - \delta, x_0 + \delta)$, we know $u \in C^2(x_0 - \delta, x_0 + \delta)$. Also,

$$u'' = y_2' = y_1(1 + y_2^2)^{3/2} = u(1 + u'^2)^{3/2}.$$

That is,

$$\frac{u''}{(1 + u'^2)^{3/2}} = u.$$

Finally, $u(x_0) = y_1(x_0) = u_0$ and $u'(x_0) = y_2(x_0) = u_0'$. One conclusion is this:

For every triple of real numbers x_0, u_0 and u_0', there exists some $\delta > 0$ for which the second order IVP

$$\frac{u''}{(1 + u'^2)^{3/2}} = u, \qquad u(x_0) = u_0, \qquad u'(x_0) = u_0'$$

has a solution $u \in C^2(x_0 - \delta, x_0 + \delta)$.

On the other hand, if $u \in C^2(x_0 - \delta, x_0 + \delta)$ is any solution of the second order IVP, then setting $y_1 = u$ and $y_2 = u'$ gives a solution $\mathbf{y} = (y_1, y_2)^T \in C^1((x_0 - \delta, x_0 + \delta) \to \mathbb{R}^2)$ of the equivalent system satisfying the associated IVP

$$\begin{cases} \mathbf{y}' = \mathbf{F}(\mathbf{y}) \\ \mathbf{y}(x_0) = (u_0, u_0')^T. \end{cases}$$

Since the first order system has a unique solution $\mathbf{y} \in C^1((x_0 - \delta, x_0 + \delta) \to \mathbb{R}^2)$ we know also that $u \equiv y_1$ is uniquely determined by the unique first component function. Thus, we can make the more comprehensive conclusion:

For every triple of real numbers x_0, u_0 and u_0', there exists some $\delta > 0$ for which the second order IVP

$$\frac{u''}{(1 + u'^2)^{3/2}} = u, \qquad u(x_0) = u_0, \qquad u'(x_0) = u_0'$$

has a unique solution $u \in C^2(x_0 - \delta, x_0 + \delta)$.

Problem 6 Consider the autonomous system of first order ordinary differential equations

$$\mathbf{x}' = \mathbf{x}^{\perp} \tag{6}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^{\perp} = (-x_2, x_1)$ and the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{\perp}, & t \in \mathbb{R} \\ \mathbf{x}(0) = (1, 0). \end{cases} \tag{7}$$

(a) Find the general solution of (6).

(b) Plot the **image** $\{(x_1(t), x_2(t)) : 0 \leq t \leq 3\pi/2\}$ of the solution of (7).

(c) Plot the **graph** $\{(t, x_1(t), x_2(t)) : t \in \mathbb{R}\}$ of the solution of (7).

Recall the following existence and uniqueness theorem for ODEs:

Theorem 3 (existence and uniqueness for linear ODEs) If $A = (a_{ij}) : (a,b) \to \mathbb{R}^{n\times n}$ is a matrix valued function satisfying $a_{ij} \in C^0(a,b)$ for $i,j = 1,2,\dots,n$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) : (a,b) \to \mathbb{R}^n$ is a vector valued function satisfying $b_j \in C^0(a,b)$ for $j - 1, 2, \dots, n$, then for each $(t_0, \mathbf{p}) \in (a,b) \times \mathbb{R}^n$ the IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y} + \mathbf{b}, & t \in (a,b) \\ \mathbf{y}(t_0) = \mathbf{p} \end{cases}$$

has a unique solution $\mathbf{y} \in C^1((a,b) \to \mathbb{R}^n)$.

Problem 7 (linear ODEs)

(a) In what ways is Theorem 3 "weaker" than Theorem 2?

(b) In what ways is Theorem 3 "stronger" than Theorem 2?

Solution:

(a) The requirements of Theorem 3 on the structure of the equation are much more restrictive than those of Theorem 2 which makes the theorem "weaker." Specifically Theorem 3 only applies to systems $\mathbf{y}' = \mathbf{F}(\mathbf{y}, t)$ for which the structure function $\mathbf{F}$ has the very special form

$$\mathbf{F}(\mathbf{y}, t) = A\mathbf{y} + \mathbf{b}$$

with the coefficients a_{ij} and b_j all continous. In particular, $\mathbf{F} \in C^0(\mathbb{R}^n \times (a,b) \to \mathbb{R}^n)$ and

$$\frac{\partial F_i}{\partial y_j} = \frac{\partial}{\partial y_j}\left(\sum_{k=1}^{n} a_{ik} y_k\right) = a_{ij} \in C^0(a,b).$$

In particular $\partial F_i / \partial y_j$ is independent of $\mathbf{y}$ and

$$\frac{\partial F_i}{\partial y_j} \in C^0(\mathbb{R}^n \times (a,b)).$$

What we have shown here is that all the hypotheses of Theorem 2 are required (and rather much more) by the hypotheses of Theorem 3. In this sense, Theorem 3 is much weaker than Theorem 2.

(b) In contrast, the conclusion of Theorem 3 is much stronger than the conclusion of Theorem 2. Notice one obtains **global** existence and uniqueness valid for the entire interval of regularity (a, b) rather than merely a local subinterval $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. In this way, if you happen to have a linear ODE, Theorem 3 is much stronger than Theorem 2 in the sense that Theorem 2 gives a much stronger conclusion.

Problem 8 (linear ODEs) Consider the linear homogeneous IVP

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{0} \in \mathbb{R}^2.$$

(a) What is the maximal interval of existence for this IVP?

(b) Solve this IVP.

Solution:

(a) The coefficent functions are all constant valued functions, so one has existence (and uniqueness) for all $t \in \mathbb{R}$ giving a maximal interval of existence and uniqueness of

$$(T_1, T_2) = \mathbb{R} = (-\infty, \infty)$$

irrespective of initial condition(s).

(b) The constant solution with $y_1(t) \equiv 0$ and $y_2(t) \equiv 0$ or $\mathbf{y}(t) \equiv \mathbf{0} \in \mathbb{R}^2$ clearly solves the problem, so this is the unique solution given by Theorem 3.

There is no sweeping theory of existence and uniqueness for partial differential equations (PDE). Most people are introduced to the subject through consideration of particular examples of PDEs. At least this is a way to get some idea of what can happen if one starts to think about partial differential equations. Very often the first PDE a person considers is Laplace's equation

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Given an open set $\Omega \subset \mathbb{R}^n$, a natural set of functions on which to consider the Laplace operator Δ and this PDE is $C^2(\Omega)$. Then we can write

$$\Delta : C^2(\Omega) \to C^0(\Omega)$$

and the Laplacian Δ is linear. This is not a bad choice of a place to start.

One source for a rich family of solutions of $\Delta u = 0$ when $\Omega \subset \mathbb{R}^2$ is the collection of **complex differentiable functions**. See Problem 7 of Assignment 2.

Problem 9 Consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$.

(a) Write $z = x+iy$ and find the real and imaginary parts $u, v \in C^\infty(\mathbb{R}^2)$ of $f = u+iv$.

(b) Compute Δu and Δv.

Solution:

(a) $(x + iy)^2 = x^2 - y^2 + 2xyi$.

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy.$$

(b)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 0 = 0.$$

Notice of course that these are harmonic conjugates, so the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

are somewhat less trivial.

Problem 10 (hanging slinky) Interpret your relation from part **(b)** of Problem 1 above involving the stretch function σ and the linear density function ρ as an ordinary differential equation with two unknown functions σ and ρ.

(a) What else would you need in order to apply an existence and uniqueness theorem for ODEs to potentially determine the stretch σ and the linear mass density ρ?

(b) What natural restriction or constraint should apply to the derivative σ' of the stretch function and why?

(c) With a view to applying an existence and uniqueness theorem for ODEs to some ODEs for ρ and σ, what can you say about the potential boundary and/or initial values.