MATH 6702 Assignment 3
Due Monday March 8, 2021

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The following problems fall into four categories: calculus of variations, multivariable calculus, abstract spaces, and gradient flow.

Calculus of Variations

Problem 1 *(Boas 9.58-9) Sometimes in Newtonian particle mechanics, i.e., when using the equation* $\mathbf{f} = m\mathbf{a}$*, you find all the forces using some sort of force diagram and sum up the forces to write down the ODE. Now that you know the calculus of variations, there is a powerful method to write down forms of Newton's second law in very complicated situations, and with respect to coordinates, where it might not be so easy to determine all the forces.*

(a) *Look up* **Hamilton's principle of mechanics** *and state it in your own words using correctly the framework of the calculus of variations. Hint: You'll need to define a particular functional and mention the role played by extremals of the functional.*

(b) *Introduce appropriate time dependent variables/quantities to describe the motion of two point masses, one of which "falls" down the negative* z*-axis from the tip of of a cone*

$$\{(x, y, z) : z = \sqrt{x^2 + y^2}\}$$

and the other of which slides (without friction) on the surface of the cone, assuming the two point masses are connected with an inextensible (and massless) string of fixed positive length. Hint: See Example 3 of section 9.5 of Boas and use spherical coordinates for the point mass on the cone.

(c) *Use Hamilton's principle to write down the equations of motion, i.e., an appropriate initial value problem, in terms of your variables as Euler-Lagrange equations.*

(d) *Find some explicit solutions in an interesting special case—well, maybe not that interesting.*

(e) *Find and animate a numerical solution when both masses are assumed to be unit masses, the hanging mass starts from rest at the vertex of the cone, the string has length* $\sqrt{2}$*, and the mass on the cone starts at* $(1, 0, 1)$ *with velocity* $(0, 1, 0)$*.*

Problem 2 *(9.8.1; first integral for an autonomous Lagrangian) If* $\mathcal{F}$ *is a Lagrangian integral functional with Lagrangian* $F = F(z, p)$*, independent of the variable* $x \in (a, b)$*, then an extremal* $u \in C^2(a, b)$ *satisfies a first order ODE, called the* **first integral equation** *or* **Erdmann's equation***.*

(a) *Show that if* $u \in C^2(a, b)$ *is a weak extremal for* $\mathcal{F}$ *in the situation described above, then there exists a constant* $c \in \mathbb{R}$ *for which*

$$u' \frac{\partial F}{\partial p}(u, u') - F(u, u') = c.$$

This is Erdmann's equation. Hint: $E : C^1(a, b) \to C^0(a, b)$ *by* $E[u] = u' F_p(u, u') - F(u, u')$ *is called the Erdmann operator. Differentiate the Erdmann operator (with respect to* x*).*

(b) *Consider* $\sigma : C^1[a, b] \to \mathbb{R}$ *by*

$$\sigma[u] = \int_a^b (u'^2 + u^2 - u)\, dx.$$

Show that for this functional there exist solutions of Erdmann's equation which are **not** *extremals for* σ*. What does this tell you about the use of Erdmann's equation as a first integral of the Euler-Lagrange equation?*

Solution:

(a)

$$\frac{d}{dx}[u' F_p - F] = u'' F_p + u'(F_p)' - [u' F_z + u'' F_p] = u'[(F_p)' - F_z] = 0.$$

The derivative of F is computed using the chain rule:

$$\frac{d}{dx}F = \frac{\partial F}{\partial z}(u, u')\,\frac{du}{dx} + \frac{\partial F}{\partial p}(u, u')\,\frac{du'}{dx} = u'F_z + u''F_p.$$

The last equality is from the Euler-Lagrange equation $(F_p)' - F_z = 0$.

(b) C^2 weak extremals of

$$\sigma[u] = \int_a^b (u'^2 + u^2 - u)\,dx$$

satisfy the Euler-Lagrange equation

$$2u'' = 2u - 1.$$

This linear second order (nonhomogeneous) equation has solutions

$$u(x) = \frac{1}{2} + c_1 e^x + c_2 e^{-x}.$$

The first integral equation associated with this same functional is

$$2u'^2 - u'^2 - u^2 + u = u'^2 - u^2 + u = c.$$

This nonlinear autonomous equation admits constant solutions satisfying the algebraic condition $u^2 - u - c = 0$ or

$$u \equiv \frac{1}{2}(1 \pm \sqrt{1 + 4c}).$$

Since the Euler-Lagrange equation has only the constant solution $u \equiv 1/2$ (corresponding to $c = -1/4$, and we know every solution of the Euler-Lagrange equation is a solution of Erdmann's equation, we conclude that Erdmann's equation can have additional extraneous solutions. To be very specific, in this example taking $c = 0$, we find extraneous constant solutions $u \equiv 0$ and $u \equiv 1$.

This means that when we use the Erdmann first integral equation, we should (perhaps) be careful to go back and check the values of the functional on all solutions we obtain. Again, a strong physical intuition (and being correct about what to expect for minimizers) is usually important in this regard. There are more sophisticated techniques to determine precisely when the Erdmann equation will give you extra solutions, but in practice such considerations are often unnecessary. See the next problem for example.

Problem 3 *(hanging chain; Assignment 1 Problems 3 and 9) Let a, b, y_1 and y_2 be given real numbers with $a < b$. Also, let L be a constant with*

$$L > \sqrt{(b-a)^2 + (y_2 - y_2)^2}.$$

(a) *Use the method of Lagrange multipliers to obtain an Euler-Lagrange ODE satisfies by a minimizer $u \in \mathcal{A} = \{w \in C^2[a,b] : w(a) = y_1,\ w(b) = y_2\}$ of the functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ by*

$$\mathcal{F}[w] = \int_a^b w\sqrt{1 + w'^2}\, dx$$

subject to the constraint

$$\mathcal{L}[w] = \int_a^b \sqrt{1 + w'^2}\, dx = L.$$

(b) *The equation given by Boas to model the shape of a hanging chain is/was*

$$(y'')^2 = k^2[1 + (y')^2].$$

Is this your Euler-Lagrange equation? If not, can you say which equation provides a better model? Are they somehow the same? Hint: You may wish to use the previous problem.

Solution:

(a) The method of Lagrange multpliers applied to the hanging chain problem involves the consideration of the augmented functional

$$\mathcal{F} - \lambda\mathcal{L} = \int_a^b (u - \lambda)\sqrt{1 + u'^2}\, dx.$$

The Euler-Lagrange equation associated to this functional is

$$\left((u - \lambda)\frac{u'}{\sqrt{1 + u'^2}}\right)' = \sqrt{1 + u'^2}.$$

As we know, $u'/\sqrt{1 + u'^2}$ is the sine of the inclination angle and

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' = \frac{u''}{(1 + u'^2)^{3/2}}$$

is the curvature of the graph of u. Thus, using the product rule we get

$$\frac{u'^2}{\sqrt{1+u'^2}} + (u-\lambda)\frac{u''}{(1+u'^2)^{3/2}} = \sqrt{1+u'^2}.$$

Multiplying by the length scaling $\sqrt{1+u'^2}$, we find a simplification:

$$(u-\lambda)\frac{u''}{1+u'^2} = 1. \tag{1}$$

This equation is (or might be) somewhat tricky to solve as it stands, though there is an obvious nominal simplifcation obtained by writing $v = u - \lambda$ so that the Lagrange multiplier is temporarily eliminated and we face

$$\frac{v''}{1+v'^2} = \frac{1}{v}.$$

Multiplying both sides by v', we find happily

$$\frac{1}{2}\frac{d}{dx}\ln(1+v'^2) = \frac{d}{dx}\ln v$$

assuming, at least, that $v > 0$. Let us assume the cases $v = u - \lambda = 0$ and $v < 0$ can be handled separately. Proceeding to integrate we find

$$\sqrt{1+v'^2} = \alpha v \qquad \text{and} \qquad v' = \pm\sqrt{\alpha^2 v^2 - 1}.$$

With another change of variables $w = \alpha v$, we can take (for example)

$$\frac{w'}{\sqrt{w^2-1}} = \alpha$$

and

$$\cosh^{-1} w = \alpha x + c \qquad \text{and} \qquad v = \frac{1}{\alpha}\cosh(\alpha x + c).$$

That is,

$$u(x) = \frac{1}{\alpha}\cosh(\alpha x + c) + \lambda.$$

It may be noted that this is essentially the same solution obtained from Boas' equation

$$y'' = c\sqrt{1+y'^2}$$

in Problem 3 of Assignment 1. The Euler-Lagrange equation, however, seemed quite different from this one. Where did Boas get her equation?

(b) Erdmann's equation for the augmented functional above (which is, for all its faults, autonomous) is given by

$$(u-\lambda)\frac{u'^2}{\sqrt{1+u'^2}} - (u-\lambda)\sqrt{1+u'^2} = c.$$

A simplification in this first order equation occurs immediately giving

$$(u-\lambda) = -c\sqrt{1+u'^2}.$$

Substututing this in (1) for $u-\lambda$, we find

$$-c\frac{u''}{\sqrt{1+u'^2}} = 1 \qquad \text{or} \qquad u'' = -\frac{1}{c}\sqrt{1+u'^2}$$

which is essentially Boas' equation. Since we know Erdmann's equation may admit extraneous solutions which are not solutions of the Euler-Lagrange equation, and we have used Erdmann's equation to get Boas' equation, we might imagine we will find extraneous solutions. We know this is not the case, however, because we have solved Boas' equation, and it gives just exactly the same solutions.

The two ODEs are essentially equivalent as models for the shape of a hanging chain, though Boas' equation is perhaps a bit easier (or at least less complicated) to solve. The solutions are the same.

As something of a side note: The real work actually begins in determining the parameters/constants in $u = c\cosh[(x-\mu)/c] + \lambda$ in terms of the endpoints (a, y_1) and (b, y_2) and the prescribed length L. Perhaps since our primary goal is to illustrate the techniques of the calculus of variations (rather than model hanging chains) we can set this determination of constants aside, and leave it for another time. But if you're interested in this particular question, I can probably suggest something for you to read.

Problem 4 *(Poisson's equation) Let $\mathcal{U}$ be an open bounded subset of $\mathbb{R}^2$.*

(a) *Given a point $\mathbf{p} \in \mathcal{U}$ and an open ball $B_\delta(\mathbf{p}) \subset \mathcal{U}$, find a function $\phi \in C_c^\infty(\mathcal{U})$ with $\phi \ge 0$ and $\phi(\mathbf{x}) > 0$ precisely for $\mathbf{x} \in B_\delta(\mathbf{p})$.*

(b) *State and prove a version of the fundamental lemma of the calculus of variations for continuous functions $f \in C^0(\mathcal{U})$.*

(c) *Find the partial differential equation satisfied by extremals* $u \in C^2(\mathcal{U})$ *of the Dirichlet energy*

$$\mathcal{D}[u] = \int_{\mathcal{U}} |Du|^2$$

subject to the constraint

$$\mathcal{V}[u] = \int_{\mathcal{U}} u = V.$$

Multivariable Calculus

Recall that continuity on a function $f : X \to Y$ with domain a metric space X and co-domain a metric space Y is defined as follows:

A function $f : X \to Y$ is **continuous** at $x_0 \in X$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \epsilon \qquad \text{whenever } x \in B_\delta(x_0).$$

The function is said to be **continous on** X is f is continuous at each point $x_0 \in X$. In this case we write $f \in C^0(X \to Y)$. That is, $C^0(X \to Y)$ is the collection of all continuous functions from the metric space X to the metric space Y. In the special case where $Y = \mathbb{R}$ and f is a real valued function, we write simply $f \in C^0(X)$.

Problem 5 *(differentiability implies continuity) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x_0 \in (a, b)$, then f is continuous at x_0.*

Recall that the partial derivatives of a function $u : \mathcal{U} \to \mathbb{R}$ defined on an open set $\mathcal{U} \subset \mathbb{R}^2$ are defined at $\mathbf{x} \in \mathcal{U}$ by

$$\frac{\partial u}{\partial x}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_1) - u(\mathbf{x})}{h} \qquad \text{and} \qquad \frac{\partial u}{\partial y}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_2) - u(\mathbf{x})}{h}$$

if these limits exist. Remember that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. If the partial derivatives of a function of several variables exists, we say the function is **partially differentiable**.

Problem 6 *(partial differentiability and continuity)*

(a) *Give an example of a function $u : \mathbb{R}^2 \to \mathbb{R}$ for which both partial derivatives exist (at all points in $\mathbb{R}^2$) but the function is discontinuous at at least one point.*

(b) *A partially differentiable function $u : \mathcal{U} \to \mathbb{R}$ is said to be* **differentiable** *at $\mathbf{p} \in \mathcal{U}$ if there exists a linear function $L : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{|\mathbf{x} - \mathbf{p}|} = 0.$$

Show that a function $u : \mathcal{U} \to \mathbb{R}$ which is differentiable at $\mathbf{p}$ is continuous at $\mathbf{p}$.

(c) *Give an example of a function which is partially differentiable but not differentiable. Hint: Look at part (a) and part (b).*

(d) *If a function* $u : \mathcal{U} \to \mathbb{R}$ *is differentiable at a point* $\mathbf{p} \in \mathcal{U}$*, then the linear function* $L : \mathbb{R}^2 \to \mathbb{R}$ *given in the definition is called the differential of* u *at* $\mathbf{p}$*. Show the differential is given by the dot product with the vector of first partials:*

$$L(\mathbf{v}) = du_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \left(\frac{\partial u}{\partial x}(\mathbf{p}), \frac{\partial u}{\partial y}(\mathbf{p}) \right).$$

The vector of first partials

$$\left(\frac{\partial u}{\partial x}(\mathbf{p}), \frac{\partial u}{\partial y}(\mathbf{p}) \right)$$

in this case is called the **gradient** *or* **total derivative** *and is denoted by* $Du = Du(\mathbf{p})$*. Note that* $Du : \mathcal{U} \to \mathbb{R}^2$*. This makes* Du *a vector field on* $\mathcal{U}$*.*

(e) *A function* $u : \mathcal{U} \to \mathbb{R}$ *where* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^2$ *is said to be in* $C^1(\mathcal{U})$ *if the partial derivatives satisfy*

$$\frac{\partial u}{\partial x} \in C^0(\mathcal{U}) \qquad \text{and} \qquad \frac{\partial u}{\partial y} \in C^0(\mathcal{U}).$$

Show that if $u \in C^1(\mathcal{U})$*, then* u *is differentiable at each point in the open set* $\mathcal{U}$*.*

(f) *(directional derivatives) Given a function* $u \in C^1(\mathcal{U})$ *with* $\mathcal{U}$ *an open subeset of* $\mathbb{R}^2$ *and a unit vector* $\mathbf{u} \in \mathbb{R}^2$ *the* **directional derivative** *of* u *in the direction* $\mathbf{u}$ *at* $\mathbf{p} \in \mathcal{U}$ *is defined by*

$$D_{\mathbf{u}} u(\mathbf{p}) = \lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{u}) - u(\mathbf{p})}{h}.$$

Assume $u \in C^1(\mathcal{U})$ *as above with* $\mathbf{p} \in \mathcal{U}$ *and* $\mathbf{u}$ *a unit vector in* $\mathbb{R}^2$*. Use the chain rule to show the following*

(i) *If* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ *by* $\mathbf{r}(t) = \mathbf{p} + t\mathbf{u}$*, then*

$$D_{\mathbf{u}} u(\mathbf{p}) = \frac{d}{dt} u \circ \mathbf{r}(t)\Big|_{t=0}.$$

(ii) $D_{\mathbf{u}} u(\mathbf{p}) = Du(\mathbf{p}) \cdot \mathbf{u}$*.*

(i) *If* $\mathbf{r} : (a, b) \to \mathcal{U}$ *is any path in* $\mathcal{U}$ *satisfying* $\mathbf{r}(0) = \mathbf{p}$ *and* $\mathbf{r}'(0) = \mathbf{u}$*, then*

$$D_{\mathbf{u}} u(\mathbf{p}) = \frac{d}{dt} u \circ \mathbf{r}(t)\Big|_{t=0}.$$

Problem 7 *(Boas 5.2.22) Use an area integral to find the volume above the triangle with vertices* $(0, 2, 0)$*,* $(1, 1, 0)$*, and* $(2, 2, 0)$ *but below the graph of* $u(x, y) = xy$*.*

Problem 8 *(Lagrange multipliers) Let* c *be a constant, and let* $u, g \in C^1(\mathcal{U})$ *with* $\mathcal{U} \subset \mathbb{R}^2$*. If* $\mathbf{p} \in \mathcal{U}$ *is a point for which* $g(\mathbf{p}) = c$ *and we have* $u(\mathbf{p}) \leq u(\mathbf{x})$ *for all* $\mathbf{x} \in \mathcal{U}$ *with* $g(\mathbf{x}) = c$*, then show one of the two following conditions holds:*

(i) $Dg(\mathbf{p}) = \mathbf{0}$*, or*

(ii) *There exists some* $\lambda \in \mathbb{R}$ *with*

$$D(u - \lambda g)(\mathbf{p}) = \mathbf{0}.$$

Problem 9 *(weak maxiumum principle) Given a function* $u \in C^2(\mathcal{U})$ *where* $\mathcal{U}$ *is an open set in* $\mathbb{R}^2$*, define the* **Hessian matrix** *to be the matrix of second partial derivatives of* u*:*

$$D^2u = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x \partial y} \\ & \\ \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial^2 u}{\partial y^2} \end{pmatrix}.$$

Given a unit vector $\mathbf{u} \in \mathbb{R}^2$ *and a point* $\mathbf{p} \in \mathcal{U}$*, the* **second directional derivative** *of* u *at* $\mathbf{p}$ *in the direction* $\mathbf{u}$ *is defined by*

$$D_{\mathbf{uu}}u(\mathbf{p}) = \frac{d^2}{dt^2} u(\mathbf{p} + t\mathbf{u})\Big|_{t=0}.$$

(a) *If* $u, v \in C^2(\mathcal{U})$ *with* $u(\mathbf{x}) \leq v(\mathbf{x})$ *for every* $\mathbf{x} \in \mathcal{U}$ *and* $u(\mathbf{p}) = v(\mathbf{p})$ *for some* $\mathbf{p} \in \mathcal{U}$*, then Show*

$$Du(\mathbf{p}) = Dv(\mathbf{p}) \qquad \text{and} \qquad \langle D^2u(\mathbf{p})\mathbf{w}, \mathbf{w}\rangle \leq \langle D^2v(\mathbf{p})\mathbf{w}, \mathbf{w}\rangle \quad \textit{for every } \mathbf{w} \in \mathbb{R}^2.$$

(b) *Show the* **second directional derivative** *in the unit direction* $\mathbf{u}$ *is given by*

$$D_{\mathbf{uu}}u(\mathbf{p}) = \langle D^2u(\mathbf{p})\mathbf{u}, \mathbf{u}\rangle.$$

(c) *Show that if* A *is a bounded subset of* $\mathbb{R}^2$*, then* $\overline{A}$ *is bounded in* $\mathbb{R}^2$*.*

(d) *Let $\mathcal{U}$ be an open bounded subset of $\mathbb{R}^2$. Show that if $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ is a solution of Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{on } \mathcal{U},$$

then

$$u(\mathbf{p}) \le \max_{x \in \partial\mathcal{U}} u(\mathbf{x}) \qquad \text{for all } \mathbf{p} \in \mathcal{U}.$$

Hint: Assume there is some $\mathbf{p}_0 \in \mathcal{U}$ with

$$u(\mathbf{p}_0) > \max_{x \in \partial\mathcal{U}} u(\mathbf{x}),$$

then consider an appropriate vertical translation of $v(x, y) = -\epsilon|\mathbf{x} - \mathbf{p}|^2$. Calculate Δv.

Gradient Flow

Problem 10 *Let $u \in C^1(\mathcal{U})$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, and define the* **gradient flow** *or* **gradient descent** *determined by u as the collection of initial value problems*

$$\begin{cases} \mathbf{x}' = -Du(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{p} \end{cases}$$

determined by the points $\mathbf{p} \in \mathcal{U}$.

(a) *Let $\mathbf{p} \in \mathcal{U}$ be fixed. Use the properties of the Euclidean inner product to find the minimum value of the directional derivative*

$$D_{\mathbf{u}} u(\mathbf{p})$$

which can be attained by taking $\mathbf{u} \in \mathbb{R}^2$ with $|\mathbf{u}| = 1$.

(b) *Find (and solve) all gradient flows determined by $u(x, y) = \sin x + \cos y$.*

Abstract Spaces Recall that we discussed metric spaces and normed spaces last semester. Here is another kind of abstract space with even more structure.

Problem 11 *(inner product space) Given a vector space V, a function*

$$\langle \cdot\, ,\, \cdot \rangle : V \times V \to \mathbb{R}$$

is called a **real inner product** *if the following conditions hold*

(i) $\langle v, w\rangle = \langle w, v\rangle$ *for all* $v, w \in V$. *(symmetric)*

(ii) $\langle av + bw, z\rangle = a\langle v, z\rangle + b\langle w, z\rangle$ *for all* $v, w, z \in V$ *and* $a, b \in \mathbb{R}$. *(bilinearity)*

(iii) $\langle v, v\rangle \geq 0$ *with equality if and only if* $v = \mathbf{0}$. *(positive definite)*

In words, an inner product is a bilinear, symmetric positive definite function on a vector space. A vector space with an inner product is called an **inner product space**.

(a) *Show every inner product space is a normed space with norm given by*

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

If you do not know (or do not remember) the definition of a normed space, you'll need to look it up.

(b) *Show the Cauchy-Schwarz inequality holds in any inner product space:*

$$|\langle v, w\rangle| \leq \|v\|\,\|w\| \qquad \text{for all } v, w \in V,$$

with equality if and only if v is parallel to w, that is to say, either $w = \mathbf{0}$ or there is some scalar λ for which $v = \lambda w$. Hint: Compute $\|v + \lambda w\|^2$ and then complete the square with respect to λ.

(c) *Show* $\langle \cdot\, ,\, \cdot \rangle : C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ *by*

$$\langle u, v\rangle = \int_{\mathbb{R}^n} uv$$

defines an inner product on $C_c^\infty(\mathbb{R}^n)$.